# Geometric aspects of integrable systems

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**Abstract**

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## Contents

# 1 Preliminaries

One of the branches of contemporary mathematical physics that has been widely investigated in the last three decades is the theory of integrable nonlinear dynamical systems. The infinite-dimensional systems of that type are described by nonlinear partial differential equations with the distinguished independent variable – the evolution (time) parameter. The origin of the theory dates back to the 19th century when the famous Korteweg-de Vries (KdV) equation was derived for the description of long solitary waves in the shallow water. However, significant progress in the development of theory was made only at the turn of the sixties and seventies of the last century when the pioneering articles by Lax [28], Gardner, Greene, Kruskal, Miura [22] and Zakharov, Shabat [53] appeared. In these articles the inverse scattering transform method was introduced. This method is closely related to the so-called Lax representations of integrable field and lattice soliton systems characterized by infinite hierarchies of symmetries and conservation laws. In 1978 Magri [31] introduced a remarkable concept of bi-Hamiltonian structures for integrable systems. From the geometrical point of view, it means that there exists a pair of compatible Poisson tensors and on the basis of a recursion chain one can generate infinite (in the infinite-dimensional case) hierarchy of constants of motion, being in involution with respect to the above Poisson tensors and commuting symmetries.

Nonlinear evolution equations describe many physical phenomena. However, most of the former manifest chaotic behaviour. The study of integrable nonlinear systems is of particular importance for understanding at least some aspects of nonlinear equations. Integrable systems are interesting not only on their own right but also because they yield exact solutions for many problems of very advanced modern mathematics and theoretical physics like topological quantum field theories, Gromov-Witten invariants and quantum cohomology, string theory, etc. Of course, there is feedback from these fields to the theory of integrable systems. All these fields of research have been developed for a relatively short time and there are still many intriguing open problems. It may be worth mentioning that Witten and Kontsevich were awarded the Fields Medals in 1990 and 1998, respectively, for the results closely related to these problems.

## 1.1 Infinite-dimensional evolution systems

In these lecture notes we are going to deal with the systems of partial differential equations (PDE's) of the form

$$\mathbf{u}_t = \mathbf{K}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \dots), \tag{1.1}$$

where $\mathbf{u} := (u_1, u_2, \dots, u_N)^{\mathrm{T}}$ is an $N$-tuple of unknown smooth functions of independent variables $x$ and $t$ taking values in a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Here and below the respective subscripts denote partial derivatives, i.e.,

$$u_x := \frac{\partial u}{\partial x} \qquad u_{2x} \equiv u_{xx} := \frac{\partial^2 u}{\partial x^2} \qquad \ldots .$$

We will understand the system (1.1) as an equation for the flow (or integral curve) on some formal infinite-dimensional smooth manifold. Then the right-hand side of (1.1) represents a vector field on this manifold and $t$ is a formal 'evolution' parameter (time) that belongs to some subinterval of $\mathbb{R}$. In this terminology $x$ is a spatial variable from the space that must be appropriately specified. Hence, (1.1) represents a (1+1)-dimensional evolution system.

We will be interested in the equations (1.1) that are nonlinear and integrable. Nonlinearity here means that right-hand side of (1.1) depends on the variables $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \ldots$ in a nonlinear way, i.e.

$$\mathbf{K}(\lambda\mathbf{u} + \mu\mathbf{v}, \lambda\mathbf{u}_x + \mu\mathbf{v}_x, \ldots) \neq \lambda\mathbf{K}(\mathbf{u}, \mathbf{u}_x, \ldots) + \mu\mathbf{K}(\mathbf{v}, \mathbf{v}_x, \ldots) + \ldots \qquad \lambda, \mu \in \mathbb{K}.$$

Notice that for the nonlinear systems the principle of linear superposition for solutions is violated. The concept of integrability is more vague, so we will get back to it lather.

The following Sections (1.2) and (1.3) are a formal introduction to the subject, so we assume existence of all the functions and that they are differentiable appropriate many times.

## 1.2 Wave phenomena

We will now explain some notions used in the theory of integrable systems on the basis of wave phenomena. We will restrict ourselves to the simplest case where we have one wave amplitude $u(x,t)$ in two-dimensional time-space.

### 1.2.1 Linear wave equation

The linear wave equation has the form

$$u_{tt} - c^2 u_{xx} = 0, \tag{1.2}$$

where $c$ is the phase velocity. Eq.(1.2) has a general solution

$$u(x,t) = f(x - ct) + g(x + ct)$$

being sum of two waves propagating to the right and left with constant speed $c$, respectively. The main property of (1.2) is that the shape of wave is preserved and that the linear superposition principle is valid. This means that the sum of two arbitrary solutions is again a solution.

One can factorize (1.2) as

$$(\partial_t \pm c\partial_x)(\partial_t \mp c\partial_x)u = 0.$$

Then the propagation to the right is given by first order PDE

$$u_t + u_x = 0, \tag{1.3}$$

where we rescaled the independent variables in a way to have unit phase velocity, i.e., $c = 1$, and then the general solution is $u(x,t) = f(x - t)$. Nevertheless, let us assume the solution in the form

$$u(x,t) = \exp(i(kx - \omega t)), \tag{1.4}$$

where $k$ is a so-called wave vector (in this case one-dimensional) and $\omega$ is a frequency. Then, we find the so-called dispersion relation for (1.3):

$$\omega(k) = k.$$

Hence, the phase velocity defined as

$$c := \frac{\omega}{k} = 1$$

is obviously constant.

### 1.2.2 Dissipation process

Let us consider the heat equation

$$u_t - u_{xx} = 0.$$

For the solution (1.4) one finds the dispersion relation

$$\omega(k) = -ik^2.$$

Hence,

$$u = \exp(i(kx - \omega t)) = \exp(ikx)\exp(-k^2 t)$$

is a solution with the amplitude decaying exponentially with time. This is known as a dissipation process, physically this means that the energy is not conserved and dissipates from the (physical) system.

### 1.2.3 Dispersion phenomenon

Now, let us take the Airy equation which is a third-order PDE of the form

$$u_t + u_{xxx} = 0.$$

For (1.4) we have

$$u_t + u_{xxx} = (i\omega - ik^3)u = 0 \qquad \Longleftrightarrow \qquad \omega = k^3$$

and thus

$$u = \exp(i(kx - k^3 t)).$$

The phase velocity $c = \frac{\omega}{k} = k^2$ is nonlinear in $k$. This means that the waves with different frequencies disperse with different velocities. This is known as a *dispersion*.

The general solution then can be given in the form of a wave packet

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(k)\exp(i\,[kx - \omega(k)t])dk,$$

where $A(k)$ is a Fourier transform of $u(x,0)$, i.e.

$$A(k) = \int_{-\infty}^{+\infty} u(x,0)\exp(-ikx)dx.$$

The group velocity, i.e., the velocity of energy propagation, is given by

$$c_g = \frac{d\omega}{dk}.$$

### 1.2.4 Wave breaking

Let us analyse the simplest possible nonlinear equation

$$u_t = uu_x$$

and postulate the solution in the implicit form

$$u(x,t) = v(x + u(x,t)t),$$

where $v$ is arbitrary smooth function. Thus, we have

$$u_t = (u_t t + u)v' \qquad u_x = (1 + u_x)v'$$

and so

$$(u_t t + u)v' = u(1 + u_x)v' \quad \Longleftrightarrow \quad u_t = uu_x.$$

Hence, existence of such an implicit wave solution means that the velocity of a point of the wave is proportional to the amplitude of this point. The higher the amplitude is, the faster the point moves. Hence, this leads to the phenomenon known as the 'breaking' of the wave. Moreover, nonlinear terms in most cases cause chaotic behavior.

## 1.3 Completely integrable systems

### 1.3.1 The Korteweg-de Vries equation

The famous Korteweg-de Vries equation (KdV) has the form

$$u_t = u_{3x} + 6uu_x. \tag{1.5}$$

This is the best known and simplest example of a completely integrable (1+1)-dimensional dispersive equation. Complete integrability means here (not quite precisely) that it can be solve for almost all arbitrary boundary conditions. The KdV equation was originally deduced for the description of long solitary waves, moving in one direction, in the shallow water. Notice that the constant coefficients at $u_{3x}$ and $6uu_x$ in (1.5) can be made completely arbitrary using rescaling of dependent and independent variables. However, this does not typically apply to more general integrable equations.

### 1.3.2 Solitons

We have already seen that the term $u_{3x}$ is responsible for the dispersion and the nonlinear therm $uu_x$ for the wave breaking phenomenon. In general when such two effects meet then the chaotic behaviour appears. The miracle in the case of KdV equation and similar completely integrable systems is that both of these effects, i.e., the dispersion and the wave breaking, compensate each other allowing for the solutions describing the combination of the solitary waves.

Let us look for a traveling-wave solution for (1.5) of the form

$$u(x,t) = v(x+ct) \qquad s := x+ct.$$

Then, assuming rapidly decreasing boundary conditions $u, u_, u_{2x}, \ldots \to 0$ for $|x| \to \infty$, one finds

$$\begin{aligned}
&u_t - u_{3x} + 6uu_x = 0 \quad \Longrightarrow \quad cv_s - v_{3s} - 6vv_s = 0 \quad \Longrightarrow \\
&v_s \cdot \big|\ cv - v_{2s} - 3v^2 = const = 0 \quad \Longrightarrow \quad cvv_s - v_s v_{2s} - 3v^2 v_s = 0 \quad \Longrightarrow \\
&\frac{1}{2}cv^2 - \frac{1}{2}{v_s}^2 - v^3 = const = 0 \quad \Longrightarrow \quad {v_s}^2 = cv^2 - 2v^3 = v^2(c-2v),
\end{aligned}$$

where we integrated several times with respect to $s$. As result we reduced the problem to an easy-to-integrate ordinary differential equation

$$\begin{aligned}
\frac{dv}{ds} = \pm v\sqrt{c-2v} \quad \Longrightarrow \quad & s = \mp\frac{2}{\sqrt{c}} \tanh^{-1}\left(\sqrt{1-\frac{2}{c}u}\right) \quad \Longrightarrow \\
& u = \frac{c}{2}\left[1 - \tanh^2\left(\frac{\sqrt{c}}{2}s\right)\right].
\end{aligned}$$

Now, using the relation $\operatorname{sech}^2\xi = 1 - \tanh^2\xi$ one finds the so-called one-soliton solution of the KdV equation

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x+ct)\right]$$

describing the solitary wave. The solitary wave is a single hump traveling in time, with constant speed proportional to amplitude, without changing the shape. The KdV and similar completely integrable systems also possess the so-called $N$-soliton solutions, that for $t \to \pm\infty$ decompose asymptotically into linear sum of $N$ solitary waves called solitons. In the finite time these solitons interact and, amazingly, they eventually recover their shape after collisions. The result of interaction (collision) between two solitons is only the phase transition. This particle-like behavior is responsible for the name of the $N$-soliton solutions.

### 1.3.3 Lax equations and isospectral problem

One of the most characteristic features of integrable systems is that one can associate with them the so-called Lax equation

$$L_t = [A, L] \tag{1.6}$$

in some algebra with a Lie bracket $[\cdot,\cdot]$.

Assume that we have two linear equations

$$L\psi = \lambda\psi \tag{1.7}$$

$$\psi_t = A\psi, \tag{1.8}$$

where $L, A$ are linear operators in some Hilbert space and $\psi$ is an eigenfunction, $\lambda$ ($\lambda_t = 0$) is an eigenvalue (spectral parameter). The first equation represents the spectral equation for $L$ and the second one defines the evolution of the eigenfunction $\psi$. Differentiating (1.7) with respect to $t$

$$L_t\psi + L\psi_t = \lambda\psi_t$$

and applying (1.8) we have

$$L_t\psi + LA\psi = \lambda A\psi = AL\psi \quad\Longleftrightarrow\quad (L_t - [A, L])\,\psi = 0,$$

where $[A, L] = AL - LA$ is the commutator. Thus, the compatibility of the linear equations (1.7) and (1.8) yields the Lax equation (1.6).

For the KdV equation we can take

$$L = \partial_x^2 + u \tag{1.9}$$

and

$$A = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x, \tag{1.10}$$

where $u = u(x, t)$ is a smooth dynamical field. The symbol $\partial_x$ means a differential operator whose action on arbitrary smooth function $v$, due to the Leibniz rule, is

$$\partial_x v = v\partial_x + v_x.$$

Thus we have that

$$L_t = u_t \quad \text{and} \quad [A, L] = \frac{1}{4}u_{3x} + \frac{3}{2}uu_x$$

and hence

$$L_t = [A, L] \quad\Longleftrightarrow\quad u_t = \frac{1}{4}u_{3x} + \frac{3}{2}uu_x.$$

For a given system (1.1) the main objective is to solve the Cauchy problem with well-posed boundary and/or initial conditions, so that a unique exact solution $u(x, t)$ exists. Of key importance in solving the Cauchy problem for completely integrable systems that possess appropriate Lax pairs (1.7-1.8) is the so-called inverse scattering transform method. This method is typical to particle physics and has its origin in quantum field theory. In the case of the KdV equation the linear problem (1.7) for (1.9) is nothing but the stationary Schrödinger equation in quantum mechanics, where the dynamical field $u(x,t)$ plays a role of the potential and $\lambda$ of the energy. The method is illustrated on the following diagram.

$$\begin{tikzcd}
u(x,t=0) \arrow[r, "\text{direct scattering}"] \arrow[d] & \psi(x\to\infty,t=0) \arrow[d, "\text{time evolution}"] \\
u(x,t>0) & \psi(x\to\infty,t>0) \arrow[l, "\text{inverse scattering}"]
\end{tikzcd}$$

First one has to employ (1.7) to compute the scattering data, i.e., the asymptotic $\psi(x \to \infty, t = 0)$, for the initial potential $u(x,t=0)$. Then one determines from (1.8) their time evolution $\psi(x \to \infty, t > 0)$. Finally, applying to (1.7) the inverse scattering method (this is the difficult part), one computes the potential $u(x,t>0)$ from the scattering data $\psi(x \to \infty, t > 0)$. In general the inverse scattering transform method, when applicable, leads to wide classes of solutions for completely integrable systems. Nevertheless, in practice the calculations are very complex, even in the case of simplest equations like KdV, and one can only find restricted classes of solutions in the explicit form. For example, the discrete spectrum of (1.7) in the case of KdV equation leads to multi-soliton solutions.

### 1.3.4 Symmetries and constants of motion

Another characteristic feature of completely integrable systems is the existence of wide classes of symmetries and constants of motion.

Informally, a symmetry of dynamical system (1.1) is a one-parameter group of transformation $\phi_\epsilon$, such that if $\mathbf{u}$ is an arbitrary solution then $\phi_\epsilon \mathbf{u}$ is also a solution of the same equation. In other words, this means that $\phi_\epsilon$ maps arbitrary integral curve $u(x,t)$ onto another integral curve $u'(x,t) = \phi_\epsilon u(x,t)$ of (1.1).

Actually we are interested in the symmetries that can be completely determined by their infinitesimal generators which in turn can be identified with the right-hand sides of the dynamical systems of the form

$$\mathbf{u}_\tau = \sigma(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \dots). \tag{1.11}$$

What does it mean that (1.1) has a symmetry of the form (1.11)? Assume that a solution $u(x,t,\tau=0)$ of (1.1) is an initial condition for (1.11). Solving (1.11) one gets $u(x,t,\tau=\delta\tau)$ at some time $\tau = \delta\tau$. Now, if (1.11) is a symmetry of (1.1), then $u(x,t,\tau=\delta\tau)$ must be another solution of (1.1).

Consider an arbitrary common initial condition $u(x,t=0,\tau=0)$. If (1.11) is a symmetry of (1.1) then the following diagram

$$\begin{tikzcd}
u(x,t=0,\tau=0) \arrow[r, "(1.1)"] \arrow[d, "(1.11)"'] & u(x,t=\delta t,\tau=0) \arrow[d, "(1.11)"] \\
u(x,t=0,\tau=\delta\tau) \arrow[r, "(1.1)"] & u(x,t=\delta t,\tau=\delta\tau)
\end{tikzcd}$$

must be commutative. Thus, from the initial condition one can get to $u(x,t=\delta t,\tau=\delta\tau)$ in two ways. Passing to the case when $\delta t$, $\delta\tau$ are infinitesimally small and considering the corresponding Taylor expansions one finds that the above diagram is commutative whenever

$$\frac{\partial \mathbf{K}}{\partial \tau} = \frac{\partial \sigma}{\partial t} \qquad \Longleftrightarrow \qquad (\mathbf{u}_t)_\tau = (\mathbf{u}_\tau)_t$$

holds. This means that the vector fields $\mathbf{K}$ and $\sigma$ commute. Notice that the symmetry relation is reflexive, so if $\sigma$ is symmetry for $\mathbf{K}$, then so is $\mathbf{K}$ for $\sigma$.

Let us show that the KdV equation $u_t = \mathbf{K} = u_{3x} + 6uu_x$ has a symmetry $u_\tau = \sigma = u_x$. Thus

$$\frac{\partial \mathbf{K}}{\partial \tau} = (u_{3x} + 6uu_x)_\tau = (u_\tau)_{3x} + 6u_\tau u_x + 6u(u_\tau)_x = u_{4x} + 6u_x^2 + 6uu_{2x}$$

and

$$\frac{\partial \sigma}{\partial t} = (u_x)_t = (u_t)_x = u_{4x} + 6{u_x}^2 + 6uu_{2x}.$$

This means that the KdV equation is invariant under the translations of the spatial variable since

$$\sigma = \left.\frac{d\phi_\epsilon u}{d\epsilon}\right|_{\epsilon=0} = u_x \quad \text{where} \quad \phi_\epsilon u(x,t) = u(x+\epsilon, t).$$

Actually, the KdV equation possesses a hierarchy of infinitely many pairwise commuting symmetries:

$$\begin{aligned}
u_{t_1} &= u_x \\
u_{t_3} &= \frac{1}{4}\left(u_{3x} + 6uu_x\right) \\
u_{t_5} &= \frac{1}{16}\left(u_{5x} + 10uu_{3x} + 20u_xu_{2x} + 30u^2u_x\right) \\
u_{t_7} &= \frac{1}{64}\big(u_{7x} + 14uu_{5x} + 42u_xu_{4x} + 70u_{2x}u_{3x} \\
&\qquad + 70u^2u_{3x} + 280uu_xu_{2x} + 70{u_x}^3 + 140u^3u_x\big) \\
&\ \ \vdots
\end{aligned} \tag{1.12}$$

This hierarchy is called the KdV hierarchy because of the first nontrivial member thereof. This is a common feature of completely integrable systems: we have not a single integrable system but a whole family of pairwise commuting completely integrable systems.

We say that a scalar field given by the functional

$$F = \int_\Sigma f(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \ldots)\, dx$$

(where we assumed boundary conditions such that the integral of total derivative with respect to $x$ vanish) is a conserved quantity for a vector field (1.1) if

$$\frac{dF}{dt} = \int_\Sigma \frac{df}{dt}\, dx = 0,$$

where $t$ is the evolution parameter related to (1.1). This means that $F$ is constant along the integral curve of (1.1). For this reason the conserved quantity is often referred to as an integral (or a constant) of motion.

In addition to the above hierarchy of symmetries the KdV equation also has an infinite hierarchy of conserved quantities:

$$\begin{aligned}
H_{-1} &= \int_\Sigma u\ dx \\
H_1 &= \int_\Sigma \frac{1}{4}u^2\ dx \\
H_3 &= \int_\Sigma \frac{1}{16}\left(2u^3 - {u_x}^2\right) dx \\
H_5 &= \int_\Sigma \frac{1}{64}\left(5u^4 - 10u{u_x}^2 + u_{2x}^2\right) dx \\
H_7 &= \int_\Sigma \frac{1}{256}\left(14u^5 - 70u^2{u_x}^2 + 14uu_{2x}^2 - u_{3x}^2\right) dx \\
&\ \vdots\ .
\end{aligned} \tag{1.13}$$

Actually, all the functionals from the above hierarchy are constants of motion for all members of the KdV hierarchy.

### 1.3.5 Complete integrability

The notion of complete integrability is best understood in the case of finite-dimensional dynamical systems. Consider the so-called Hamiltonian equations, having origin in the classical mechanics,

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad i = 1, \dots, N, \tag{1.14}$$

living in $2N$-dimensional phase space, where the function $H = H(p, q)$ is called Hamiltonian. This systems has $N$ degrees of freedom. Due to the classical theorem of Liouville and its modern version given by Arnold [3], we say that the above system is completely integrable if it has $N$ functionally independent integrals of motion $H_1 = H, H_2, \dots, H_N$ being in involution with respect to the canonical Poisson bracket. This means that the completely integrable system (1.14) can be integrated in quadratures. Moreover, by the Noether theorem then this system has $N$ mutually commuting symmetries.

One can extend this notion to the field systems (1.1), like the KdV equation, that can be interpreted as having infinitely many degrees of freedom. Thus we say that a dynamical system (1.1) is integrable if it has a hierarchy of infinitely many pairwise commuting symmetries and/or an infinite hierarchy of conserved quantities. We say that a system is completely integrable if it further has infinitely many exact solutions, for example the multi-soliton ones, or e.g. if the inverse scattering transform method is applicable.

There are also several other different definitions of complete integrability, and many systems simultaneously satisfy the conditions of several of these definitions. The definitions in question include e.g. existence of an appropriate Lax pair and classical $R$-matrix formalism (see Chapter 3), the aforementioned inverse scattering transform method, existence of bi-Hamiltonian structures (see Chapter 2), Bäcklund and Darboux transformations, or e.g. the so-called bilinear Hirota equations. For the above notions of integrability, and more, see the following already classical references [1], [4], [13], [21], [32] and [34].

## 1.4 Some useful algebraic concepts

### 1.4.1 Lie algebras

We will recall some basic definitions.

**Definition 1.1** A linear (or vector) space $\mathcal{V}$ over the commutative field $\mathbb{K}$ endowed with a bilinear product $[\cdot,\cdot]:\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ which is *antisymmetric*

$$[a,b]=-[b,a] \qquad a,b,c\in\mathcal{V}$$

and satisfies the so-called *Jacobi identity*

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0 \qquad a,b,c\in\mathcal{V}$$

is called a *Lie algebra* and the product $[\cdot,\cdot]$ is called the *Lie bracket*.

Notice that any algebra $\mathcal{A}$ endowed with associative multiplication $\cdot$, is a Lie algebra with Lie bracket defined by the *commutator*:

$$[a,b]:=a\cdot b-b\cdot a \qquad a,b\in\mathcal{A}. \tag{1.15}$$

Of course, if this multiplication is commutative, the Lie algebra structure is trivial.

### 1.4.2 Derivations

**Definition 1.2** A *derivation* of an algebra $\mathcal{A}$ is a mapping (morphism) $\partial:\mathcal{A}\to\mathcal{A}$ such that it is linear

$$\partial(\alpha a+\beta b)=\alpha\partial(a)+\beta\partial(b) \qquad \alpha,\beta\in\mathbb{K}\quad a,b\in\mathcal{A}$$

and satisfied the Leibniz rule

$$\partial(ab)=\partial(a)b+a\partial(b) \qquad a,b\in\mathcal{A}.$$

By $\mathrm{Der}\mathcal{A}$ we will denote the space of all derivations on $\mathcal{A}$. Then

**Proposition 1.3** $\mathrm{Der}\mathcal{A}$ *is a Lie algebra with respect to the commutator*

$$[\partial_1,\partial_2]=\partial_1\partial_2-\partial_2\partial_1 \qquad \partial_1,\partial_2\in\mathrm{Der}\mathcal{A}. \tag{1.16}$$

The proof is left as an exercise for the reader. Notice that the composition of two derivations is not a derivation.

### 1.4.3 The left module and the associated complex

**Definition 1.4** Let $\mathcal{V}$ be a Lie algebra endowed with the Lie bracket $[\cdot,\cdot]$. A *left $\mathcal{V}$-module* is a linear space $\Omega^0$ such that elements of $\mathcal{V}$ act on $\Omega^0$ as left linear operators

$$\mathcal{V} \times \Omega^0 \to \Omega^0 \quad (v, f) \mapsto vf$$

and the following requirement is satisfied

$$\forall\, v, w \in \mathcal{V} \quad \forall\, f \in \Omega^0 \qquad (vw - wv)\, f = [v, w]\, f.$$

Notice that the Lie bracket in $\mathcal{V}$ does not have to be necessarily in the form of a commutator.

Now we can define $q$-forms as totally antisymmetric $q$-linear mappings

$$\omega : \mathcal{V} \times \cdots \times \mathcal{V} \to \Omega^0. \tag{1.17}$$

The space of all forms of degree $q$ is denoted by $\Omega^q$. Then the *exterior differential* (also known as the exterior derivative) $d : \Omega^q \to \Omega^{q+1}$ is defined as

$$\begin{aligned} d\omega(v_1, \ldots, v_{q+1}) = &\sum_i (-1)^{i+1} v_i \left(\omega(v_1, \ldots, \hat{v}_i, \ldots, v_{q+1})\right) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{q+1}). \end{aligned} \tag{1.18}$$

The hat $\hat{}$ over $v_i$ means that $v_i$ is omitted. The *inner product* $i_v : \Omega^{q+1} \to \Omega^q$ is

$$i_v\omega(v_1, \ldots, v_q) = \omega(v, v_1, \ldots, v_q),$$

where $v, v_1, \ldots, v_{q+1} \in \mathcal{V}$.

**Proposition 1.5** *The square power of the exterior differential vanish, i.e.* $d^2 = 0$.

**Proof.** Exercise. □

Let $\Omega = \bigoplus_{q=0}^{\infty} \Omega^q$. Then $(\Omega, d)$ is called $\mathcal{V}$-complex associated with left $\mathcal{V}$-module $\Omega^0$. This complex can be equipped with a bilinear map called the *exterior product*

$$\wedge : \Omega^p \times \Omega^q \to \Omega^{p+q} \quad (\omega, \eta) \mapsto \omega \wedge \eta.$$

It is required that this product is anti-commutative:

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega \qquad \omega \in \Omega^p \quad \eta \in \Omega^q,$$

and associative

$$(\omega \wedge \eta) \wedge \xi = \omega \wedge (\eta \wedge \xi).$$

The action of one-forms $\eta \in \Omega^1$ on vectors $v \in \mathcal{V}$ can be given through the bilinear *duality map*:

$$\langle \cdot, \cdot \rangle : \Omega^1 \times \mathcal{V} \to \Omega^0 \quad (\eta, v) \mapsto \langle \eta, v \rangle \equiv \eta(v).$$

Then, the generating rule for the exterior product of $k$ one-forms $\gamma_i \in \Omega^1$ acting on $k$ vectors can be given by the determinantal formula

$$(\gamma_1 \wedge \cdots \wedge \gamma_k)(v_1, \ldots, v_k) = \det(\langle \gamma_i, v_j \rangle) \qquad 1 \leqslant i, j \leqslant k. \tag{1.19}$$

Moreover, for an arbitrary $q$-form $\omega$ the following relations hold:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^q \omega \wedge d\eta$$
$$i_v(\omega \wedge \eta) = i_v\omega \wedge \eta + (-1)^q \omega \wedge i_v\eta,$$

The vector space $\Omega$ endowed with the wedge product is called an *exterior algebra* over $\mathbb{K}$.

**Example 1.6** Consider the Lie algebra of derivations on an algebra $\mathcal{A}$, so let $\mathcal{V} = \mathrm{Der}\mathcal{A}$. Then it is clear that $\Omega^0 = \mathcal{A}$ is a $\mathcal{V}$-left module, and hence according to the above procedure we can construct complex $(\Omega, d)$ completely determined by a given algebra $\mathcal{A}$.

Later in these lecture notes we will be interested in the linear operators $\theta : \mathcal{V} \to \Omega^1$ and $\pi : \Omega^1 \to \mathcal{V}$. The so-called adjoint representations of these operators with respect to the duality map are given through $\theta^\dagger : \mathcal{V} \to \Omega^1$ and $\pi^\dagger : \Omega^1 \to \mathcal{V}$ such that the following equalities hold

$$\left\langle \theta^\dagger \mathbf{v}, \mathbf{w} \right\rangle = \langle \theta \mathbf{w}, \mathbf{v} \rangle \qquad \mathbf{v}, \mathbf{w} \in \mathcal{V}$$
$$\left\langle \alpha, \pi^\dagger \beta \right\rangle = \langle \beta, \pi\alpha \rangle \qquad \alpha, \beta \in \Omega^1.$$

Then the operators satisfying $\theta^\dagger = -\theta$ (or respectively $\pi^\dagger = -\pi$) are called *skew-adjoint* operators.

**Remark 1.7** In practice it is rather difficult to work with all linear functionals acting on some infinite-dimensional linear vector space. Thus, applying the above scheme we do not assume that $\Omega^q$ contains all linear functionals (1.17) but we require that the spaces $\Omega^q$ are rich enough. Therefore, we require that

(i) the duality map is *nondegenerate*, i.e.,

$$\forall\ \eta \in \Omega^1 \quad \eta \neq 0 \quad \exists\ v \in \mathcal{V} \quad \text{such that} \quad \langle \eta, v \rangle \neq 0$$
$$\forall\ v \in \mathcal{V} \quad v \neq 0 \quad \exists\ \eta \in \Omega^1 \quad \text{such that} \quad \langle \eta, v \rangle \neq 0;$$

(ii) and the images of the exterior differential and the inner product remain within the appropriate spaces, i.e., $d(\Omega^q) \subset \Omega^{q+1}$ and $i_v(\Omega^{q+1}) \subset \Omega^q$ for all $v \in \mathcal{V}$.

**Example 1.8** The space of smooth vector fields on a smooth finite-dimensional manifold $M$ can be identified with the space of all derivations on the algebra of smooth functions. Thus it is clear that this space is a left module over the Lie algebra of vector fields. However, in the case of the well-known de Rham complex on smooth manifold the space of differential one-forms $\Omega^1$ is not the space of all linear functionals on the Lie algebra of vector fields. Actually, the space $\Omega^1$ consists of smooth sections from the cotangent bundle $T^\star M$, which is a union of finite-dimensional cotangent vector spaces at all points of $M$. Accordingly, all other differential forms are constructed as smooth sections of the appropriate (antisymmetrized) tensor products of several copies of cotangent bundle.

The operator

$$L_v = i_v d + d i_v$$

on the exterior algebra given by complex $\Omega$ will be called the *Lie derivative* along the vector (field) $v \in \mathcal{V}$. Thus, it is immediate from the definitions of the exterior differential and the interior product that the Lie derivative is a linear operator $L_v : \Omega^q \to \Omega^q$ such that

$$L_v \eta(v_1, \ldots, v_q) = v(\eta(v_1, \ldots, v_q)) - \sum_i \eta(v_1, \ldots, [v, v_i], \ldots, v_q), \tag{1.20}$$

where $v, v_1, \ldots, v_q \in \mathcal{V}$. The Lie derivative is a derivation on the exterior algebra

$$L_v(\omega \wedge \eta) = L_v \omega \wedge \eta + \omega \wedge L_v \eta,$$

that commutes with the exterior differential

$$L_v d = d L_v, \tag{1.21}$$

for all $v \in \mathcal{V}$.

Furthermore, from the above relations one can obtain the following one:

$$L_v L_w - L_w L_v = L_{[v,w]},$$

where $v, w \in \mathcal{V}$.

## 1.5 Exercises

1. Instead of (1.9-1.10) consider the following pair of operators

   $$L = \partial_x^3 + u\partial_x + v \qquad A = \partial_x^2 + a\partial_x + b,$$

   where $u, v$ are dynamical fields and $a, b$ are some auxiliary functions. Find such $a$ and $b$ that the Lax equation (1.6) yield consistent two-component evolution system. Find the system in question. This is the completely integrable *Boussinesq system* which describes long waves of shallow water moving in two directions.

2. Show that the first three dynamical systems from the KdV hierarchy (1.12) pairwise commute.

3. Show that the first four functionals from (1.13) are integrals of motion of all the three first symmetries from the KdV hierarchy (1.12).

4. Show that the commutator (1.15) on the algebra with associative multiplication is a well defined Lie bracket.

5. Prove Proposition 1.3.

6. Prove Proposition 1.5. It suffices to use the definition of the exterior differential (1.18).

7. Show that (1.20) holds.

# 2 Theory of infinite-dimensional Hamiltonian systems

The nonlinear integrable dynamical systems can be considered as evolution vector fields on some infinite dimensional functional manifolds. Therefore, in this chapter we present an indispensable set of definitions needed for understanding at least part of the issues connected with the geometro-algebraic theory of the integrable systems described by the PDE's. In particular, we introduce the concepts of Hamiltonian and bi-Hamiltonian structures.

We assume that the reader is at least familiar with the concept of finite-dimensional differential geometry and Lie algebras. This chapter may be considered as a compiled version of the theories presented in [37],[16],[13] and [4]. For the general theory of infinite-dimensional manifolds see e.g. [12].

## 2.1 Infinite-dimensional differential calculus

The construction of infinite-dimensional differential calculus, needed for the theory of soliton systems using the rigorous definition of infinite-dimensional manifolds is a rather cumbersome task. There are two best known approaches. The first one is presented in [37] and relies on an appropriate generalization (or prolongation) of finite-dimensional ideas. The second one is a (very) abstract rigorous algebraic approach, in which the specific properties of the phase space are irrelevant, see [16]. Here we will try to tread between them.

### 2.1.1 Infinite dimensional phase space and (1+1)-dimensional vector fields

Consider a linear space $\mathcal{U}$ of $N$-tuples

$$\mathbf{u} := (u^1, u^2, \ldots, u^N)^{\mathrm{T}}$$

of smooth functions

$$u^i : \Sigma \to \mathbb{K} \qquad x \mapsto u^i(x)$$

with values in a commutative field of complex or real numbers, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We have $\Sigma = \mathbb{S}^1$ if the boundary conditions imposed on $u^i$ are periodic, or $\Sigma = \mathbb{R}$ if the functions $u^i$ are Schwartz functions, i.e., $u^i$ and all their derivatives rapidly vanish as $|x| \to \infty$.

We can introduce topology on $\mathcal{U}$ turning the latter into a topological linear space, on which one can introduce differential calculus with full rigor, see for example [52]. Nevertheless, in these lecture notes we avoid discussions related to the issues of functional analysis, as they are irrelevant unless we consider general classes of solutions, and we concentrate only on algebraic aspects of differential calculus related to the theory of integrable systems.

Introduce 'formal' local coordinates $\{\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \dots\}$ on $\mathcal{U}$ defining our infinite-dimensional phase space, where

$$u^i_{jx} := \frac{\partial^j u^i}{\partial x^j} \qquad j = 0, 1, 2, \dots .$$

We will refer to $\mathcal{U}$ as to an infinite-dimensional manifold but only in a rather formal sense. Let $\mathcal{A}$ be an algebra of *smooth differential functions*

$$f[\mathbf{u}] := f(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \dots)$$

of a finite number of the above coordinates; we consider $\mathcal{A}$ as an algebra over the field $\mathbb{K}$ only. Notice that we do not assume explicit dependence of these functions on $x$.

This algebra can be extended into a differential algebra $\mathcal{A}[\![\partial_x]\!]$[1], where $\partial_x$ is a linear operator of total derivative with respect to $x$, i.e.,

$$\partial_x : \mathcal{A} \to \mathcal{A} \qquad f[\mathbf{u}] \mapsto \partial_x f[\mathbf{u}] = \sum_{n,i} \frac{\partial f}{\partial u^i_{nx}} u^i_{(n+1)x}.$$

Then, each element $A \in \mathcal{A}[\![\partial_x]\!]$ has the form

$$A = \sum_{i \geqslant 0} a_i \partial^i_x \qquad a_i \in \mathcal{A}.$$

Later we will be interested in linear differential operators

$$\Phi : \mathcal{A}^N \to \mathcal{A}^N \quad \eta \mapsto \Phi\eta$$

such that

$$(\Phi\eta)^i = \sum_{j=1}^{N} \Phi_{ij}\eta^j \qquad i = 1, \dots, N,$$

and

$$\Phi_{ij} \in \mathcal{A}[\![\partial_x]\!].$$

Nevertheless, the above notion of linear differential operators is often, in particular in the case of applications, not sufficient. Thus, we allow the extension of the above notion to

$$\Phi_{ij} \in \mathcal{A}[\![\partial_x, \partial_x^{-1}]\!],$$

i.e., to the case of non-local operators. Here, $\partial_x^{-1}$ is a formal inverse of $\partial_x$, which for example could be defined as $\partial_x^{-1} f(x) := \int_{-\infty}^{x} f(x')dx'$. The non-local linear differential operator $\Phi$ cannot be defined on the whole $N$-tuple of differential functions. Thus, in the non-local case we assume that $\Phi$ is defined on the appropriate subspace of $\mathcal{A}^N$ such that it image lies within $\mathcal{A}^N$, see e.g. [41] and references therein for more details. The assertions presented in the forthcoming sections will be formulated using local linear differential operators. Nevertheless, the results presented remain valid for nonlocal operators as well.

[1] $\mathcal{A}[\![\lambda, \mu, \dots]\!]$ means the ring of all polynomials in $\lambda, \mu, \dots$ with coefficients from $\mathcal{A}$

The scalar fields on $\mathcal{U}$ are functionals $F:\mathcal{A}\to\mathbb{K}$:

$$F(\mathbf{u})=\int_\Sigma f[\mathbf{u}]\;dx, \tag{2.1}$$

where densities $f[\mathbf{u}]$ belong to the quotient space $\mathcal{A}/\partial_x\mathcal{A}$, since due to the above boundary conditions the integrals (2.1) vanish if $f[\mathbf{u}]$ are total derivatives. Thus, one can integrate by parts staying in the same equivalence class

$$\int_\Sigma f_x g\;dx=-\int_\Sigma f g_x\;dx.$$

Let us denote the space of all above functionals (2.1) by $\mathcal{F}$. It is important to mention that $\mathcal{F}$ is a vector space, but, unlike the finite-dimensional case, does not have the structure of algebra due to the lack of multiplication law.

A smooth vector field on $\mathcal{U}$ is given by a system of partial differential equations

$$\mathbf{u}_t=\mathbf{K}(\mathbf{u}), \tag{2.2}$$

of first order in some evolution parameter $t\in I$, where $I$ is an subinterval of $\mathbb{R}$ and $u_t:=\frac{\partial u}{\partial t}$. Coefficients of

$$\mathbf{K}(\mathbf{u}):=(K^1[\mathbf{u}],K^2[\mathbf{u}],\dots,K^N[\mathbf{u}])^{\mathrm{T}}$$

are assumed to belong to $\mathcal{A}$. Then, system (2.2) represents a (1+1)-dimensional dynamical system as we can treat $t$ as an temporal parameter (time) and $x$ as a spatial coordinate. The space of smooth vector fields $\mathbf{K}(\mathbf{u})$ on $\mathcal{U}$, which we will denote by $\mathcal{V}$, is a linear space over the field $\mathbb{K}$ and can be identified with the space of $N$-tuples of differential functions $\mathcal{A}^N$.

**Definition 2.1** A scalar field on $\mathcal{U}$, i.e., a functional $F(\mathbf{u})\in\mathcal{F}$, is said to be an *integral of motion* (or a conserved quantity) of (2.2), if the total derivative of $F$ with respect to evolution parameter of (2.2) vanishes, i.e.,

$$\frac{dF(\mathbf{u})}{dt}=0.$$

**Definition 2.2** We will refer to the vector field $\mathbf{v}\in\mathcal{V}$, associated with the dynamical system

$$\mathbf{u}_\tau=\mathbf{v}(\mathbf{u}),$$

as to the *symmetry* of (2.2), if the flows of the vector fields $\mathbf{v}$ and $\mathbf{K}$ commute, i.e.,

$$(\mathbf{u}_t)_\tau=(\mathbf{u}_\tau)_t. \tag{2.3}$$

**Lemma 2.3** *All vector fields* $\mathbf{K}\in\mathcal{V}$ *of the form* (2.2) *commute with so-called* $x$*-translation symmetry*

$$\mathbf{u}_\tau=\mathbf{u}_x. \tag{2.4}$$

**Proof.** For $i = 1, \dots, N$ we have

$$(u^i_t)_\tau = K^i_\tau = \sum_{n=0}^{\infty} \sum_{j=1}^{N} u^j_{nx,\tau} \frac{\partial K^i}{\partial u^j_{nx}} = \sum_{n=0}^{\infty} \sum_{j=1}^{N} u^j_{(n+1)x} \frac{\partial K^i}{\partial u^j_{nx}} = \frac{dK^i}{dx} \equiv K^i_x$$

and $(u^i_\tau)_t = u^i_{x,t} = K^i_x$. Hence, $(u^i_t)_\tau = (u^i_\tau)_t$. □

### 2.1.2 Evolution derivations

Consider the space $\mathrm{Der}\mathcal{A}$ of all derivations of the algebra of differential functions $\mathcal{A}$. This space has the Lie algebra structure given by the commutator (1.16). Then one can show that, under certain technical assumptions, the general form of a derivation on $\mathcal{A}$ is given by

$$\partial f = \sum_{n=0}^{\infty} \sum_{i=1}^{N} h^i_n \frac{\partial f}{\partial u^i_{nx}} \qquad f \in \mathcal{A},$$

where also $h^i_n \in \mathcal{A}$, then

$$\partial = \sum_{n,i} h^i_n \frac{\partial}{\partial u^i_{nx}}. \tag{2.5}$$

We will skip the summation ranges whenever they are obvious. The coefficients of (2.5) can be obtained from the action on local coordinates, i.e. $h^i_n = \partial u^i_{nx}$. There is a special derivation in $\mathrm{Der}\mathcal{A}$, namely, the total derivative with respect to $x$

$$\partial_x = \sum_{n,i} u^i_{(n+1)x} \frac{\partial}{\partial u^i_{nx}}.$$

We will be interested only in the derivations from $\mathrm{Der}\mathcal{A}$ that can be identified with vector fields from $\mathcal{V}$. Consider $\partial$ that commutes with $\partial_x$, i.e. such that

$$[\partial, \partial_x] = 0. \tag{2.6}$$

Then,

$$h^i_{n+1} = \partial u^i_{(n+1)x} = \partial \partial_x u^i_{nx} \overset{\text{by (2.6)}}{=} \partial_x \partial u^i_{nx} = \partial_x h^i_n = (h^i_n)_x.$$

So, all the coefficients can be calculated from $h^i_0$, i.e., $h^i_n = (h^i_0)_{nx}$, and thus the general form of derivation commuting with $\partial_x$ is

$$\partial = \sum_{n,i} (h^i_0)_{nx} \frac{\partial}{\partial u^i_{nx}}.$$

Now, this derivation can be identified with vector field $\mathbf{K} \in \mathcal{V}$ upon setting $h^i_0 = K^i$.

**Definition 2.4** A derivation $\partial_{\mathbf{K}}$ from $\mathrm{Der}\mathcal{A}$ commuting with total derivative $\partial_x$ is called an *evolution derivation*, as it can be identified with a vector field $\mathbf{K}$

$$\mathrm{Der}\mathcal{A} \ni \partial_{\mathbf{K}} = \sum_{n,i} K^i_{nx} \frac{\partial}{\partial u^i_{nx}} \quad \rightleftarrows \quad \mathbf{K} \in \mathcal{V}.$$

The linear space of all evolution derivations will be denoted by $\mathrm{Der}_{\mathcal{V}}\mathcal{A} \cong \mathcal{V}$.

**Proposition 2.5** *The space of all evolution derivations* $\mathrm{Der}_{\mathcal{V}}\mathcal{A}$ *is a Lie subalgebra of* $\mathrm{Der}\mathcal{A}$ *with respect to the commutator* (1.16), *i.e., the commutator of two evolution derivations is a evolution derivation.*

**Proof.** Let $\partial_{\mathbf{v}} = \sum_{n,i} v^i_{nx} \frac{\partial}{\partial u^i_{nx}}$ and $\partial_{\mathbf{w}} = \sum_{n,i} w^i_{nx} \frac{\partial}{\partial u^i_{nx}}$. Then we have

$$\begin{aligned}
[\partial_{\mathbf{v}}, \partial_{\mathbf{w}}] &= \partial_{\mathbf{v}}\partial_{\mathbf{w}} - \partial_{\mathbf{w}}\partial_{\mathbf{v}} \\
&= \sum_{n,i} \left[\partial_{\mathbf{v}}(w^i_{nx})\frac{\partial}{\partial u^i_{nx}} + w^i_{nx}\partial_{\mathbf{v}}\frac{\partial}{\partial u^i_{nx}} - \partial_{\mathbf{w}}(v^i_{nx})\frac{\partial}{\partial u^i_{nx}} - v^i_{nx}\partial_{\mathbf{w}}\frac{\partial}{\partial u^i_{nx}}\right] \\
&= \sum_{n,i} \left[\partial_{\mathbf{v}}(w^i_{nx}) - \partial_{\mathbf{w}}(v^i_{nx})\right] \frac{\partial}{\partial u^i_{nx}} \\
&\overset{\text{by (2.6)}}{=} \sum_{n,i} \left[\partial_{\mathbf{v}}(w^i) - \partial_{\mathbf{w}}(v^i)\right]_{nx} \frac{\partial}{\partial u^i_{nx}}.
\end{aligned}$$

Thus, the commutator preserves the evolution form. □

Therefore, if $\partial_{\mathbf{v}}, \partial_{\mathbf{w}} \in \mathrm{Der}_{\mathcal{V}}\mathcal{A}$ then

$$\partial_{[\mathbf{v},\mathbf{w}]} := [\partial_{\mathbf{v}}, \partial_{\mathbf{w}}]\,,$$

where $[\mathbf{v}, \mathbf{w}] \in \mathcal{V}$ is a vector field with coefficients

$$[\mathbf{v}, \mathbf{w}]^i := \partial_{\mathbf{v}}(w^i) - \partial_{\mathbf{w}}(v^i) \qquad i = 1, \dots, N.$$

Hence, the evolution derivations induce the Lie algebra structure on the vector space $\mathcal{V}$.

Moreover, we can extend the action of evolution derivatives to the functionals $\mathcal{F}$ according to the rule

$$\mathbf{K}(F) := \int_{\Sigma} \partial_{\mathbf{K}} f\, dx \qquad F \in \mathcal{F},$$

where vector field $\mathbf{K} \in \mathcal{V}$ is treated as a linear operator $\mathbf{K} : \mathcal{F} \to \mathcal{F}$. Important is fact that the action of vector fields on functionals is compatible with the quotient structure. This means that if two densities $f$ and $g$ differ by an exact derivative then so do $\partial_{\mathbf{K}} f$ and $\partial_{\mathbf{K}} g$. This is immediate from the fact that the evolution derivations commute with total derivative with respect to $x$. Then, the Lie bracket in $\mathcal{V}$ is consequently defined as a commutator

$$[\mathbf{v}, \mathbf{w}] := \mathbf{v}\ \mathbf{w} - \mathbf{w}\ \mathbf{v} \qquad \mathbf{v}, \mathbf{w} \in \mathcal{V}, \tag{2.7}$$

as then

$$[\mathbf{v}, \mathbf{w}]\,(F) = \int_{\Sigma} \partial_{[\mathbf{v},\mathbf{w}]} f\, dx \qquad F \in \mathcal{F}.$$

Notice that the composition of two vector fields $\mathbf{v}\ \mathbf{w}$ by itself is not a well defined vector field. Why?

Notice that the total derivative itself is an evolution derivation and since

$$\partial_x \equiv \partial_{\mathbf{u}_x}$$

it can be identified with the translational symmetry (2.4). Thus, showing that the above Lie bracket in $\mathcal{V}$ is compatible with the commutation of flows one can see that the definition of evolution derivations is consistent with Lemma 2.3.

**Example 2.6** Consider two one-component vector fields $\mathbf{v} = uu_x$ and $\mathbf{w} = u_{2x}$. Then

$$[\mathbf{v}, \mathbf{w}] = \partial_{\mathbf{v}}(u_{2x}) - \partial_{\mathbf{w}}(uu_x) = 2u_x u_{2x}.$$

### 2.1.3 Variational calculus

From the previous section it is obvious that the space of functionals $\mathcal{F}$ on $\mathcal{U}$ is a $\mathcal{V}$-left module with respect to the Lie algebra of vector fields $\mathcal{V} \cong \mathrm{Der}_{\mathcal{V}}\mathcal{A}$. Thus, let $\Omega^0 := \mathcal{F}$. Now taking into account the requirements from Remark 1.7 we can construct differential forms according to Section 1.4.3.

A (functional) differential $q$-form reads

$$\omega = \int_\Sigma dx \{ \sum_{i,n} \omega^{i_1,\dots,i_q}_{n_1,\dots,n_q} du^{i_1}_{n_1 x} \wedge \cdots \wedge du^{i_q}_{n_q x} \}, \tag{2.8}$$

where the sum is in fact over a finite number of terms (diffferent for different forms!), $\omega^{i_1,\dots,i_q}_{n_1,\dots,n_q} \in \mathcal{A}$ are differential functions and $du^i_{nx}$ are 'formal' dual objects to $\frac{\partial}{\partial u^j_{mx}}$ such that

$$du^i_{nx}\left(\frac{\partial}{\partial u^j_{mx}}\right) = \delta_{i,j}\delta_{n,m}.$$

Then, the space $\Omega^q$ consists of all elements of the form (2.8) for fixed $q = 0, 1, 2, \dots$.

A one-form $\eta \in \Omega^1$

$$\eta = \int_\Sigma dx \{ \sum_{i,n} \eta^i_n du^i_n \}$$

acts on an evolution derivation $\mathbf{v} \cong \partial_{\mathbf{v}} = \sum_{n,i} v^i_{nx} \frac{\partial}{\partial u^i_{nx}}$ as follows:

$$\langle \eta, \mathbf{v} \rangle = \int_\Sigma \sum_{i,n} \eta^i_n v^i_{nx}\, dx = \int_\Sigma \sum_{i,n} (-1)^n (\eta^i_n)_{nx} v^i\, dx, \tag{2.9}$$

where we integrated by parts. Then, an arbitrary $q$-form (2.8) acts on evolution derivations according to the determinant formula (1.19). Of course, the functional differential $q$-forms (2.8) act on evolution derivations within equivalence classes of $\mathcal{A}/\partial_x\mathcal{A}$. This means that two such functional differential forms are equivalent when their densities differ modulo total derivatives of $x$. Using (1.20) and the fact that evolution derivatives commute with $\partial_x$ we have

$$L_{\partial_x}\eta(\mathbf{v}_1, \dots, \mathbf{v}_q) = \partial_x(\eta(\mathbf{v}_1, \dots, \mathbf{v}_q)).$$

Hence, the above equivalence classes in $\Omega^q$ constitute the following quotient space $\Omega^q / L_{\partial_x}\Omega^q$.

Upon inspection of (2.9) it is clear that one can make the following identification

$$\partial_x^n du^i \equiv du^i_{nx}$$

and hence we can use the following integration by parts formula

$$\int_\Sigma dx\{\omega \wedge \partial_x \eta\} = -\int_\Sigma dx\{\partial_x \omega \wedge \eta\}.$$

Moreover, (2.9) implies that an arbitrary one-form $\eta \in \Omega^1$ can be represented in the following canonical form

$$\eta = \int_\Sigma dx \sum_i \eta_i du^i \qquad \eta_i \in \mathcal{A}. \tag{2.10}$$

Hence the space of differential one-forms $\Omega^1/L_{\partial_x}\Omega^1$ can be identified with the space of $N$-tuples of differential functions $\mathcal{A}^N$. Then the action of an arbitrary one-form in the canonical representation (2.10) on an arbitrary vector field $\mathbf{v} \cong \partial_\mathbf{v} = \sum_{n,i} v^i_{nx} \frac{\partial}{\partial u^i_{nx}}$ is given by

$$\langle \eta, \mathbf{v} \rangle = \int_\Sigma \sum_{i=1}^N \eta_i v_i \; dx = \int_\Sigma \eta^{\mathrm{T}} \cdot \mathbf{v} \; dx,$$

where $\eta := (\eta_1, \eta_2, \ldots, \eta_N)^{\mathrm{T}}$.

To an arbitrary differential two-form $\omega \in \Omega^2$ one can associate a skew-adjoint linear operator

$$\theta : \mathcal{V} \to \Omega^1 \qquad \theta = (\theta_{ij}),$$

where $\theta_{ij} \in \mathcal{A}[\![\partial_x]\!]$, such that

$$\omega(\mathbf{v}, \mathbf{w}) \equiv \langle \theta \mathbf{w}, \mathbf{v} \rangle = -\langle \theta \mathbf{v}, \mathbf{w} \rangle = -\omega(\mathbf{w}, \mathbf{v}) \qquad \mathbf{v}, \mathbf{w} \in \mathcal{V}.$$

Therefore, $\omega$ can be represented in the following canonical form

$$\omega = \frac{1}{2}\int_\Sigma \{\sum_{i,j} du^i \wedge \theta_{ij} \; du^j\},$$

for which we have

$$\begin{aligned}\omega(\mathbf{v}, \mathbf{w}) &= \frac{1}{2}\int_\Sigma \{\sum_{i,j} du^i \wedge \theta_{ij} \; du^j\}(\mathbf{v}, \mathbf{w}) \overset{\text{by (1.19)}}{=} \int_\Sigma \sum_{i,j} v^i \theta_{ij}(w^j) \; dx \\ &= \int_\Sigma (\theta \mathbf{w})^{\mathrm{T}} \cdot \mathbf{v} \; dx \equiv \langle \theta \mathbf{w}, \mathbf{v} \rangle .\end{aligned}$$

The skew-adjointness in this setting means that

$$\theta^\dagger = -\theta \quad \Leftrightarrow \quad (\theta_{ij})^\dagger = -\theta_{ji}.$$

Thus, the space of differential two-forms $\Omega^2/L_{\partial_x}\Omega^2$ can be identified with the space of the skew-adjoint differential operators whose coefficients are $N \times N$ differential square matrices.

**Example 2.7** Consider the one-component ($N = 1$) two-form

$$\begin{aligned}
\omega &= \int_\Sigma dx\{u_x du \wedge du_{2x}\} = -\int_\Sigma dx\{(u_x du)_x \wedge du_x\} \\
&= -\int_\Sigma dx\{u_{2x} du \wedge du_x + u_x du_x \wedge du_x\} \\
&= \int_\Sigma dx\{du \wedge (-u_{2x}\partial_x) du\} = \int_\Sigma dx\{du \wedge (u_{2x}\partial_x)^\dagger du\} \\
&= \frac{1}{2}\int_\Sigma dx\{du \wedge \left((u_{2x}\partial_x)^\dagger - u_{2x}\partial_x\right) du\}.
\end{aligned}$$

The corresponding skew-adjoint operator reads

$$\theta = (u_{2x}\partial_x)^\dagger - u_{2x}\partial_x = -\partial_x u_{2x} - u_{2x}\partial_x.$$

**Proposition 2.8** *The local formula for the exterior differential* $d : \Omega^q \to \Omega^{q+1}$ *defined by* (1.18) *is given by*

$$d\omega = \int_\Sigma dx\{\sum_{i,n}\sum_{k,m} \frac{\partial \omega^{i_1,\ldots,i_q}_{n_1,\ldots,n_q}}{\partial u^k_{mx}} du^k_{mx} \wedge du^{i_1}_{n_1 x} \wedge \cdots \wedge du^{i_q}_{n_q x}\},$$

*where* $\omega \in \Omega^q$ *has the form* (2.8)*. Moreover, it is consistent with the quotient structures in* $\Omega^q / L_{\partial_x}\Omega^q$ *for all* $q = 0, 1, \ldots$.

Then, the differential (or gradient) of a functional $F = \int_\Sigma f\ dx \in \mathcal{F}$, i.e., of a zero-form, is given by

$$\begin{aligned}
dF &= \int_\Sigma dx\{\sum_{i,n} = \int_\Sigma dx \frac{\partial f}{\partial u^i_{nx}} du^i_{nx}\} \\
&= \int_\Sigma dx\{\sum_{i,n} (-\partial_x)^n \left(\frac{\partial f}{\partial u^i_{nx}}\right) du^i\} =: \int_\Sigma dx\{\sum_i \frac{\delta F}{\delta u^i} du^i\},
\end{aligned}$$

where integrating by parts we brought $dF \in \Omega^1$ into its canonical form. Hence, $dF$ can be represented as an $N$-tuple

$$dF(\mathbf{u}) = \left(\frac{\delta F}{\delta u_1}, \ldots, \frac{\delta F}{\delta u_N}\right)^{\mathrm{T}},$$

where

$$\frac{\delta F}{\delta u_i} := \sum_{n \geqslant 0} (-\partial_x)^n \frac{\partial f}{\partial u^i_{nx}} = \frac{\partial f}{\partial u^i} - \partial_x \frac{\partial f}{\partial u^i_x} + \partial_x^2 \frac{\partial f}{\partial u^i_{2x}} - \ldots$$

is the usual *variational derivative.*

### 2.1.4 Directional and Lie derivatives

In general, tensor fields on $\mathcal{U}$ of $(r,s)$-type ($r$ times contravariant and $s$ times covariant) are multi-linear maps

$$\underbrace{\mathcal{V}\times\cdots\times\mathcal{V}}_{s}\times\underbrace{\Omega^1\times\cdots\times\Omega^1}_{r}\rightarrow\Omega^0.$$

Thus, the differential $q$-forms are totally anti-symmetric $s$-covariant tensor fields. The vector fields $\mathcal{V}$ are once-contravariant tensor fields. We are not going to consider general theory of such tensor fields; we will focus only on the cases of interest to us.

The first case of interest is that of twice-contravariant tensor fields $\pi:\Omega^1\times\Omega^1\rightarrow\Omega^0$, the ones to which we can, through the duality map, associate linear differential operators $\pi:\Omega^1\rightarrow\mathcal{V}$ such that

$$\pi(\xi,\eta)=\langle\eta,\pi\xi\rangle=\int_\Sigma\eta^{\mathrm{T}}\cdot\pi\xi\;dx, \tag{2.11}$$

where $\xi,\eta\in\Omega^1$. Thus, $\pi=(\pi_{ij})$ are $N\times N$ matrices with coefficients from $\mathcal{A}[\![\partial_x]\!]$ or $\mathcal{A}[\![\partial_x,\partial_x^{-1}]\!]$ in the nonlocal case.

**Definition 2.9 ([37])** A linear differential operator $\pi:\Omega^1\rightarrow\mathcal{V}$ is said to be *degenerate* if there exists a nonzero differential operator $\widetilde{\pi}:\mathcal{V}\rightarrow\Omega^1$ such that $\widetilde{\pi}\cdot\pi\equiv 0$.

Therefore, for a nondegenerate linear differential operator $\pi:\Omega^1\rightarrow\mathcal{V}$ we can construct its inverse $\pi:\mathcal{V}\rightarrow\Omega^1$, defining a twice-covariant tensor field such that

$$\pi\pi^{-1}=\pi^{-1}\pi=I,$$

where $I$ is an $N\times N$ identity matrix. But in general $\pi^{-1}$ will be nonlocal.

The second case of interest are linear differential operators $\Phi:\mathcal{V}\rightarrow\mathcal{V}$ defining tensor fields of type $(1,1)$

$$\Phi(\mathbf{v},\eta)=\langle\eta,\Phi\mathbf{v}\rangle=\int_\Sigma\eta^{\mathrm{T}}\cdot\Phi\mathbf{v}\;dx\qquad\mathbf{v}\in\mathcal{V}\quad\eta\in\Omega^1. \tag{2.12}$$

**Definition 2.10** For an arbitrary tensor field $T(\mathbf{u})$ on $\mathcal{U}$ its *directional* (or Fréchet) *derivative* in the direction of a vector field $\mathbf{K}\in\mathcal{V}$ is defined by the formula

$$T'(\mathbf{u})[\mathbf{K}]=\left.\frac{dT(\mathbf{u}+\varepsilon\mathbf{K})}{d\varepsilon}\right|_{\varepsilon=0}.$$

Important is fact that the directional derivative does not change the type of the tensor field. The procedure of finding directional derivatives of $T$ is quite simple as it reduces to the differentiation of all coefficients of $T$ with respect to the evolution parameter associated with $\mathbf{K}$.

Consider directional derivative of a functional $F(\mathbf{u})=\int_\Sigma f[\mathbf{u}]\;dx$ in the direction of a vector field $\mathbf{u}_t=\mathbf{K}(\mathbf{u})$. Then

$$\begin{aligned}F'(\mathbf{u})[\mathbf{K}]&\equiv\int_\Sigma\frac{df}{dt}\;dx=\int_\Sigma\sum_{i,n}\frac{\partial f}{\partial u^i_{nx}}(u^i_t)_{nx}\;dx=\int_\Sigma\sum_i\frac{\delta F}{\delta u^i}K^i\;dx\\&=\int_\Sigma dF^{\mathrm{T}}\cdot\mathbf{K}\;dx=\langle dF,\mathbf{K}\rangle=\mathbf{K}(F).\end{aligned} \tag{2.13}$$

Thus we see that the directional derivative of a given functional is the same as the action of a vector field on this functional. On the other hand, the above formula yields another method for defining the differential $dF$.

Moreover, if $F$ is an integral of motion of $\mathbf{K}$, the time derivative of $F$ vanishes:

$$\frac{dF}{dt} = F'[\mathbf{K}] = \mathbf{K}(F) = \langle dF, \mathbf{K}\rangle = 0,$$

as it naturally should.

**Proposition 2.11** *An equivalent formula for the Lie bracket* (2.7) *in* $\mathcal{V}$ *can be defined by means of directional derivatives as follows:*

$$[\mathbf{v}, \mathbf{w}] := \mathbf{w}'[\mathbf{v}] - \mathbf{v}'[\mathbf{w}], \tag{2.14}$$

*where* $\mathbf{v}, \mathbf{w}$ *are arbitrary vector fields.*

**Proof.** Left as an exercise for the reader. □

Thus we have two alternative formulas for the Lie bracket of vector fields that have their own advantages and disadvantages. We will use both of these formulas. When the Lie bracket of two vector fields vanish we say that these vector fields commute. Notice that commutativity with respect to the Lie bracket is equivalent to commutativity of the respective flows (2.3).

Using (2.14) we can rewrite the coordinate-free formula for the exterior differential (1.18) as

$$d\omega(v_1, \dots, v_{q+1}) = \sum_i (-1)^{i+1}\omega'[v_i](v_1, \dots, \hat{v}_i, \dots, v_{q+1}), \tag{2.15}$$

where $\omega \in \Omega^q$ and $v_1, \dots, v_{q+1} \in \mathcal{V}$.

**Lemma 2.12** *The following relation*

$$\langle dF'[\mathbf{v}], \mathbf{w}\rangle = \langle dF'[\mathbf{w}], \mathbf{v}\rangle \tag{2.16}$$

*holds for arbitrary* $F \in \mathcal{F}$ *and* $\mathbf{v}, \mathbf{w} \in \mathcal{V}$.

**Proof.** The above result is equivalent to the vanishing of square power of the exterior differential, i.e. $d^2F = 0$. Hence, by (2.15) we have

$$0 = d^2F(\mathbf{v}, \mathbf{w}) = dF'[\mathbf{v}](\mathbf{w}) - dF'[\mathbf{w}](\mathbf{v}) = \langle dF'[\mathbf{v}], \mathbf{w}\rangle - \langle dF'[\mathbf{w}], \mathbf{v}\rangle$$

and the result follows. □

Yet another important operator is the Lie derivative. It is defined by the formula (1.20) within the exterior algebra of functional differentials. However, it can be uniquely extended to arbitrary tensor fields assuming that it preserves the Leibniz rule. Now, we will calculate the formulae for the Lie derivative along the vector field $\mathbf{K} \in \mathcal{V}$ for the tensor fields of interest to us. From the definition (1.20) it follows that for an arbitrary functional $F \in \mathcal{F}$, i.e., a zero-form, we have

$$L_{\mathbf{K}}F = \mathbf{K}(F) = F'[\mathbf{K}] = \langle dF, \mathbf{K}\rangle .$$

Therefore, for any $\mathbf{v} \in \mathcal{V}$ we have

$$L_{\mathbf{K}} \langle dF, \mathbf{v}\rangle = \langle dL_{\mathbf{K}}F, \mathbf{v}\rangle + \langle dF, L_{\mathbf{K}}\mathbf{v}\rangle ,$$

where we used the formula (1.21). Hence,

$$\begin{aligned} \langle dF, L_{\mathbf{K}}\mathbf{v}\rangle = L_{\mathbf{K}} \langle dF, \mathbf{v}\rangle - \langle dL_{\mathbf{K}}F, \mathbf{v}\rangle \quad &\Longleftrightarrow \\ (L_{\mathbf{K}}\mathbf{v})(F) = L_{\mathbf{K}}(\mathbf{v}(F)) - \mathbf{v}(L_{\mathbf{K}}F) = (\mathbf{K}\mathbf{v} - \mathbf{v}\mathbf{K})(K) & \end{aligned}$$

and

$$L_{\mathbf{K}}\mathbf{v} = [\mathbf{K}, \mathbf{v}] .$$

In a similar fashion one can show that for a linear operator $\pi : \Omega^1 \to \mathcal{V}$ such that (2.11) holds we have

$$\begin{aligned} \langle \eta, L_{\mathbf{K}}\pi\ \xi\rangle = \langle \eta, \pi'[\mathbf{K}]\xi\rangle - \langle \eta, \mathbf{K}'[\pi\xi]\rangle - \left\langle \eta, \pi\ \mathbf{K}'^{\dagger}[\xi]\right\rangle \quad &\Longleftrightarrow \\ L_{\mathbf{K}}\pi = \pi'[\mathbf{K}] - \mathbf{K}'\pi - \pi\ \mathbf{K}'^{\dagger}, & \end{aligned} \tag{2.17}$$

where the adjoint directional derivative is defined through the duality map

$$\left\langle \mathbf{K}'^{\dagger}[\xi], \mathbf{v}\right\rangle := \langle \xi, \mathbf{K}'[\mathbf{v}]\rangle .$$

For a linear differential operator $\Phi : \mathcal{V} \to \mathcal{V}$ such that (2.12) holds we have

$$\begin{aligned} \langle \eta, L_{\mathbf{K}}\Phi\ \mathbf{v}\rangle = \langle \eta, \Phi'[\mathbf{K}]\mathbf{v}\rangle + \langle \eta, \Phi\ \mathbf{K}'[\mathbf{v}]\rangle - \langle \eta, \mathbf{K}'[\Phi\mathbf{v}]\rangle \quad &\Longleftrightarrow \\ L_{\mathbf{K}}\Phi = \Phi'[\mathbf{K}] + \Phi\ \mathbf{K}' - \mathbf{K}'\Phi. & \end{aligned} \tag{2.18}$$

The Lie derivative gives us information on how the tensor fields change locally along an integral curve of a given dynamical system. Thus the following definition is very convenient.

**Definition 2.13** A tensor field $T$ is an *invariant* of a vector field $\mathbf{K} \in \mathcal{V}$, if

$$L_{\mathbf{K}}T = 0.$$

Hence, invariant scalar fields (functionals) are nothing but integrals of motion, and invariant vector fields are symmetries. As we will see in the following sections, invariant tensor fields play major role in the theory of integrable systems.

## 2.2 Hamiltonian theory

### 2.2.1 Hamiltonian systems

**Definition 2.14** A bilinear product $\{\cdot,\cdot\} : \mathcal{F}\times\mathcal{F}\to\mathcal{F}$ such that

$$\{H,F\}_\pi = \langle dF, \pi dH\rangle = \int_\Sigma dF^{\mathrm{T}}\cdot \pi dH\ dx \qquad F,H\in\mathcal{F}, \tag{2.19}$$

where $\pi : \Omega^1\to\mathcal{V}$ is a linear differential operator, is called a *Poisson bracket*, when it defines a Lie algebra structure on $\mathcal{F}$. The operator $\pi$ is called a *Poisson* (or Hamiltonian) *operator* if the bracket is a Poisson bracket.

In contrast with the usual definition of the Poisson bracket in finite-dimensional case, the Leibniz rule is missing from the above definition. There is no counterpart for this rule, as $\mathcal{F}$ no longer has an algebra structure. Because of this, some further properties must be proved in a more sophisticated way than it is usually done for the case of finite-dimensional manifolds. The first question is when a given differential operator $\pi : \Omega^1\to\mathcal{V}$ yields a Poisson bracket.

**Theorem 2.15** *The bracket* (2.19) *defines a Lie algebra on $\mathcal{F}$ if*

*(i) $\pi$ is a bi-vector, i.e., it is skew-symmetric with respect to the duality map*

$$\pi^\dagger = -\pi \qquad \Longleftrightarrow \qquad \langle \eta, \pi\xi\rangle = -\langle \xi,\pi\eta\rangle \qquad \eta,\xi\in\Omega^1;$$

*(ii) and for arbitrary one-forms $\alpha,\beta,\gamma\in\Omega^1$ the following identity holds:*

$$\langle \alpha, \pi'[\pi\beta]\gamma\rangle + \langle \beta, \pi'[\pi\gamma]\alpha\rangle + \langle \gamma, \pi'[\pi\alpha]\beta\rangle = 0. \tag{2.20}$$

**Proof.** The first condition is clear, as it is due to the anti-symmetry property of the Poisson bracket. The second one follows from the Jacobi identity, as we have

$$\begin{aligned}
\{F,\{G,H\}\} + c.p. &= \langle d\{G,H\}, \pi dF\rangle + c.p. \\
&= \{G,H\}'[\pi dF] + c.p. = \langle dH, \pi dG\rangle'[\pi dF] + c.p. \\
&= \langle dH'[\pi dF], \pi dG\rangle + \langle dH, \pi'[\pi dF]dG\rangle + \langle dH, \pi dG'[\pi dF]\rangle + c.p. \\
&\overset{\text{by } c.p. \text{ and } (i)}{=} \langle dG'[\pi dH], \pi dF\rangle + \langle dH, \pi'[\pi dF]dG\rangle - \langle dG'[\pi dF], \pi dH\rangle + c.p. \\
&\overset{\text{by (2.16)}}{=} \langle dH, \pi'[\pi dF]dG\rangle + c.p. = 0,
\end{aligned}$$

and from the fact that the last equality does not depend on the form of one-forms. This can be seen upon rewriting the condition explicitly. □

To check whether a skew-adjoint differential operator satisfies (2.20) is in general a cumbersome task. The right-hand side of (2.20) is a tri-vector; the condition of vanishing of the latter can be rewritten in an explicit 'coordinate' form, thus simplifying the above task, see Theorem 7.8 in [37].

**Definition 2.16** A vector field $\mathbf{K} \in \mathcal{V}$ is said to be *Hamiltonian* (with respect to $\pi$) if there exists a Poisson operator $\pi : \Omega^1 \to \mathcal{V}$ and scalar field $H \in \mathcal{F}$ such that

$$\mathbf{K} = \pi dH.$$

Then, the functional $H$ is called a Hamiltonian functional or simply a *Hamiltonian* (for $\mathbf{K}$), and the evolution system (2.2) is said to be Hamiltonian with respect to $\pi$ with a Hamiltonian $H$.

Two functionals are said to be in involution with respect to a Poisson bracket if this Poisson bracket of these two functionals vanishes. Hence, if $\mathbf{K}$ is a Hamiltonian vector field, i.e., $\mathbf{K} = \pi dH$, then $F$ is an integral of motion for $\mathbf{K}$ if it is in involution with Hamiltonian $H$, since

$$F'[\mathbf{K}] = \langle dF, \mathbf{K}\rangle = \langle dF, \pi dH\rangle = \{H, F\}_\pi = 0.$$

Of course, any Hamiltonian itself is an integral of motion of the Hamiltonian systems it generates. The following proposition is very important.

**Proposition 2.17** *Let*

$$X_H = \pi dH \qquad \textit{for} \qquad H \in \mathcal{F}$$

*stand for the Hamiltonian vector fields with respect to some Poisson operator $\pi$. Then, the linear map*

$$\pi d : \mathcal{F} \to \mathcal{V} \quad F \mapsto \pi dF,$$

*being composition of the exterior differential with Poisson operator, is a Lie algebra homomorphism, i.e.,*

$$\pi d \{F, G\}_\pi = [\pi dF, \pi dG] \qquad \Longleftrightarrow \qquad X_{\{F,G\}_\pi} = [X_F, X_G].$$

**Proof.** We have

$$\{H, F\}_\pi = \langle dF, \pi dH\rangle = \langle dF, X_H\rangle = X_H(F),$$

and thus

$$\{\{F, G\}_\pi, H\}_\pi = X_{\{F,G\}_\pi}(H).$$

Then, by the Jacobi identity for the respective Poisson bracket one finds that

$$\begin{aligned}\{\{F, G\}_\pi, H\}_\pi &= \{F, \{G, H\}_\pi\}_\pi + \{G, \{H, F\}_\pi\}_\pi = \{F, X_G(H)\}_\pi - \{G, X_F(H)\}_\pi \\ &= (X_F X_G - X_G X_F)(H) = [X_F, X_G](H),\end{aligned}$$

what ends the proof. □

From this proposition it follows that if two functionals are in involution then the related Hamiltonian vector fields (and dynamical systems) commute. Thus, if we have fixed a Hamiltonian dynamical system with respect to some Poisson tensor $\pi$, the Lie algebra homomorphism $\pi d : \mathcal{F} \to \mathcal{V}$ maps integrals of motion into symmetries and thus we have a Hamiltonian version of the well-known Noether theorem. Moreover, conserved quantities and symmetries constitute Lie subalgebras of a suitable Poisson algebra of scalar fields and of the Lie algebra of vector fields, respectively.

**Proposition 2.18** *A given Poisson tensor $\pi : \Omega^1 \to \mathcal{V}$ is invariant under its Hamiltonian vector fields $\mathbf{K} = \pi dH$, i.e.,*

$$L_{\mathbf{K}}\pi = L_{\pi dH}\pi = 0$$

*for all $H \in \mathcal{F}$.*

**Proof.** Left as an exercise for the reader. □

It often happens that we need to restrict the dynamics under study to a 'submanifold' defined trough some constraints. In such a case, a question arises whether and how one can reduce Poisson tensors. The simplest posible case is considered in the following lemma.

**Lemma 2.19** *Assume that the linear phase space $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$ is spanned by $\mathbf{u}_1 \in \mathcal{U}_1$ and $\mathbf{u}_2 \in \mathcal{U}_2$, i.e. $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)^{\mathrm{T}}$, and let*

$$\pi(\mathbf{u}_1, \mathbf{u}_2) = \begin{pmatrix} \pi_{11}(\mathbf{u}_1, \mathbf{u}_2) & \pi_{12}(\mathbf{u}_1, \mathbf{u}_2) \\ \pi_{21}(\mathbf{u}_1, \mathbf{u}_2) & \pi_{22}(\mathbf{u}_1, \mathbf{u}_2) \end{pmatrix}$$

*be a Poisson operator on $\mathcal{U}$. Also let $\pi_{22}$ be nondegenerate and hence invertible. Then, for an arbitrary constant $\mathbf{c} \in \mathcal{U}_2$ and the constraint $\mathbf{u}_2 = \mathbf{c}$ the operator*

$$\pi^{red}(\mathbf{u}_1) := \pi_{11}(\mathbf{u}_1, \mathbf{c}) - \pi_{12}(\mathbf{u}_1, \mathbf{c}) \cdot [\pi_{22}(\mathbf{u}_1, \mathbf{c})]^{-1} \cdot \pi_{21}(\mathbf{u}_1, \mathbf{c})$$

*is a Poisson operator on the affine space $\mathcal{U}_1 + \mathbf{c}$.*

We skip the proof of the above lemma as it consists of a pages of a tedious and cumbersome, yet rather straightforward calculations, see [15].

### 2.2.2 Recursion operators

**Definition 2.20** A linear differential operator $\Phi : \mathcal{V} \to \mathcal{V}$ that upon acting on a symmetry produces another symmetry of a given vector field $\mathbf{K} \in \mathcal{V}$ is called a *recursion operator* for $\mathbf{K}$.

The following simple proposition holds.

**Proposition 2.21** *A linear differential operator $\Phi : \mathcal{V} \to \mathcal{V}$ is a recursion operator for $\mathbf{K} \in \mathcal{V}$ if and only if it is invariant under $\mathbf{K}$, i.e., $L_{\mathbf{K}}\Phi = 0$.*

**Proof.** Let $\sigma \in \mathcal{V}$ be a symmetry of $\mathbf{K}$, i.e., $L_{\mathbf{K}}\sigma = 0$. Then

$$L_{\mathbf{K}}(\Phi\sigma) = (L_{\mathbf{K}}\Phi)\sigma + \Phi(L_{\mathbf{K}}\sigma) = (L_{\mathbf{K}}\Phi)\sigma = 0.$$

Hence, $\Phi\sigma$ is symmetry of $\mathbf{K}$ if and only if $L_{\mathbf{K}}\Phi = 0$. □

Notice that in most cases recursion operators are nonlocal integro-differential operators, see e.g. [41] and references therein for the discussion of the structure of corresponding nonlocalities. The most important are recursion operators possessing the so-called heredity property.

**Definition 2.22** Let $\Phi : \mathcal{V} \to \mathcal{V}$ be a linear (integro-)differential operator such that for an arbitrary vector field $\mathrm{v} \in \mathcal{V}$

$$L_{\Phi\mathbf{v}}\Phi = \Phi\ L_{\mathbf{v}}\Phi.$$

Then $\Phi$ is called a *hereditary operator* (or a regular operator, or a Nijenhuis operator).

A straightforward corollary of the definition is the following theorem.

**Theorem 2.23** *Let $\Phi : \mathcal{V} \to \mathcal{V}$ be a hereditary operator and $\mathrm{K}$ be a vector field such that $L_K\Phi = 0$. Let $\mathbf{K}_n = \Phi^n\mathbf{K}$ for $n = 0, 1, 2, \ldots$. Then*

*(i) $\Phi$ is invariant under $\mathrm{K}_n$ for all $n$, i.e., $L_{\mathbf{K}_n}\Phi = 0$;*

*(ii) the vector fields $\mathrm{K}_n$ pairwise commute, i.e.*

$$[K_m, K_n] = 0.$$

**Proof.** The first part follows from the definition of a hereditary operator and assumptions in the theorem, thus

$$L_{\mathbf{K}_n}\Phi = \Phi^n L_{\mathbf{K}}\Phi = 0.$$

For the second part we have

$$\begin{aligned}[][\mathbf{K}_m, \mathbf{K}_n] &= L_{\mathbf{K}_m}\mathbf{K}_n = L_{\Phi^m\mathbf{K}}(\Phi^n\mathbf{K}) = L_{\Phi^m\mathbf{K}}(\Phi^n)\mathbf{K} + \Phi^n\ L_{\Phi^m\mathbf{K}}\mathbf{K}\\ &= \Phi^m\ L_{\mathbf{K}}(\Phi^n)\ \mathbf{K} + \Phi^{m+n}\ L_{\mathbf{K}}\mathbf{K} = 0,\end{aligned}$$

where we made further use of the Leibniz chain rule. □

Almost all known today recursion operators are hereditary. Thus, if we have a dynamical system with a hereditary recursion operator which generates an infinite hierarchy of independent vector fields, then we can refer to this system as to an integrable one. Actually all the systems from this hierarchy will pairwise commute, i.e., one system will be a symmetry to another, and all these systems will be integrable in the above sense.

### 2.2.3 Bi-Hamiltonian systems

The dynamical systems that can be represented as Hamiltonian ones in two distinct ways possesses important properties inherent to the completely integrable systems. The remarkable concept of bi-Hamiltonian evolution equations was first introduced by F. Magri [31].

**Definition 2.24** A vector field $\mathrm{K} \in \mathcal{V}$ is called *bi-Hamiltonian* with respect to Poisson operators $\pi_0$ and $\pi_1$ if there exists functionals $H_0, H_1 \in \mathcal{F}$ such that

$$\mathbf{K} = \pi_0 dH_1 = \pi_1 dH_0. \tag{2.21}$$

For the bi-Hamiltonian system (2.21) one can immediately construct two additional Hamiltonian vector fields, as

$$\begin{aligned}\mathbf{K}_0 &= \pi_0 dH_0 = \pi_1 d(\cdot)\\ \mathbf{K} &= \pi_0 dH_1 = \pi_1 dH_0\\ \mathbf{K}_2 &= \pi_0 d(\cdot) = \pi_1 dH_1,\end{aligned}$$

which yield a 'pre-beginning' of the so-called bi-Hamiltonian chain. Of course the Hamiltonians of (2.21) are integrals of motion of $\mathbf{K}$ and hence are in involution with respect to the Poisson brackets generated by $\pi_0$ and $\pi_1$. Hence, by Proposition 2.17 the vector fields $\mathbf{K}_0$ and $\mathbf{K}_2$ commute with $\mathbf{K}$. Now, several questions arise. When $\mathbf{K}_1$ and $\mathbf{K}_2$ commute? When there exist Hamiltonians making them into bi-Hamiltonian systems? And, when the above chain can be continued? The answers are given in the following two theorems.

**Definition 2.25** A pair of Poisson tensors $\pi_0$ and $\pi_1$ is said to be compatible if the linear combination $\pi_0 + \epsilon\pi_1$ is also Poisson for any scalar $\epsilon$.

Actually, for compatibility between two Poisson tensors it is enough to check whether $\pi_0 + \pi_1$ is a Poisson tensor.

**Theorem 2.26** *Let $(\pi_0, \pi_1)$ be a pair of compatible Poisson tensors. Assume that $\pi_0$ is nondegenerate. Let $\mathbf{K} \in \mathcal{V}$ be such that*

$$L_{\mathbf{K}}\pi_0 = L_{\mathbf{K}}\pi_1 = 0.$$

*Then*

*(i) the operator $\Phi = \pi_1\, \pi_0^{-1}$ is hereditary;*

*(ii) all operators $\pi_n = \Phi^n \pi_0$, for $n = 0, 1, \ldots,$ are Poisson tensors.*

**Proof.** For the proof of this theorem see [4]. □

Of course, $\pi_0$ and $\pi_1$ are invariants of bi-Hamiltonian systems (2.21) associated with them. Hence, if the vector field $\mathbf{K}$ is bi-Hamiltonian with respect to two compatible Poisson tensors, one of which (say, $\pi_0$) is non-degenerate, then the hierarchy of vectors fields

$$\mathbf{K}_n = \Phi^n \mathbf{K}$$

yields a hierarchy of pairwise commuting dynamical systems. This is the case, as by the above theorem $\Phi = \pi\pi_0^{-1}$ is a hereditary operator invariant under $\mathbf{K}$ since

$$L_{\mathbf{K}}\Phi = L_{\mathbf{K}}(\pi_1\pi_0^{-1}) = L_{\mathbf{K}}\pi_1\; \pi_0^{-1} - \pi_1\pi_0^{-1} L_{\mathbf{K}}\pi_0\; \pi_0^{-1} = 0.$$

Moreover, if some additional technical assumptions hold, all these systems are bi-Hamiltonian.

The main properties of bi-Hamiltonian systems are contained in the following theorem, whose original version is due to Magri [31], and the version given below is the one of Olver [37].

**Theorem 2.27** *Let*

$$\mathbf{u}_{t_1} = \mathbf{K}_1[\mathbf{u}] = \pi_0 dH_1 = \pi_1 dH_0$$

*be a bi-Hamiltonian system of evolution equations. Assume that the operator* $\pi_0$ *is nondegenerate. Let*

$$\Phi := \pi_1\, \pi_0^{-1} : \mathcal{V} \mapsto \mathcal{V},$$

*be the so-called recursion operator. Assume that for each* $n = 1, 2, \ldots$ *we can recursively define*

$$\mathbf{u}_{t_0} = \mathbf{K}_0[\mathbf{u}] := \pi_0 dH_0 \qquad \Longrightarrow \qquad \mathbf{K}_n = \Phi\, \mathbf{K}_{n-1}$$

*meaning that for each* $n$, $\mathbf{K}_{n-1}$ *lies in the image of* $\pi_0$. *Then there exists a sequence of functionals* $H_0, H_1, H_2, \ldots$ *such that*

*(i) for each* $n \geqslant 1$, *the evolution equation*

$$\mathbf{u}_{t_n} = \mathbf{K}_n[\mathbf{u}] = \pi_0 dH_n = \pi_1 dH_{n-1} \tag{2.22}$$

*is a bi-hamiltonian system;*

*(ii) the corresponding evolutionary vector fields* $\mathbf{K}_n$ *all pairwise commute:*

$$[\mathbf{K}_m, \mathbf{K}_n] = 0 \qquad m, n \geqslant 0;$$

*(iii) the Hamiltonian functionals* $H_n$ *are all in involution with respect to both of the Poisson brackets:*

$$\{H_m, H_n\}_{\pi_0} = \{H_m, H_n\}_{\pi_1} = 0 \qquad m, n \geqslant 0,$$

*and hence provide an infinite collection of conserved quantities for each of the bi-Hamiltonian systems* (2.22).

Notice that the assumption that $\mathbf{K}_{n-1}$ lies in the range of $\pi_0$ is essential for the recursion procedure. In a more general setting, we should require that $\mathbf{K}_{n-1}$ lie in the domain of definition of the recursion operator $\Phi$ for all $n$; see [41] and references therein for the methods of proving this. There are known examples of bi-Hamiltonian systems for which this assumption is violated. From the fact that $\Phi$ is hereditary it is clear that $\mathbf{K}_n$ pairwise commute. But this is not the place where one needs the assumption about the compatibility of Poisson tensors, since the commutativity of $\mathbf{K}_n$ follows by Proposition 2.17 from the involutivity of the Hamiltonians $H_n$. One uses the compatibility assumption to show the existence of Hamiltonians $H_n$ and this is the non-trivial part of this theorem. For the complete proof see [37].

It is well known that a finite-dimensional Hamiltonian system with $N$ degrees of freedom is completely integrable if it has $N$ independent integrals of motion in involution. Likewise, in infinitely many dimensions we can consider the system to be integrable if it has an infinite hierarchy of symmetries or independent functionally conserved quantities. Hence, a bi-Hamiltonian system from Theorem 2.27 is integrable or even completely integrable, but still one needs to show that the Hamiltonians $H_n$ are functionally independent. Notice also that there are known examples of completely integrable systems that are bi-Hamiltonian with respect to a non-compatible pair of Poisson tensors. As a closing remark, in order to stress the importance of the bi-Hamiltonian structures for evolution systems let us quote Dickey [2]

> "The existence of two compatible Poisson (or Hamiltonian) structures is a remarkable feature of the most, if not all, integrable systems, sometimes it is considered as the essence of the integrability."

**Example 2.28** The Korteweg-de Vries equation has bi-Hamiltonian structure [31]

$$u_{t_3} = \frac{1}{4}u_{3x} + \frac{3}{2}uu_x = \pi_0 dH_3 = \pi_1 dH_1,$$

where the Poisson tensors

$$\begin{aligned} \pi_0 &= 2\partial_x \\ \pi_1 &= \frac{1}{2}\partial_x^3 + u\partial_x + \partial_x u \end{aligned}$$

and Hamiltonians are

$$\begin{aligned} H_1 &= \int_\Sigma \frac{1}{4}u^2 \, dx \\ H_3 &= \int_\Sigma \frac{1}{16}\left(2u^3 - {u_x}^2\right) dx. \end{aligned}$$

Hence, the form of the (hereditary) recursion operator is

$$\Phi = \pi_1 \pi_0^{-1} = \frac{1}{4}\left(\partial_x^2 + 4u + 2u_x\partial_x^{-1}\right). \tag{2.23}$$

We find the KdV hierarchy (1.12) with bi-Hamiltonian structure

$$u_{t_n} = \Phi^{\frac{n-1}{2}} u_x = \pi_0 dH_n = \pi_1 dH_{n-2} \tag{2.24}$$

given for Hamiltonians (1.13).

**Example 2.29** Consider the following $2$-component system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -u_{2x} + 2v_x \\ -\frac{2}{3}u_{3x} + v_{2x} - \frac{2}{3}uu_x \end{pmatrix} = \pi_0 dH_1 = \pi_1 dH_0.$$

Eliminating the field $v$ from the above system one finds the Boussinesq equation

$$u_{2t} + \frac{1}{3}\left(u_{3x} + 4uu_x\right)_x = 0.$$

The related bi-Hamiltonian structure is defined by the following Poisson tensors

$$\pi_0 = \begin{pmatrix} 0 & 3\partial_x \\ 3\partial_x & 0 \end{pmatrix}$$

[2] [13], page 43.

and

$$\pi_1 = \begin{pmatrix} 2\partial_x^3 + \partial_x u + u\partial_x & -\partial_x^4 - \partial_x^{-2} u + 2\partial_x v + v\partial_x \\ \partial_x^4 + u\partial_x^2 + \partial_x v + 2v\partial_x & -\frac{2}{3}\left(\partial_x^5 + \partial_x^3 u + u\partial_x^3 + u\partial_x u\right) + \partial_x^2 v - v\partial_x^2 \end{pmatrix}.$$

The related Hamiltonians are

$$H_0 = \int_\Sigma v \; dx$$
$$H_1 = \int_\Sigma \frac{1}{135}\left(-5u^3 + 45v^2 - 45u_x v - 15uu_{2x} - 3u_{4x}\right) \; dx.$$

**Example 2.30** The next example is the Nonlinear Schrödinger equation

$$u_t = iu_{2x} + 2iu\,|u|^2\,,$$

where $i$ is the imaginary unit. Taking the complex conjugation the above equatuion can be turn into bi-hamiltonian one

$$\begin{pmatrix} u \\ \bar{u} \end{pmatrix}_t = \begin{pmatrix} iu_{2x} + 2iu\,|u|^2 \\ -i\bar{u}_{2x} - 2i\bar{u}\,|u|^2 \end{pmatrix} = \pi_0 dH_1 = \pi_0 dH_0,$$

where

$$\pi_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\pi_1 = \begin{pmatrix} 2u\partial_x^{-1} & -\partial_x - 2u\partial_x^{-1}\bar{u} \\ -\partial_x - 2\bar{u}\partial_x^{-1} u & 2\bar{u}\partial_x^{-1}\bar{u} \end{pmatrix}$$

and

$$H_0 = \int_\Sigma \frac{i}{2}\left(\bar{u}_x u - u_x \bar{u}\right) \; dx$$
$$H_1 = \int_\Sigma \left(|u_x|^2 - |u|^4\right) \; dx.$$

**Remark 2.31** There is a very important subclass of dynamical systems that we have not mentioned yet, namely the so-called integrable dispersionless (or equivalently hydrodynamic) systems. The $(1+1)$-dimensional dispersionless systems are described by the first-order partial differential equations of the form

$$u_t^i = \sum_j A_i^j(u) u_x^j \qquad i, j = 1, \ldots, n.$$

The theory describing these equations belongs to the most recent ones and has been systematically developed from the 1980's. In the beginning the theory was mainly developed by the Russian school, represented by such researchers as: S. Novikov, B. Dubrovin, S. Tsarev, O. Mokhov, E. Ferapontov, M. Pavlov and others, and presently it is developed at academic centers all over the world. Intensive research into integrable dispersionless systems started since S. Tsarev

[49] proposed the technique of linearization for a certain class of dispersionless systems, written in the so-called Riemann invariants, through the so-called generalized hodograph method, and then showed the way of finding solutions using quadratures.

The study of the Poisson structures of dispersionless systems was initiated by B. Dubrovin and S. Novikov [20]. They established a remarkable result that Poisson tensors of hydrodynamic type can be generated by contravariant nondegenerate flat Riemannian metrics. These Poisson tensors have the form (2.25) with $c = 0$, where $g^{ij}$ is a contravariant nondegenerate flat metric and $\Gamma^{ij}_k$ are the components of the contravariant Levi-Civita connection. This result was later widely investigated and the general theory of Poisson tensors of hydrodynamic type generated by Riemannian metrics of constant curvature was presented by O. Mokhov and E. Ferapontov [33]. These Poisson tensors of hydrodynamic type are nonlocal and have the form

$$\pi_{ij} = g^{ij}(u)\partial_x - \sum_k \Gamma^{ij}_k(u)u^k_x + c\, u^i_x\partial_x^{-1}u^j_x, \qquad (2.25)$$

where $g^{ij}$ is non-degenerate metric of constant curvature $c$. Nevertheless, the condition of nondegeneracy of $g^{ij}$ for the above Poisson tensors is not necessary. The degenerate hydrodynamic Poisson tensors were considered by Grinberg [24] and Dorfman [16].

A natural geometric setting of related bi-Hamiltonian structures (Poisson pencils) is the theory of Frobenius manifolds based on the geometry of pencils of contravariant Riemannian metrics [19]. The Frobenius manifolds were introduced by B. Dubrovin [18] as a coordinate-free form of the associativity equations appearing in the context of deformations of 2-dimensional topological field theories (TFT) studied in the early 90s by E. Witten, R. Dijkgraaf, E. Verlinde and H. Verlinde [50, 14]. These equations are called the WDVV equations and can be identified with hydrodynamic-type systems. Thus, solutions of dispersionless systems can be understood as particular solutions of the TFT. In the same period E. Witten formulated his famous conjecture that the free energy of 2-dimensional gravity coincides with the $\tau$-function of the KdV hierarchy [51]. This conjecture was later proved by M. Kontsevich in [26]. The theory of Frobenius manifolds as well as the WDVV equations play a major role in the quantum cohomology and the theory of Gromov-Witten invariants [27, 19].

## 2.3 Exercises

1. Prove Proposition 2.8. Hint: you can use (2.15) instead of (1.18).

2. Prove Proposition 2.11. It suffices to show that

$$[\mathbf{v},\mathbf{w}]^i := \mathbf{w}'[v^i] - \mathbf{v}'[w^i] = \partial_{\mathbf{v}}(w^i) - \partial_{\mathbf{w}}(v^i) \qquad i = 1,\dots,N.$$

3. Show that the commutativity of two vector fields with respect to the Lie bracket (2.14) (or (2.7)) is equivalent to the commutativity of the respective flows (2.3).

4. Show the formulae (2.17) and (2.18).

5. Prove Proposition 2.18. Hint: compute the Lie derivative along $\mathbf{K} = \pi dH$ of the Poisson bracket (2.19) of two arbitrary functionals different from Hamiltonian $H$. Then to show the proposition first use the Leibniz rule for the Lie derivative and then the Jacobi identity for the Poisson bracket.

6. Show by explicit computation that (2.23) is a hereditary operator.

7. Check if the bi-Hamiltonian hierarchy (2.24) agrees with the hierarchies of symmetries (1.12) and Hamiltonians (1.13).

8. Find the recursion operators for the systems from Examples 2.29 and 2.30 and using the translational symmetry 2.4 as a starting point, construct first three symmetries from the related hierarchies.

9. Consider Theorem 2.27. Assuming existence of functionals $H_0, H_1, H_2, \ldots$ show that they are pairwise in involution with respect to both Poisson brackets associated with $\pi_0$ and $\pi_1$.

10. It is well known that one can construct integrable dispersionless systems by taking the so-called quasi-classical or dispersionless limit (if it exists) of the soliton systems. To go to the dispersionless limit one first has to make the following transformation of independent variables

$$\begin{array}{c} t \mapsto \frac{1}{\hbar} t \\ x \mapsto \frac{1}{\hbar} x \end{array} \quad \Longrightarrow \quad \begin{array}{c} \frac{\partial}{\partial t} \mapsto \hbar \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \mapsto \hbar \frac{\partial}{\partial x} \end{array},$$

where $\hbar$ is a deformation parameter, and then take the limit $\hbar \to 0$. Find the dispersionless limit of integrable soliton systems from Examples 2.28 and 2.29. What is the bi-Hamiltonian structure for these dispersionless systems?

# 3 Classical $R$-matrix theory

In the theory of evolutionary systems one of the most important issues is a systematic method for construction of integrable systems. It is well known that a very powerful tool, called the classical R-matrix formalism [39], proved to be very fruitful in the systematic construction of the field and lattice soliton systems as well as dispersionless systems. The crucial point of the formalism is the observation that integrable dynamical systems can be obtained from the Lax equations on appropriate Lie algebras. The greatest advantage of this formalism, besides the possibility of systematic construction of the integrable systems, is the construction of bi-Hamiltonian structures and infinite hierarchies of symmetries and conserved quantities.

In this chapter we will present a unified approach to the construction of integrable evolution equations together with their (multi-)Hamiltonian structures. This approach is based on the classical $R$-matrix formalism that originated from the pioneering article [23] by Gelfand and Dickey, where they presented the construction of Hamiltonian soliton systems of KdV type by means of pseudo-differential operators. Next, Adler [2] showed how to construct, within classical $R$-matrix formalism, the bi-Hamiltonian structures for the above soliton systems. This scheme, well-known today, is now called the Adler-Gelfand-Dickey (AGD) scheme. Later the abstract formalism of classical $R$-matrices, applicable to appropriate Lie algebras, was formulated in [38, 17, 39]. In [29, 36] it was shown that there are in fact three natural Poisson brackets associated with classical $R$-structures. Quite recently Li [30] considered the classical $R$-matrix theory on the so-called (commutative) Poisson algebras. This approach leads to the construction of dispersionless multi-Hamiltonian systems.

In this chapter we illustrate the theory of classical $R$-matrices only on example of the pseudo-differential operators algebra and the simplest Gelfand-Dickey Lax operators with the related Lax hierarchies. For more general applications of the theory see [4]-[11], [43]-[48] and references therein. Many important subjects related to the theory of $R$-matrices like factorization problem or central extensions are not presented here. For a nice review about these and even more see [40].

## 3.1 Classical $R$-matrix formalism

Let $\mathfrak{g}$ be an algebra with respect to some multiplication, over a commutative field of complex or real numbers, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Further assume that $\mathfrak{g}$ is equipped with an additional bilinear product given by a Lie bracket $[\cdot,\cdot] : \mathfrak{g} \to \mathfrak{g}$, which is skew-symmetric and satisfies the Jacobi identity.

**Definition 3.1** *A linear map $R : \mathfrak{g} \to \mathfrak{g}$ such that the bracket*

$$[a,b]_R := [Ra,b] + [a,Rb] \tag{3.1}$$

*is another Lie bracket on $\mathfrak{g}$ is called the classical $R$-matrix.*

The skew-symmetry of (3.1) is obvious. As for the Jacobi identity for (3.1) one finds that

$$
\begin{aligned}
0 = [a,[b,c]_R]_R + \text{c.p.} &= [Ra,[Rb,c]] + [Ra,[b,Rc]] + [a,R\,[b,c]_R] + \text{c.p.}\\
&= [Rb,[Rc,a]] + [Rc,[a,Rb]] + [a,R\,[b,c]_R] + \text{c.p.}\\
&= [a,R\,[b,c]_R - [Rb,Rc]] + \text{c.p.},
\end{aligned}
\tag{3.2}
$$

where c.p. stands for cyclic permutations in $\{a,b,c\}$ and the last equality follows from the Jacobi identity for $[\cdot,\cdot]$. Hence, a sufficient condition for $R$ to be a classical $R$-matrix is to satisfy the following so-called Yang-Baxter equation, YB($\alpha$),

$$
[Ra,Rb] - R\,[a,b]_R + \alpha\,[a,b] = 0,
\tag{3.3}
$$

where $\alpha$ is some number from $\mathbb{K}$. There are only two relevant cases of YB($\alpha$), namely $\alpha \neq 0$ and $\alpha = 0$, as Yang-Baxter equations for $\alpha \neq 0$ are equivalent through reparameterization.

## 3.2 Lax hierarchy

Assume now that the Lie bracket $[\cdot,\cdot]$ is a derivation with respect to the multiplication, i.e., this bracket satisfies the Leibniz rule

$$
[a,bc] = b\,[a,c] + [a,b]\,c.
\tag{3.4}
$$

Notice that this condition is satisfied automatically in the case of a commutative algebra $\mathfrak{g}$ when the Lie bracket is given by a Poisson bracket as well as in the case of a non-commutative algebra $\mathfrak{g}$ and the Lie bracket given by the commutator. Then, any well-defined smooth map

$$
X:\mathfrak{g}\to\mathfrak{g} \qquad L\mapsto X(L)
\tag{3.5}
$$

is an invariant of the Lie bracket, i.e.

$$
[X(L),L] = 0 \qquad L\in\mathfrak{g}.
\tag{3.6}
$$

Moreover, the following relation holds

$$
dX(L)\circ[L',L] = [L',X(L)] \qquad L,L'\in\mathfrak{g},
$$

where $dX$ is the differential of the smooth map (3.5), i.e.

$$
dX(L):\mathfrak{g}\to\mathfrak{g} \qquad L_t\mapsto dX(L)\circ L_t = (X(L))_t\,.
$$

The power functions $X(L)=L^n$ are always well defined on $\mathfrak{g}$ and are invariant functions of the Lie bracket. One can consider less trivial functions, for example the logarithmic ones, like $X(L)=\ln L$, but only when they have proper interpretation in $\mathfrak{g}$.

**Example 3.2** For the power function $X(L) = L^n$, where $n = 1, 2, \dots$, which is a smooth function, one finds that

$$dX(L) : \mathfrak{g} \to \mathfrak{g} \qquad L' \mapsto dX(L) \circ L' = \sum_{k=1}^{n} L^{k-1} L' L^{n-k}.$$

In the case of a commutative algebra $\mathfrak{g}$ simply $dX(L) = nL^{n-1}$ and $\circ$ is substituted by a multiplication from $\mathfrak{g}$.

Smooth functions $X_n(L)$ generate a hierarchy of vector fields on $\mathfrak{g}$ of the form

$$L_{t_n} = \left[RX_n(L), L\right], \tag{3.7}$$

where $t_n$ are evolution parameters. These vector fields yield self-consistent evolutions on $\mathfrak{g}$ when the left- and right-hand sides of (3.7) span the same subspace of $\mathfrak{g}$. So, the element $L$ of $\mathfrak{g}$ has to be properly chosen. The directional derivative of a smooth function $F : \mathfrak{g} \to \mathfrak{g}$ in the direction of (3.7) is given by

$$F(L)_{t_n} = dF(L) \circ L_{t_n} = dF(L) \circ [RX_n(L), L] = [RX_n(L), F(L)]. \tag{3.8}$$

There is an important issue of whether the vector fields (3.7) commute. One finds that

$$\begin{aligned}
&(L_{t_m})_{t_n} - (L_{t_n})_{t_m} = [RX_m(L), L]_{t_n} - [RX_n(L), L]_{t_m} \\
&= \left[(RX_m(L))_{t_n} - (RX_n(L))_{t_m}, L\right] + [RX_m(L), [RX_n(L), L]] - [RX_n(L), [RX_m(L), L]] \\
&= \left[(RX_m(L))_{t_n} - (RX_n(L))_{t_m} + [RX_m(L), RX_n(L)], L\right].
\end{aligned} \tag{3.9}$$

Hence, the vector fields (3.7) mutually commute if the so-called zero-curvature equations

$$(RX_m(L))_{t_n} - (RX_n(L))_{t_m} + [RX_m(L), RX_n(L)] = 0 \tag{3.10}$$

are satisfied. In this case the hierarchy (3.7) is called the Lax hierarchy and $L$ is called the Lax operator or the Lax function depending on the nature of a given Lie algebra $\mathfrak{g}$. Now we have to additionally assume that $R$-matrices commute with directional derivatives and hence with the derivatives with respect to evolution parameters, i.e.

$$(RL)_t = RL_t. \tag{3.11}$$

This property is equivalent to the assumption that $R$ commute with differentials of smooth maps $\mathfrak{g} \to \mathfrak{g}$. This property is important although is not explicitly stressed in most works on the $R$-matrices. Then

$$\begin{aligned}
&R\,(X_m(L))_{t_n} - R\,(X_n(L))_{t_m} + [RX_m(L), RX_n(L)] = \\
&\overset{\text{by (3.11) and (3.8)}}{=} R\,[RX_n(L), X_m(L)] - R\,[RX_m(L), X_n(L)] + [RX_m(L), RX_n(L)] \\
&= [RX_m(L), RX_n(L)] - R\,[X_m(L), X_n(L)]_R.
\end{aligned} \tag{3.12}$$

Now, if an $R$-matrix satisfies the Yang-Baxter equation (3.3) the last expression is equal to $-\alpha\,[X_m(L), X_n(L)] = 0$. Hence, the following proposition is valid.

**Proposition 3.3** *The Yang-Baxter equation is a sufficient condition for the pairwise commutativity of the vector fields from the Lax hierarchy* (3.7).

It is natural to ask when the abstract Lax hierarchy (3.7) represents a "real" hierarchy of integrable evolution systems on a suitable functional space, i.e., on the related infinite-dimensional phase space $\mathcal{U}$. This occurs when one can construct an embedding map $\iota$ from $\mathcal{U}$ to $\mathfrak{g}$, which induces the smooth manifold structure on $\mathfrak{g}$ such that

$$\begin{aligned} \iota : \mathcal{U} \to \mathfrak{g} \qquad & \mathbf{u} \mapsto \iota(\mathbf{u}) = L \\ d\iota : \mathcal{V} \to \mathfrak{g} \qquad & \mathbf{u}_t \mapsto d\iota(\mathbf{u}_t) = L_t, \end{aligned}$$

where $d\iota$ is the differential of $\iota$. (Notices that for $\iota$ being an embedding its differential $d\iota$ is an injective map.) In such a case the Lax hierarchy (3.7) can be pulled back to the original functional space by $(d\iota)^{-1}$. The symmetries from the Lax hierarchy (3.7) represent consistent evolution systems when the left- and right-hand sides of (3.7) span the same subspace of $\mathfrak{g}$. So, the Lax element $L$ of $\mathfrak{g}$ has to be chosen in a suitable fashion.

## 3.3 Simplest $R$-matrices

Assume that the Lie algebra $\mathfrak{g}$ from the previous section can be split into a direct sum of Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, i.e.,

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \qquad [\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm \qquad \mathfrak{g}_+ \cap \mathfrak{g}_- = \emptyset.$$

Upon denoting the projections onto these subalgebras by $P_\pm$, we define a linear map $R : \mathfrak{g} \to \mathfrak{g}$ as

$$R = \frac{1}{2}(P_+ - P_-). \tag{3.13}$$

Using the equality $P_+ + P_- = 1$, (3.13) can be represented in the following equivalent forms:

$$R = P_+ - \frac{1}{2} = \frac{1}{2} - P_-.$$

Let $a_\pm := P_\pm(a)$ for $a \in \mathfrak{g}$. Then

$$[a,b]_R = [a_+, b_+] - [a_-, b_-] \qquad \Longrightarrow \qquad R\,[a,b]_R = \frac{1}{2}\,[a_+, b_+] + \frac{1}{2}\,[a_-, b_-] \tag{3.14}$$

and

$$[Ra, Rb] = \frac{1}{4}\,[a_+, b_+] - \frac{1}{4}\,[a_+, b_-] - \frac{1}{4}\,[a_-, b_+] + \frac{1}{4}\,[a_-, b_-]\,. \tag{3.15}$$

Hence, the map (3.13) solves the Yang-Baxter equation (3.3) for $\alpha = \frac{1}{4}$ and is a well-defined classical $R$-matrix. This is the simplest and most important example of a well-defined $R$-matrix.

The Lax hierarchy (3.7) for the $R$-matrix (3.13) can be written in two equivalent ways:

$$L_{t_n} = [P_+\left(X_n(L)\right), L] = -\left[P_-\left(X_n(L)\right), L\right]. \tag{3.16}$$

Consider now more general situation when a given Lie algebra $\mathfrak{g}$ contains a set of $N$ disjoint Lie subalgebras $\mathfrak{g}_i \subset \mathfrak{g}$ such that their complements $\overline{\mathfrak{g}}_i$ to $\mathfrak{g}$ are also Lie subalgebras, i.e.,

$$\mathfrak{g} = \overline{\mathfrak{g}}_i \oplus \mathfrak{g}_i \qquad [\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_i \qquad [\overline{\mathfrak{g}}_i, \overline{\mathfrak{g}}_i] \subset \overline{\mathfrak{g}}_i \qquad \mathfrak{g}_i \cap \mathfrak{g}_j = \emptyset \quad i \neq j \tag{3.17}$$

for $i, j \in \{1, \ldots, N\}$. Then $\mathfrak{g}$ can be decomposed in the following way:

$$\mathfrak{g} = \overline{\mathfrak{g}} \oplus \left( \bigoplus_{i=1}^{N} \mathfrak{g}_i \right),$$

where $\overline{\mathfrak{g}}$ stands for the complement of $\left(\bigoplus_{i=1}^{N} \mathfrak{g}_i\right)$ to $\mathfrak{g}$. We make here an additional assumption that

$$[\overline{\mathfrak{g}}, \mathfrak{g}_i] \subset \overline{\mathfrak{g}} \oplus \mathfrak{g}_i \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_i \oplus \mathfrak{g}_j \qquad i \neq j. \tag{3.18}$$

The simplest situation occurs when $\overline{\mathfrak{g}} = \emptyset$. Let $\overline{P}, P_i$ be projections on $\overline{\mathfrak{g}}, \mathfrak{g}_i$, respectively. Then we obviously have

$$1 = \overline{P} + \sum_{k=1}^{N} P_k \qquad \Longrightarrow \qquad P_i = 1 - \overline{P} - \sum_{k \neq i} P_k. \tag{3.19}$$

In view of the considerations presented earlier in this section, the linear maps

$$R = P_i - \frac{1}{2} \qquad i = 1, 2, \ldots, N \tag{3.20}$$

are well-defined classical $R$-matrices, and we have a family of $N$ Lax hierarchies

$$L_{t_{i,n}} = \left[P_i \left(X_n(L)\right), L\right] \tag{3.21}$$

generated by smooth functions $X_n(L)$. We are asking whether these Lax hierarchies mutually commute. Proceeding in analogy with Section 3.2, one finds that

$$\begin{aligned}\left(L_{t_{i,m}}\right)_{t_{j,n}} - \left(L_{t_{j,n}}\right)_{t_{i,m}} = \\ &= \left[\left[P_i X_m(L), P_j X_n(L)\right] - P_j \left[P_i X_m(L), X_n(L)\right] - P_i \left[X_m(L), P_j X_n(L)\right], L\right],\end{aligned}$$

for $i \neq j$. Next, it follows from (3.18) and (3.19) that

$$\begin{aligned}\left[P_i X_m(L), P_j X_n(L)\right] &= P_i \left[P_i X_m(L), P_j X_n(L)\right] + P_j \left[P_i X_m(L), P_j X_n(L)\right] \\ &= P_i \left[X_m(L), P_j X_n(L)\right] + P_j \left[P_i X_m(L), X_n(L)\right] \qquad i \neq j.\end{aligned}$$

Hence, the following proposition is proved.

**Proposition 3.4** *If a given Lie algebra $\mathfrak{g}$ contains disjoint Lie subalgebras $\mathfrak{g}_i$ that satisfy the conditions* (3.17) *and* (3.18)*, then all vector fields from all Lax hierarchies* (3.21) *mutually commute.*

For applications of the above proposition see [43].

## 3.4 The algebra of pseudo-differential operators

We will illustrate the theory of classical $R$-matrices by considering the algebra of pseudo-differential operators (PDO)

$$\mathfrak{g} = \left\{ L = \sum_{i \geqslant -\infty}^{N} u_i(x)\partial_x^i \right\}, \tag{3.22}$$

where the smooth functions $u_i(x)$ are dynamical fields, hence $u_i$ further depend on evolution parameters. The $\partial_x$ is operator related to the total derivative with respect to $x$. Thus, the multiplication in $\mathfrak{g}$ is defined through the so-called generalized Leibniz rule

$$\partial^m u(x) = \sum_{n \geqslant 0} \binom{m}{n} u(x)_{nx} \partial^{m-n}, \tag{3.23}$$

where $\binom{m}{n}$ stands for the standard binomial coefficient, and

$$\binom{m}{n} = (-1)^n \binom{-m+n-1}{n} \tag{3.24}$$

for $m < 0$. From (3.23) it follows that

$$\begin{aligned} \partial_x u &= u\partial_x + u_x \\ \partial^{-1} u &= u\partial_x^{-1} - \partial_x^{-1} u_x \partial_x^{-1} \\ &= u\partial_x^{-1} - u_x\partial_x^{-2} + u_{2x}\partial_x^{-3} - \dots \ , \end{aligned}$$

where $u$ is some smooth function. The algebra (3.22) with the multiplication defined through (3.23) is an associative and noncommutative algebra. Therefore, we have a well-defined Lie algebra structure on $\mathfrak{g}$ with the natural commutator

$$[A, B] = AB - BA \qquad A, B \in \mathfrak{g}.$$

Consider the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{\geqslant k} \oplus \mathfrak{g}_{<k} := \left\{ \sum_{i \geqslant k} u_i \partial_x^i \right\} \oplus \left\{ \sum_{i<k} a_i \partial_x^i \right\}. \tag{3.25}$$

Then, $\mathfrak{g}_{\geqslant k}$ and $\mathfrak{g}_{<k}$ are Lie subalgebras of $\mathfrak{g}$ only for $k = 0, 1, 2$. In these cases the classical $R$-matrices (3.13) are given by

$$R = \frac{1}{2}(P_{\geqslant k} - P_{<k}) = P_{\geqslant k} - \frac{1}{2} = \frac{1}{2} - P_{<k}, \tag{3.26}$$

where $P_{\geqslant k}$ and $P_{<k}$ are projections onto $\mathfrak{g}_{\geqslant k}$ and $\mathfrak{g}_{<k}$, respectively.

Consider an element $L$ from $\mathfrak{g}$ of the form

$$L = u_N\partial_x^N + u_{N-1}\partial_x^{N-1} + u_{N-2}\partial_x^{N-2} + \dots, \tag{3.27}$$

where $N > 0$. Then its $N$-th root

$$L^{\frac{1}{N}} = a_1\partial_x + a_0 + a_{-1}\partial_x^{-1} + a_{-2}\partial_x^{-2} + \dots,$$

where coefficients $a_i$ are differential function of $u_i$, can be constructed, solving recursively for the functions $a_i$, from the equality

$$\left(L^{\frac{1}{N}}\right)^N = \overbrace{L^{\frac{1}{N}} \cdot \ldots \cdot L^{\frac{1}{N}}}^{N} = L.$$

Hence, we can take the fractional powers of (3.27)

$$L^{\frac{n}{N}} = \overbrace{L^{\frac{1}{N}} \cdot \ldots \cdot L^{\frac{1}{N}}}^{n},$$

where $n = 1, 2, \ldots$, for the invariants (3.6). The fractional powers of $L$ generate the following Lax hierarchies (3.7) related to classical $R$-matrices (3.26):

$$L_{t_n} = \left[\left(L^{\frac{n}{N}}\right)_{\geqslant k}, L\right] = -\left[\left(L^{\frac{n}{N}}\right)_{<k}, L\right] \qquad k = 0, 1, 2. \tag{3.28}$$

Nevertheless, we are interested in the construction of finite-component integrable evolution systems. Hence, we need to consider some restrictions of (3.27) yielding consistent Lax equations (3.28) such that right- and left-hand sides of (3.28) span the same subspace of $\mathfrak{g}$. We will consider only the simplest case $k = 0$. For the operator (3.27) one finds that

$$\begin{aligned} L_t &= (u_N)_t \partial_x^N + (u_{N-1})_t \partial_x^{N-1} + (u_{N-2})_t \partial_x^{N-2} + \mathit{lower\ terms}, \\ L_t &= -\left[A_{<0}, L\right] = -\left[\gamma\partial_x^{-1} + l.t., u_N\partial_x^N + l.t.\right] = \left(N u_N \gamma_x + (u_N)_x \gamma\right)\partial_x^{N-2} + l.t.\ . \end{aligned}$$

Here $l.t.$ denotes the lower-order terms. Thus, we find that the fields $u_N$ and $u_{N-1}$ must be time-independent and hence they are not dynamical fields. Without loosing generality we can choose them to be the following constants: $u_N = 1$ and $u_{N-1} = 0$. On the other hand, the zero-order terms in (3.28) for $N > 1$ are always present. Therefore we can restrict (3.27) to the following form:

$$k = 0: \quad L = \partial_x^N + u_{N-2}\partial_x^{N-2} + \cdots + u_1\partial_x + u_0. \tag{3.29}$$

These are the well-known Gelfand-Dickey Lax operators [23] yielding consistent Lax equations. The related Lax hierarchy (3.28) for $k = 0$ with the Lax operators of the form (3.29) is called the Gelfand-Dickey hierarchy. Notice that the equations from hierarchy (3.28) for (3.29) are trivial when $n$ is a multiple of $N$ since in this case $L^{\frac{n}{N}} \in \mathfrak{g}_{\geqslant 0}$. More general theory of the application of the $R$-matrix formalism to the algebra of pseudo-differential operators dealing with the remaining values of $k$ can be found in [25] or [4].

**Example 3.5** Consider the $N = 2$ case of (3.27). Then

$$L = \partial_x^2 + u.$$

One finds that

$$\begin{aligned} L^{\frac{1}{2}} &= \partial_x + \frac{1}{2}u\partial_x^{-1} - \frac{1}{4}u_x\partial_x^{-2} + \frac{1}{8}\left(u_{2x} - u\right)\partial_x^{-3} - \frac{1}{16}\left(u_{3x} - 6uu_x\right)\partial_x^{-4} \\ &\quad + \frac{1}{32}\left(u_{4x} - 14uu_{2x} - 11{u_x}^2 + 2u^3\right)\partial_x^{-5} + \ldots \end{aligned}$$

and

$$L^{\frac{3}{2}} = L \cdot L^{\frac{1}{2}} = \partial^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x + (\ldots)\partial_x^{-1} + \ldots\ .$$

Hence

$$\left(L^{\frac{3}{2}}\right)_{\geqslant 0} = \partial^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x$$

and we recover the Lax equation for the KdV system

$$L_{t_3} = \left[\left(L^{\frac{3}{2}}\right)_{\geqslant 0}, L\right] \quad \Longleftrightarrow \quad u_{t_3} = \frac{1}{4}u_{3x} + \frac{3}{2}uu_x. \tag{3.30}$$

The whole KdV hierarchy can be constructed in a similar fashion.

**Example 3.6** The Lax operator (3.27) for $N = 3$ has the form

$$L = \partial_x^3 + u\partial_x + v.$$

We have

$$\begin{aligned} L^{\frac{1}{3}} = {} & \partial_x + \frac{1}{3}u\partial_x^{-1} - \frac{1}{3}\left(u_x - v\right)\partial_x^{-2} + \frac{1}{9}\left(2u_{2x} - 3v_x - u^2\right)\partial_x^{-3} \\ & - \frac{1}{9}\left(u_{3x} - 2v_{2x} - 4uu_x + 2uv\right)\partial_x^{-4} \\ & + \frac{1}{81}\left(3u_{4x} - 9v_{3x} - 45uu_{2x} + 36uv_x - 45{u_x}^2 + 45u_xv - 9v^2 + 5u^3\right)\partial_x^{-5} + \ldots\ . \end{aligned}$$

Then, for

$$\left(L^{\frac{2}{3}}\right)_{\geqslant 0} = \partial_x^2 + \frac{2}{3}u$$

one finds the Lax equation for the Boussinesq system

$$L_{t_2} = \left[\left(L^{\frac{2}{3}}\right)_{\geqslant 0}, L\right] \quad \Longleftrightarrow \quad \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} -u_{2x} + 2v_x \\ -\frac{2}{3}u_{3x} + v_{2x} - \frac{2}{3}uu_x \end{pmatrix}. \tag{3.31}$$

## 3.5 Lie-Poisson structures

Let $\mathfrak{g}^*$ be a dual of a given Lie algebra $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{K}$ be the usual duality pairing. The Lie bracket $[\cdot,\cdot]$ defines the adjoint action $\mathrm{ad}$ of $\mathfrak{g}$ on $\mathfrak{g}$:

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \qquad (a,b) \mapsto \mathrm{ad}_a b = [a,b]\,.$$

Then the co-adjoint action $\mathrm{ad}^*$ of $\mathfrak{g}$ on $\mathfrak{g}^*$ is defined by the relation

$$\langle \mathrm{ad}_a^*\eta, b\rangle + \langle \eta, \mathrm{ad}_a b\rangle = 0 \quad \Longleftrightarrow \quad \langle a \diamond \eta, b\rangle = \langle \eta, [a,b]\rangle \qquad a,b \in \mathfrak{g} \quad \eta \in \mathfrak{g}^*, \tag{3.32}$$

where

$$a \diamond \eta := -\mathrm{ad}_a^*\eta.$$

We will often use the above simplified notation for the co-adjoint action.

Let this time $\iota : \mathcal{U} \to \mathfrak{g}^*$ be the embedding of the original phase space into the dual Lie algebra, i.e.

$$\begin{aligned} \iota : \mathcal{U} \to \mathfrak{g}^* \qquad & \mathbf{u} \mapsto \iota(\mathbf{u}) = \eta \\ d\iota : \mathcal{V} \to \mathfrak{g}^* \qquad & \mathbf{u}_t \mapsto d\iota(\mathbf{u}_t) = \eta_t. \end{aligned}$$

Then every functional $F : \mathcal{U} \to \mathbb{K}$ can be extended to the smooth function on $\mathfrak{g}^*$. Therefore, let $\mathcal{F}(\mathfrak{g}^*)$ be the space of all smooth functions on $\mathfrak{g}^*$ of the form $F \circ \iota^{-1} : \mathfrak{g}^\star \to \mathbb{K}$, where $F \in \mathcal{F}$. Then differentials $dF(\eta)$ of $F(\eta) \in \mathcal{F}(\mathfrak{g}^*)$ at the point $\eta \in \mathfrak{g}^*$ belong to $\mathfrak{g}$, because they can be computed through the relation

$$F(\eta)'[\xi] := \left.\frac{dF(\eta + \varepsilon\xi)}{d\varepsilon}\right|_{\varepsilon=0} = \langle \xi, dF(\eta) \rangle \qquad \xi \in \mathfrak{g}^* \tag{3.33}$$

being the counterpart of (2.13). We also have an analogue of the relation (2.16):

$$\langle \xi, dF'[\eta] \rangle = \langle \eta, dF'[\xi] \rangle \qquad F \in \mathcal{F}(\mathfrak{g}^*) \quad \eta, \xi \in \mathfrak{g}^*. \tag{3.34}$$

We make an additional assumption that the Lie bracket in $\mathfrak{g}$ is such that the directional derivative along an arbitrary $\xi \in \mathfrak{g}^\star$ is a derivation with respect to the Lie bracket. This means that the following relation holds

$$[a, b]'\,[\xi] = [a'[\xi], b] + [a, b'[\xi]] \qquad a, b \in \mathfrak{g}. \tag{3.35}$$

**Theorem 3.7** *On $\mathcal{F}(\mathfrak{g}^*)$ there exists a Poisson bracket defined as follows*

$$\{H, F\}\,(\eta) := \langle \eta, [dF, dH] \rangle \qquad \eta \in \mathfrak{g}^* \quad H, F \in \mathcal{F}(\mathfrak{g}^*) \tag{3.36}$$

*This bracket is called a (natural) Lie-Poisson bracket.*

**Lemma 3.8** *The differential of* (3.36) *is given by*

$$d\,\{H, F\} = [dF, dH] - dF'[dH \diamond \eta] + dH'[dF \diamond \eta]. \tag{3.37}$$

**Proof.** By (3.33) one finds that

$$\begin{aligned} \{H, F\}'\,[\xi] &= \langle \eta'[\xi], [dF, dH] \rangle + \langle \eta, [dF'[\xi], dH] + [dF, dH'[\xi]] \rangle \\ &\overset{\text{by (3.32)}}{=} \langle \xi, [dF, dH] \rangle - \langle dH \diamond \eta, dF'[\xi] \rangle + \langle dF \diamond \eta, dH'[\xi] \rangle \\ &\overset{\text{by (3.34)}}{=} \langle \xi, [dF, dH] - dF'[dH \diamond \eta] + dH'[dF \diamond \eta] \rangle\,, \end{aligned}$$

and the result of the lemma follows. □

**Proof of Theorem 3.7.** The bilinearity and skew-symmetry of (3.36) is obvious. So, we only have to prove the Jacobi identity:

$$
\begin{aligned}
\{F,\{G,H\}\} + c.p. &= \langle \eta, [d\{G,H\}, dH] + c.p. \rangle \\
&\overset{\text{by (3.37)}}{=} \langle \eta, [[dH,dG], dF] - [dH'[dG \diamond \eta], dF] + [dG'[dH \diamond \eta], dF] + c.p. \rangle \\
&\overset{\text{by (3.32)}}{=} \langle \eta, [[dH,dG], dF] \rangle + \langle dF \diamond \eta, dH'[dG \diamond \eta] \rangle - \langle dF \diamond \eta, dG'[dH \diamond \eta] \rangle + c.p. \\
&= \langle \eta, [[dH,dG], dF] \rangle + \langle dF \diamond \eta, dH'[dG \diamond \eta] \rangle - \langle dG \diamond \eta, dH'[dF \diamond \eta] \rangle + c.p. \\
&\overset{\text{by (3.34)}}{=} \langle \eta, [[dH,dG], dF] + c.p. \rangle = 0,
\end{aligned}
$$

where the last equality follows from the Jacobi identity for $[\cdot,\cdot]$. $\square$

Now assume that we have an additional Lie bracket (3.1) on $\mathfrak{g}$ defined through the classical $R$-matrix such that (3.11) is valid. Then (3.1) satisfies the condition (3.35). As a result, on the space of scalar fields $\mathcal{F}(\mathfrak{g}^*)$ there is another well-defined (by Theorem 3.7) Lie-Poisson bracket:

$$\{H,F\}_R(\eta) := \langle \eta, [dF, dH]_R \rangle \qquad \eta \in \mathfrak{g}^* \quad H,F \in \mathcal{F}(\mathfrak{g}^*). \tag{3.38}$$

Using (3.32) one finds that the related Poisson operators at $\eta \in \mathfrak{g}^*$ for the above Lie-Poisson brackets have the form

$$
\begin{aligned}
\{H,F\} &= \langle \pi dH, dF \rangle &\iff&& \pi &: dH \mapsto \mathrm{ad}^*_{dH}\eta \\
\{H,F\}_R &= \langle \pi_R dH, dF \rangle &\iff&& \pi_R &: dH \mapsto \mathrm{ad}^*_{RdH}\eta + R^*\mathrm{ad}^*_{dH}\eta,
\end{aligned}
$$

where the adjoint of $R$ is defined by the relation

$$\langle R^*\eta, a \rangle = \langle \eta, Ra \rangle \qquad \eta \in \mathfrak{g}^* \quad a \in \mathfrak{g}.$$

The Casimir functions $\mathcal{C}_n(\eta) \in \mathcal{F}(\mathfrak{g}^*)$ of the natural Lie-Poisson bracket (3.36) satisfy the following condition

$$\forall\ F \in \mathcal{F}(\mathfrak{g}^*) \quad \{F, C_n\} = 0 \qquad \iff \qquad \mathrm{ad}^*_{dC_n}\eta = 0,$$

that is, their differentials are $\mathrm{ad}^*$-invariant. Hence, they are in involution with respect to the Lie-Poisson bracket (3.38), i.e.

$$\{C_n, C_m\}_R = 0.$$

Now, as $\pi_R d$ is a Lie algebra homomorphism, the related Hamiltonian vector fields with the Casimir functions as Hamiltonians,

$$\eta_{t_n} = \pi_R dC_n(\eta) = \mathrm{ad}^*_{RdC_n}\eta \tag{3.39}$$

pairwise commute, i.e.,

$$(\eta_{t_m})_{t_n} = (\eta_{t_n})_{t_m}.$$

**Proposition 3.9** *All evolution systems in the hierarchy* (3.39) *on* $\mathfrak{g}^*$ *generated by the Casimir functions* $C_n$ *of the natural Lie-Poisson bracket* (3.36) *pairwise commute and are Hamiltonian with respect to* (3.38)*. Moreover, any equation from* (3.39) *admits all Casimir functions* $C_n$ *as conserved quantities.*

The construction of Casimir functions $C_n$ and related dynamical systems (3.39) on the dual Lie algebra $\mathfrak{g}^*$, contrary to (3.7), is rather difficult and quite impractical. Thus, a formulation of a similar theory on $\mathfrak{g}$ instead on $\mathfrak{g}^*$ is justified. This can be done when one can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of a suitable scalar product.

## 3.6 $Ad$-invariant scalar products

We restrict our further considerations to the Lie algebras $\mathfrak{g}$ for which its dual $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ through the duality map. So, we assume the existence of a scalar product

$$(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\rightarrow\mathbb{K} \tag{3.40}$$

on $\mathfrak{g}$, and we assume this product to be symmetric,

$$(a,b)=(b,a)\qquad a,b\in\mathfrak{g},$$

and non-degenerate, that is, $a=0$ is the only element of $\mathfrak{g}$ that satisfies

$$(a,b)=0\qquad\forall\ b\in\mathfrak{g}.$$

Then, we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ ($\mathfrak{g}^*\cong\mathfrak{g}$) by setting

$$\langle\eta,b\rangle=(c,b)\qquad\forall\ b\in\mathfrak{g},$$

where $\eta\in\mathfrak{g}^*$ is identified with $c\in\mathfrak{g}$. We also make an additional assumption that the symmetric product (3.40) is $ad$-invariant, i.e.,

$$([a,c],b)+(c,[a,b])=0\qquad a,b,c\in\mathfrak{g}. \tag{3.41}$$

This is a counterpart of the relation (3.32). Thus, if $\eta\in\mathfrak{g}^*$ is identified with $c\in\mathfrak{g}$ we have

$$\langle\mathrm{ad}^*_a\eta,b\rangle=([a,c],b)\qquad a,b\in\mathfrak{g}$$

and one identifies $\mathrm{ad}^*_a\eta\in\mathfrak{g}^*$ with $\mathrm{ad}_a c\in\mathfrak{g}$.

Now consider the case when $R$ is a well defined classical $R$-matrix which does not necessarily satisfy the Yang-Baxter equation (3.3). Let $X_m(L)$ will be smooth invariant maps (3.6) generating hierarchy (3.7). Then for an arbitrary $L'\in\mathfrak{g}$ we have

$$\begin{aligned}
&(L',[[RX_m,RX_n]-R[X_m,X_n]_R,L])\overset{\text{by (3.41)}}{=}([L',[RX_m,RX_n]-R[X_m,X_n]_R],L)\\
&\overset{\text{by (3.2)}}{=}([X_m,[RL',RX_n]-R[L',X_n]_R],L)+([X_n,[RX_m,RL']-R[X_m,L']_R],L)\\
&\overset{\text{by (3.41)}}{=}([RL',RX_n]-R[L',X_n]_R,[L,X_m])+([RX_m,RL']-R[X_m,L']_R,[L,X_n])\overset{\text{by (3.6)}}{=}0.
\end{aligned}$$

Since the symmetric product (3.40) is non-degenerate, we obtain

$$[[RX_m,RX_n]-R[X_m,X_n]_R,L]=0.$$

Hence, combining (3.9) and (3.12) shows that vector fields (3.7) pairwise commute. So, if there exists a symmetric, non-degenerate and $ad$-invariant product on $\mathfrak{g}$ then the Yang-Baxter equation (3.3) is not a necessary condition for the commutativity of vector fields from the Lax hierarchy (3.7). However, if (3.3) is not satisfied then the zero-curvature equations (3.10) will not be satisfied as well.

In fact, by virtue of the scheme presented in the previous section all equations from the hierarchy (3.7) are Hamiltonian. Since $\mathfrak{g}^*\cong\mathfrak{g}$, the Lie-Poisson brackets (3.36) and (3.38) on the space of scalar fields $\mathcal{F}(\mathfrak{g})\cong\mathcal{F}(\mathfrak{g}^*)$ at $L\in\mathfrak{g}$ take the form

$$\begin{aligned}
\{H,F\}&=(L,[dF,dH])=(dF,\pi dH)&&\Longleftrightarrow&&\pi:dH\mapsto[dH,L]\\
\{H,F\}_R&=(L,[dF,dH]_R)=(dF,\pi_R dH)&&\Longleftrightarrow&&\pi_R:dH\mapsto[RdH,L]+R^*[dH,L],
\end{aligned}$$

where now $R^*$ is defined by the relation

$$(R^*a, b) = (a, Rb) \qquad a, b \in \mathfrak{g}.$$

Differentials of the Casimir functions $\mathcal{C}_n(L) \in \mathcal{F}(\mathfrak{g})$ of the natural Lie-Poisson bracket are invariants of the Lie bracket, i.e. $[dC_n(L), L] = 0$. The Casimir functions of course are still in involution with respect to the second Lie-Poisson bracket defined by $R$ and generate pairwise commuting Hamiltonian vector fields of the form

$$L_{t_n} = \pi_R dC_n(L) = [RdC_n, L] \, .$$

The simplest way to define an appropriate scalar product on some Lie algebra $\mathfrak{g}$ is to use of a trace form $\mathrm{Tr} : \mathfrak{g} \to \mathbb{K}$ such that the scalar product

$$(a, b) := \mathrm{Tr}\,(ab) \qquad a, b \in \mathfrak{g} \tag{3.42}$$

is nondegenerate. In this case the symmetry of (3.42) entails that

$$\mathrm{Tr}\,(ab) = \mathrm{Tr}\,(ba) \, . \tag{3.43}$$

**Proposition 3.10** *Let* $\mathrm{Tr} : \mathfrak{g} \to \mathbb{K}$ *be a trace form defining a symmetric and nondegenerate scalar product* (3.42) *such that the trace of Lie bracket vanishes, i.e.,*

$$\mathrm{Tr}\,[a, b] = 0 \qquad \forall\ a, b \in \mathfrak{g}.$$

*Then the condition* (3.4) *for a Lie bracket to be a derivation with respect to the multiplication is a sufficient condition for* (3.42) *to be* ad*-invariant.*

**Proof.** It is immediate, as

$$\begin{aligned}([a, c]\,, b) + (c, [a, b]) &= \mathrm{Tr}\,([a, c]\, b + c\,[a, b]) = \mathrm{Tr}\,([a, c]\, b + [a, cb] - [a, c]\, b)\\ &= \mathrm{Tr}\,[a, cb] = 0,\end{aligned}$$

where we used the assumptions from the proposition. □

Upon assuming that we have a nondegenerate trace form $\mathrm{Tr}$ on $\mathfrak{g}$, and having defined a scalar product in the fashion described above, the most natural Casimir functions $\mathcal{C}_q(L) \in \mathcal{F}(\mathfrak{g})$ are given by the traces of powers of $L$, i.e.,

$$C_q(L) = \frac{1}{q+1}\mathrm{Tr}\left(L^{q+1}\right) \quad \Longleftrightarrow \quad dC_q = L^q \qquad q \neq -1. \tag{3.44}$$

The related differentials are calculated from the expression (3.33), which can be now reduced to

$$\frac{d}{dt}F(L) = (L_t, dF) = \mathrm{Tr}\,(L_t\ dF) \qquad L \in \mathfrak{g}, \tag{3.45}$$

where $t$ is an evolution parameter related to the vector field $L_t$ on $\mathfrak{g}$.

## 3.7 The trace form on the PDO algebra

Let us study some properties of the Lie algebra of pseudo-differential operators (3.22). The first observation is the existence of a symmetric, non-degenerate and $\mathrm{ad}$-invariant product on $\mathfrak{g}$ allowing us to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$.

**Lemma 3.11** *Consider the scalar product on* $\mathfrak{g}$ *is given by a trace form*

$$(A,B) := \mathrm{Tr}\,(AB) \qquad A,B\in\mathfrak{g}, \tag{3.46}$$

*where*

$$\mathrm{Tr}L = \int_\Omega \mathrm{res}L\; dx, \qquad \mathrm{res}L := u_{-1},$$

*for* $L=\sum_i u_i\partial_x^i$. *Then* (3.46) *is symmetric, non-degenerate and* $\mathrm{ad}$*-invariant.*

**Proof.** The non-degeneracy of product (3.46) is obvious. Let $A=\sum_m u_m\partial_x^m$ and $B=\sum_n v_n\partial_x^n$, then we find

$$\begin{aligned}
(A,B) &= \mathrm{Tr}\,(A\cdot B) = \mathrm{Tr}\left(\sum_{m,n} u_m\partial_x^m v_n\partial_x^n\right) \overset{\text{by (3.23)}}{=} \mathrm{Tr}\left(\sum_{m,n}\sum_{s=0}^\infty \tbinom{m}{s} u_m (v_n)_{sx}\partial_x^{m+n-s(1-r)}\right)\\
&= \int_\Sigma \sum_n\sum_{s=0}^\infty \tbinom{s-1-n}{s} u_{s-1-n}(v_n)_{sx}\; dx \overset{i.b.p.}{=} \int_\Sigma \sum_n\sum_{s=0}^\infty (-1)^s\tbinom{s-1-n}{s}(u_{s-1-n})_{sx}v_n\; dx\\
&\overset{\text{by (3.24)}}{=} \int_\Sigma \sum_n\sum_{s=0}^\infty \tbinom{n}{s}(u_{s-1-n})_{sx}v_n\; dx = \mathrm{Tr}\left(\sum_{m,n}\sum_{s=0}^\infty \tbinom{n}{s}(u_m)_{sx}v_n\partial_x^{m+n-s}\right)\\
&= \mathrm{Tr}\left(\sum_{m,n} v_n\partial_x^n u_m\partial_x^m\right) = \mathrm{Tr}\,(B\cdot A) = (B,A)\,,
\end{aligned}$$

where $i.b.p.$ means integration by parts. Hence

$$\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA) \qquad \Longleftrightarrow \qquad \mathrm{Tr}\,[A,B]=0$$

and the $\mathrm{ad}$-invariance for (3.46) follows by Proposition 3.10. □

The adjoints of (3.26) with respect to the scalar product (3.46) such that

$$\forall A,B\in\mathfrak{g} \qquad (R^\star A,B) = (A,RB)$$

are

$$R^* = \frac{1}{2}(P^*_{\geqslant k} - P^*_{<k}) = \frac{1}{2} - P_{\geqslant -k} = P_{<-k} - \frac{1}{2} \qquad k=0,1,2.$$

Let $L=\sum_i u_i\partial_x^i$, then the vector fields $L_t$ and the related differentials $dH(L)$ are conveniently parameterized by

$$\begin{aligned}
L_t &= \sum_i (u_i)_t\partial_x^i \quad \Longrightarrow\\
dH(L) &= \frac{\delta H}{\delta L} = \sum_i \partial_x^{-1-i}\frac{\delta H}{\delta u_i},
\end{aligned} \tag{3.47}$$

where $\frac{\delta H}{\delta u_i}$ is the variational derivative of a functional $H = \int_\Sigma h(\mathbf{u})\, dx$. In this case the trace duality assumes the usual Euclidean form

$$(dH, L_t) = \mathrm{Tr}\,(dH L_t) = \sum_i \int_\Sigma \frac{\delta H}{\delta u_i}(u_i)_t\, dx.$$

## 3.8 Hamiltonian structures on Poisson algebras

**Definition 3.12** *Let $\mathcal{A}$ be a commutative, associative algebra with unit. If there is a Lie bracket on $\mathcal{A}$ such that for each element $a \in \mathcal{A}$, the operator $\mathrm{ad}_a : b \mapsto \{a, b\}$ is a derivation of the multiplication, i.e. $\{a, bc\} = \{a, b\}c + b\{a, c\}$, then $(\mathcal{A}, \{\cdot,\cdot\})$ is called a Poisson algebra and the bracket $\{\cdot,\cdot\}$ is a Poisson bracket.*

Thus, the Poisson algebras are Lie algebras, $[\cdot,\cdot] := \{\cdot,\cdot\}$, with an additional structure. Of course we should not confuse the above bracket with the Poisson brackets in the algebra of scalar fields. It will follow easily from the context which bracket is used. In the case of the Poisson algebra $\mathcal{A}$ a classical $R$-matrix defines the second Lie product on $\mathcal{A}$ but not the Poisson bracket; in general this would not be possible.

**Theorem 3.13** *[30] Let $\mathcal{A}$ be a Poisson algebra with the Poisson bracket $\{\cdot,\cdot\}$ and non-degenerate $\mathrm{ad}$-invariant scalar product $(\cdot,\cdot)$ such that the operation of multiplication is symmetric with respect to the latter, i.e., $(ab, c) = (a, bc)$, $\forall a, b, c \in \mathcal{A}$. Assume that $R$ is a classical $R$-matrix such that* (3.11) *holds, then for any integer $n \geqslant 0$, the formula*

$$\{H, F\}_n = (L, \{R(L^n dF), dH\} + \{dF, R(L^n dH)\}) \tag{3.48}$$

*where $H, F$ are smooth functions on $\mathcal{A}$, defines a Poisson structure on $\mathcal{A}$. Moreover, all brackets $\{\cdot,\cdot\}_n$ are compatible.*

An important property that classical $R$-matrices commute with differentials of smooth maps from $\mathcal{A}$ to $\mathcal{A}$ or equivalently satisfy (3.11) is used in the proof of Theorem 4.2 of [30], although it is not explicitly stressed there. In fact, the existence of scalar product being symmetric with respect to the multiplication, $(ab, c) = (a, bc)$, entails existence of a trace form on $\mathcal{A}$. Setting $c = 1$ we have $(ab, 1) = (a, b)$. Thus, the trace can be defined as $\mathrm{Tr}(a) := (a, 1) = (1, a)$.

The Poisson operators $\pi_n$ related to Poisson brackets (3.48) such that $\{H, F\}_n = (dF, \pi_n dH)$, are given by the following Poisson maps

$$\pi_n : dH \mapsto \{R(L^n dH), L\} + L^n R^* (\{dH, L\}) \qquad n \geqslant 0. \tag{3.49}$$

Notice that the bracket (3.48) with $n = 0$ is just a Lie-Poisson bracket with respect to the second Lie bracket on $\mathcal{A}$ defined by a classical $R$-matrix. Referring to the dependence on $L$, Poisson maps (3.49) are called linear for $n = 0$, quadratic for $n = 1$ and cubic for $n = 2$, respectively. The Casimir functions $C(L)$ of the natural Lie-Poisson bracket are in involution with respect to all Poisson brackets (3.49) and generate pairwise commuting Hamiltonian vector fields of the form

$$L_t = \pi_n dC = \{R\,(L^n dC)\,, L\} \qquad L \in \mathcal{A}.$$

Taking the most natural Casimir functions (3.44), defined by traces of powers of $L$, for the Hamiltonians, one finds a hierarchy of evolution equations which are multi-Hamiltonian dynamical systems:

$$L_{t_q} = \{RdC_q, L\} = \pi_0 dC_q = \pi_1 dC_{q-1} = \cdots = \pi_l dC_{q-l} = \dots \tag{3.50}$$

For any $R$-matrix any two evolution equations in the hierarchy (3.50) commute because of the involutivity of the Casimir functions $C_q$. Each equation admits all the Casimir functions as conserved quantities in involution. In this sense we will consider (3.50) as a hierarchy of integrable evolution equations.

The theory from this section can be effectively used in the systematic construction of integrable dispersionless systems (Remark 2.31) with multi-Hamiltonian structures, see [7, 43, 47].

## 3.9 Hamiltonian structures on noncommutative algebras

In this section in contrast to the previous one we will consider noncommutative associative algebra $\mathfrak{g}$ for which a Lie structure is defined as the commutator, i.e.

$$[a, b] := ab - ba \qquad a, b \in \mathfrak{g}.$$

Such Lie bracket automatically satisfies the Leibniz rule (3.4) required by us. We further assume the existence of nondegenerate, symmetric and $\mathrm{ad}$-invariant scalar product on $\mathfrak{g}$. Let $R$ be a classical $R$-matrix such that (3.11) is satisfied.

In this case the situation is more complex and only three explicit forms of Poisson brackets on the space of smooth functions $\mathcal{F}(\mathfrak{g})$ defined by related Poisson tensors are known from the literature:

$$\{H, F\}_n = (dF, \pi_n dH) \qquad n = 0, 1, 2.$$

These Poisson brackets (or related tensors) are called linear, quadratic and cubic bracket (resp. tensors) for $n = 0, 1, 2$, respectively.

The linear one is simply the Lie-Poisson bracket, with respect to the second Lie structure on $\mathfrak{g}$ defined by $R$, with Poisson tensor [36]

$$\pi_0 : dH \mapsto [RdH, L] + R^* [dH, L], \tag{3.51}$$

for which we need no additional assumptions.

In further considerations we have to assume that the scalar product is symmetric with respect to the operation of multiplication, $(ab, c) = (a, bc)$. Note that this property implies that the scalar product is automatically $\mathrm{ad}$-invariant with respect to the Lie bracket defined by the commutator.

The quadratic case is more delicate. A quadratic tensor [42]

$$\pi_1 : dH \mapsto A_1(LdH)L - LA_2(dHL) + S(dHL)L - LS^*(LdH) \tag{3.52}$$

defines a Poisson tensor if the linear maps

$$A_{1,2} : \mathfrak{g} \to \mathfrak{g}$$

are skew-symmetric,

$$A_{1,2}^{*} = -A_{1,2},$$

solutions of the YB($\alpha$) (3.3) for $\alpha \neq 0$ and the linear map $S : \mathfrak{g} \to \mathfrak{g}$ with adjoint $S^{*}$ satisfy

$$\begin{aligned} &S\left([A_2 a, b] + [a, A_2 b]\right) = [Sa, Sb] \\ &S^{*}\left([A_1 a, b] + [a, A_1 b]\right) = [S^{*}a, S^{*}b]\,. \end{aligned} \tag{3.53}$$

In the special case when

$$\widetilde{R} := \frac{1}{2}\left(R - R^{*}\right) \tag{3.54}$$

satisfies the YB($\alpha$), for the same $\alpha$ as $R$, under the substitution

$$A_1 = A_2 = R - R^{*} \qquad S = S^{*} = R + R^{*}$$

the quadratic Poisson operator (3.52) reduces to [36]

$$\pi_1 : dH \mapsto \left[R\,[dH, L]_{+}\,, L\right] + LR^{*}\,[dH, L] + R^{*}\left([dH, L]\right)L, \tag{3.55}$$

where

$$[a, b]_{+} := ab + ba$$

and the conditions (3.53) are equivalent to YB($\alpha$) for $R$ and $\widetilde{R}$.

Another special case occurs when the maps $A_{1,2}$ and $S$ originate from the decomposition of a given classical $R$-matrix satisfying YB($\alpha$) for $\alpha \neq 0$

$$R = \frac{1}{2}\left(A_1 + S\right) = \frac{1}{2}\left(A_2 + S^{*}\right),$$

where $A_{1,2}$ are skew-symmetric. Then, the conditions (3.53) imply that both $A_1$ and $A_2$ satisfy YB($\alpha$) for the same value of $\alpha$ as $R$, [35]. Hence, in this case we only have to check the conditions (3.53) for (3.52) to be a Poisson operator; these reduce to

$$\pi_1 : dH \mapsto 2R\left(LdH\right)L - 2LR\left(dHL\right) + S\left([dH, L]\right)L + LS^{*}\,[dH, L]\,. \tag{3.56}$$

Finally, the cubic tensor $\pi_2$ takes the simple form [36]

$$\pi_2 : dH \mapsto \left[R\left(LdHL\right), L\right] + LR^{*}\left([dH, L]\right)L$$

and is Poisson one without further additional assumptions.

Once again, taking the Casimir functions (3.44), defined by trace of powers of $L$, for the Hamiltonians yields a hierarchy of evolution equations which are tri-Hamiltonian dynamical systems

$$L_{t_q} = \{RdC_q, L\} = \pi_0 dC_q = \pi_1 dC_{q-1} = \pi_2 dC_{q-2}, \tag{3.57}$$

where we assumed that $\pi_2$ is given by (3.55) or (3.56). In the first case all three Poisson tensors in (3.57) are automatically compatible. In the second case this has to be checked separately.

## 3.10 The bi-Hamiltonian structure of Gelfand-Dickey hierarchies

We are going to consider Hamiltonian structures of the Gelfand-Dickey hierarchy, that is, the Lax hierarchy (3.28) for $k = 0$ generated by fractional powers of (3.29). By (3.47) the differential of a given functional $H = \int_\Sigma h \, dx$ has the form

$$dH = \partial_x^{-1} \frac{\delta H}{\delta u_0} + \partial_x^{-2} \frac{\delta H}{\delta u_1} + \ldots + \partial_x^{1-N} \frac{\delta H}{\delta u_{N-2}}.$$

Therefore, we have the following tri-Hamiltonian Lax hierarchy

$$L_{t_n} = \left[ \left( L^{\frac{n}{N}} \right)_{\geqslant 0}, L \right] = \pi_0 dH_n = \pi_1 dH_{n-N} = \pi_2 dH_{n-2N} \qquad n = 1, 2, \ldots, \tag{3.58}$$

where Hamiltonians are defined by

$$H_n(L) = \frac{1}{\frac{n}{N} + 1} \int_\Sigma \operatorname{res} \left( L^{\frac{n}{N}+1} \right) \, dx \qquad n \neq -N.$$

The linear Poisson tensor (3.51), defined by classical $R$-matrix (3.26) for $k = 0$, has two equivalent representations

$$\begin{aligned} \pi_0 dH &= [(dH)_{\geqslant 0}, L] - ([dH, L])_{\geqslant 0} \\ &= -\, [(dH)_{<0}, L] + ([dH, L])_{<0} \,. \end{aligned}$$

All Lax operators of the form (3.29) form a proper subspace of $\mathfrak{g}$ with respect to the above linear Poisson tensor, i.e., $\pi_0 dH$ span the same subspace of $\mathfrak{g}$ as (3.29). Since $(dH)_{\geqslant 0} = 0$, the linear Poisson tensor reduces to a simpler form

$$\pi_0 dH = ([L, dH])_{\geqslant 0} \,. \tag{3.59}$$

One finds that for $k = 0$ (3.54) we have $\widetilde{R} = R$, i.e., $\widetilde{R}$ solves the same YB equation as $R$. Hence, the quadratic bracket is given by (3.55)

$$\begin{aligned} \pi_1 dH &= \frac{1}{2} \left[ \left( [dH, L]^+ \right)_{\geqslant 0}, L \right] + \frac{1}{2} \left[ L, ([dH, L])_{\geqslant 0} \right]^+ \\ &= -\frac{1}{2} \left[ \left( [dH, L]^+ \right)_{<0}, L \right] - \frac{1}{2} \left[ L, ([dH, L])_{<0} \right]^+ , \end{aligned} \tag{3.60}$$

where $[A, B]^+ := AB + BA$. The quadratic Poisson bracket can be properly restricted to the space spanned by the operators of the form

$$L' = \partial_x^N + u \partial_x^{N-1} + u_{N-2} \partial_x^{N-2} + \cdots + u_1 \partial_x + u_0.$$

Thus, the Dirac reduction, see Lemma 2.19, with the constraint $u_{N-1} = 0$, is required to reduce (3.60) to the subspace of $\mathfrak{g}$ spanned by (3.29). Let

$$L := L'|_{u=0} \quad \Longrightarrow \quad L' = L + u \partial_x^{N-1} \quad \Longrightarrow \quad dH' = dH + \partial_x^{-N} \frac{\delta H'}{\delta u},$$

where $H' = H'(L')$ and $H = H(L) = H'(L')|_{u=0}$. Rewriting (3.60) we have

$$\pi_1 dH' = L'\left(L'dH'\right)_{\geqslant 0} - \left(dH'L'\right)_{\geqslant 0} L'.$$

Then, the Hamiltonian flow for $u$ is given by the coefficient at $\partial_x^{N-1}$ of $\pi_1 dH'$. So, under the constraint $u = 0$, one finds that

$$\begin{aligned} 0 = u_t|_{u=0} &= \operatorname{res}\left[dH', L'\right]|_{u=0} \\ &= \operatorname{res}\left[dH + \partial_x^{-N}\frac{\delta H'}{\delta u}, L\right] = \operatorname{res}\left[dH, L\right] - N\left(\frac{\delta H'}{\delta u}\right)_x . \end{aligned} \tag{3.61}$$

Solving (3.61) with respect to $\frac{\delta H'}{\delta u}$ one gets

$$\frac{\delta H'}{\delta u} = \frac{1}{N}\hat{\partial}_x^{-1}\operatorname{res}\left[dH, L\right],$$

where $\hat{\partial}_x^{-1}$ is a formal inverse of the derivative with respect to $x$. The hat is used to distinguish it from the pseudo-differential operator $\partial_x^{-1}$. Thus $\frac{\delta H'}{\delta u}$ can be expressed in terms of $\frac{\delta H}{\delta u_i}$. This implies

$$\begin{aligned} \pi_1^{red} dH \equiv \pi_1 dH'|_{u=0} &= L\left(LdH'\right)_{\geqslant 0} - \left(dH'L\right)_{\geqslant 0} L \\ &= L\left(LdH + L\partial_x^{-N}\frac{\delta H'}{\delta u}\right)_{\geqslant 0} - \left(dHL + \partial_x^{-N}\frac{\delta H'}{\delta u}L\right)_{\geqslant 0} L \\ &= L\left(LdH\right)_{\geqslant 0} - \left(dHL\right)_{\geqslant 0} L + L\frac{\delta H'}{\delta u} - \frac{\delta H'}{\delta u}L. \end{aligned}$$

Hence, the Dirac reduction yields the reduced quadratic Poisson tensor of the form

$$\pi_1^{red} dH = \left(LdH\right)_{\geqslant 0} L - L\left(dHL\right)_{\geqslant 0} + \frac{1}{N}\left[\hat{\partial}_x^{-1}\left(\operatorname{res}\left[dH, L\right]\right), L\right]. \tag{3.62}$$

The Poisson tensor (3.62) is local as always $\operatorname{res}\left[\cdot,\cdot\right] = (\dots)_x$. It is also compatible with the linear Poisson tensor (3.59) since

$$\pi_1^{red}(L + \epsilon) = \pi_1^{red}(L) + \epsilon\ \pi_0(L), \tag{3.63}$$

where $\epsilon$ is an arbitrary scalar.

The Lax operators in the form (3.29) do not span proper subspaces of $\mathfrak{g}$ with respect to the cubic Poisson tensor (3.9):

$$\begin{aligned} \pi_2 dH &= \big(\left(LdHL\right)_{\geqslant 0}, L\big) - L\left(\left[dH, L\right]\right)_{\geqslant 0} L \\ &= -\big[\left(LdHL\right)_{<0}, L\big] + L\left(\left[dH, L\right]\right)_{<0} L. \end{aligned}$$

Nevertheless, the Dirac reduction can be applied. Here, unlike the previous case, the number of constraints depends on $N$, so the reduction has to be considered separately for each Lax operator (3.29).

**Example 3.14** The $N = 2$ case of the Gelfand-Dickey hierarchy. One finds the following bi-Hamiltonian structure for the KdV equation (3.30)

$$L_{t_3} = \left[ \left( L^{\frac{3}{2}} \right)_{\geqslant 0}, L \right] = \pi_0 dH_3 = \pi_1^{red} dH_1,$$

where the Poisson tensors (3.59) and (3.62) are

$$\begin{aligned} \pi_0 &= 2\partial_x \\ \pi_1^{red} &= \frac{1}{2}\partial_x^3 + 2u\partial_x + u_x. \end{aligned}$$

The respective Hamiltonians read

$$\begin{aligned} H_1 &= \frac{2}{3} \int_\Sigma \mathrm{res}\ L^{\frac{3}{2}}\ dx = \int_\Sigma \frac{1}{4} u^2\ dx \\ H_3 &= \frac{2}{5} \int_\Sigma \mathrm{res}\ L^{\frac{5}{2}}\ dx = \int_\Sigma \frac{1}{16} \left( 2u^3 - {u_x}^2 \right)\ dx. \end{aligned}$$

**Example 3.15** The case of $N = 3$. The bi-Hamiltonian structure of the Boussinesq system (3.30) is

$$L_{t_2} = \left[ \left( L^{\frac{2}{3}} \right)_{\geqslant 0}, L \right] = \pi_0 dH_2 = \pi_1 dH_{-1},$$

where the linear Poisson tensor (3.59) has the form

$$\pi_0 = \begin{pmatrix} 0 & 3\partial_x \\ 3\partial_x & 0 \end{pmatrix}$$

and the quadratic Poisson tensor (3.62) is

$$\pi_1^{red} = \begin{pmatrix} 2\partial_x^3 + \partial_x u + u\partial_x & -\partial_x^4 - \partial_x^2 u + 2\partial_x v + v\partial_x \\ \partial_x^4 + u\partial_x^2 + \partial_x v + 2v\partial_x & -\frac{2}{3}\left( \partial_x^5 + \partial_x^3 u + u\partial_x^3 + u\partial_x u \right) + \partial_x^2 v - v\partial_x^2 \end{pmatrix}.$$

The respective Hamiltonians are

$$\begin{aligned} H_{-1} &= \frac{3}{2} \int_\Sigma \mathrm{res}\ L^{\frac{2}{3}}\ dx = \int_\Sigma v\ dx \\ H_2 &= \frac{3}{5} \int_\Sigma \mathrm{res}\ L^{\frac{5}{3}}\ dx = \int_\Sigma \frac{1}{135} \left( -5u^3 + 45v^2 - 45u_x v - 15uu_{2x} - 3u_{4x} \right)\ dx. \end{aligned}$$

A lot of further examples of soliton systems with their bi-Hamiltonian structures, associated with the algebra of pseudo-differential operators, including the cases of $k \neq 0$, can be found in [25] or [4].

## 3.11 Exercises

1. Establish the relations (3.14), (3.15), and prove that (3.13) solves the Yang-Baxter equation (3.3) for $\alpha = \frac{1}{4}$.

2. Show that the subspaces $\mathfrak{g}_{\geqslant k}$ and $\mathfrak{g}_{<k}$ of $\mathfrak{g}$ (3.25) are Lie subalgebras *only* for $k = 0, 1, 2$.

3. Perform all the computations from Examples 3.5 and 3.14 related to the KdV hierarchy. Find the remaining symmetries and functionals from (1.12) and (1.13), respectively.

4. Do the same for Examples (3.6) and (3.15) connected with the Boussinesq hierarchy. Moreover, observe that the Boussinesq hierarchy splits into two bi-Hamiltonian hierarchies. Why?

5. Consider the Gelfand-Dickey Lax operator (3.29) for $N = 3$. Find the first nontrivial system from the hierarchy (3.28). Construct related bi-Hamiltonian structure (3.58) of this system.

6. Establish the relation (3.63). How it proves that $\pi_0$ (3.59) and $\pi_1^{red}$ (3.62) are compatible Poisson tensors?

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