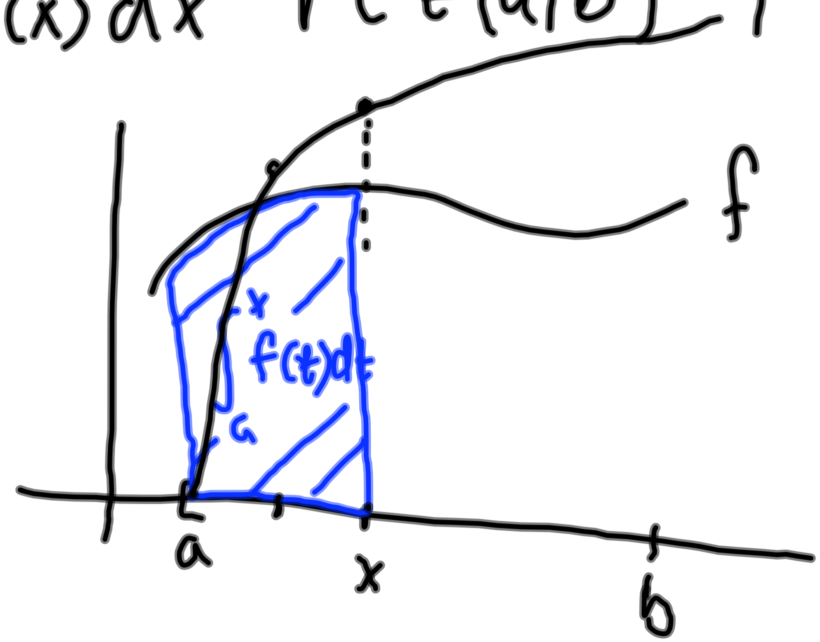
f – omezená $[a,b]$

$$\int_a^b f(x)\,dx \qquad \int_a^c f(x)\,dx \quad \forall c \in (a,b)$$

$$F(x) = \int_a^x f(t)\,dt$$



Věta 8.12 Nechť $f: [a,b] \longrightarrow \mathbb{R}$ je integrov
na $[a,b]$, $x_0 \in (a,b)$ spojitosti f, $c \in (a,b)$
Potom funkce $F(x) = \int_c^x f(t)dt$ je diferencovatelná
a platí $F'(x_0) = f(x_0)$.

Důkaz: $\lim\limits_{h \to 0} \dfrac{F(x_0+h) - F(x_0)}{h} = f(x_0)$

Dokážu $\forall \varepsilon > 0 \ \exists \delta \ \forall h \ |h| < \delta$: $\underline{f(x_0) - \varepsilon < \dfrac{F(x_0+h) - F(x_0)}{h} < f(x_0) + \varepsilon}$

Volím $\varepsilon > 0$ ze spojitosti f v x_0 existuje $\delta > 0$
$\forall x \in (x_0 - \delta, x_0 + \delta)$ $\ f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$

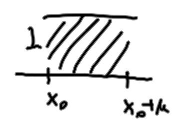
na intervalu $(x_0-\delta, x_0+\delta)$ $\ f(x_0) - \varepsilon < f|_{(x_0-\delta, x_0+\delta)} < f(x_0) + \varepsilon$

$h > 0$

$$\int_{x_0-\delta}^{x_0+\delta} f(x_0) - \varepsilon \, dx \le \int_{x_0-\delta}^{x_0+\delta} f(x)dx \le \int_{x_0-\delta}^{x_0+\delta} f(x_0) + \varepsilon \, dx$$

$$\frac{F(x_0+h) - F(x_0)}{h} = \frac{1}{h}\left(\int_c^{x_0+h} f(t)dt - \int_c^{x_0} f(t)dt \right) = \underline{\frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt}$$

$$\int_{x_0}^{x_0+h} f(x_0) - \varepsilon \, dt \le \int_{x_0}^{x_0+h} f(t)dt \le \int_{x_0}^{x_0+h} f(x_0) + \varepsilon \, dt$$



$$\frac{1}{h}\int_{x_0}^{x_0+h} \underbrace{f(x_0) - \varepsilon}\, dt \le \frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt \le \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0) + \varepsilon \, dt$$

$$\frac{f(x_0) - \varepsilon}{h} \underbrace{\int_{x_0}^{x_0+h} 1 \, dt} \le \frac{F(x_0+h) - F(x_0)}{h} \le \frac{f(x_0) + \varepsilon}{h}\int_{x_0}^{x_0+h} 1 \, dt$$

$$f(x_0) - \varepsilon \le \frac{F(x_0+h) - F(x_0)}{h} \le f(x_0) + \varepsilon$$

Důsledek 8.13 Je-li f spojitá na $[a,b]$ $c \in (a,b)$
Potom $F(x) = \int_c^x f(t)dt$ je primitivní k f.

Věta 8.14 Nechť f, g mají neurčité integrály a $c \in \mathbb{R}$. Potom

1. $$\int (f+g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$
2. $$\int (cf)(x)\,dx = c \cdot \int f(x)\,dx$$

Důkaz: Označme F – primitivní funkce k f

G – primitivní g

1. $$\underbrace{(F+G)'}_{\int (f+g)(x)\,dx} = \underbrace{F'+G'} = \underbrace{f+g}$$

$$\int f(x)\,dx = F \quad \int g(x)\,dx = G$$

2. F primit. k f $\quad F' = f \quad (\underline{c}F)' = c \cdot F' = cf$

Věta 8.15 (Metoda per-partes)

Mějme u, v funkce na $[a,b]$ - spojité derivaci.

Potom platí

$$\int u(x)\cdot v'(x)\,dx = u(x)v(x) - \int u'(x)\cdot v(x)\,dx.$$

Důkaz: $(u\cdot v)' = u'v + uv'$

$$\int (u(x)\cdot v(x))'\,dx = u(x)\cdot v(x) = \int \left(u'(x)\cdot v(x) + u(x)v'(x)\right)dx$$
$$= \int u'(x)\, v(x)\,dx + \int u(x)v'(x)\,dx$$

Příklad $\quad u = x \quad v' = e^x \quad u' = 1 \quad v = e^x$

$$\int x e^x\,dx = x\cdot e^x - \int 1\cdot e^x\,dx = x\cdot e^x - e^x = e^x(x-1)$$

Příklad

$$\int \ln(x)\,dx = \left| \begin{matrix} u = \ln x & u' = \frac{1}{x} \\ v' = 1 & v = x \end{matrix} \right| = x\ln x - \int \frac{1}{x}\cdot x\,dx =$$

$$= x\cdot \ln x - \int dx = x\ln x - x = x(\ln x - 1)$$

Věta 8.16 (Substituční metoda I druhu)
Buď $f: (a,b) \to \mathbb{R}$ spojitá a funkce $\varphi: (\alpha,\beta) \to$
$\to (a,b)$ spojitá

Potom je-li F primitivní funkce k f

$$\int f(\varphi(t))\varphi'(t)\,dt = F(\varphi(t)) + c$$

Důkaz:

$$\left(F(\varphi(t))\right)' = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t)) \cdot \varphi'(t)$$

Příklad

$f(x) = x^2 \quad \varphi(x) = \sin x \quad F(x) = \frac{x^3}{3}$

$$\int \underbrace{\sin^2 x \cdot \cos x}_{f(\varphi(x)) \cdot \varphi'(x)}\,dx = \underbrace{F(\varphi(x))}_{F(\varphi(x))} = \frac{\sin^3 x}{3}$$

$$\int \sin^2 x \cos x\,dx = \left| \begin{matrix} z = \sin x \\ dz = \cos x\,dx \end{matrix} \right| = \int z^2\,dz = \frac{z^3}{3} = \frac{\sin^3 x}{3}$$

Věta 8.17. (Substituční metoda II. druhu)
Nechť $\varphi:(\alpha,\beta)\to(a,b)$ je surjekce a φ' existuje na celém (α,β); $\psi:(a,b)\to(\alpha,\beta)$ takové, že $\varphi\circ\psi=id_{(a,b)}$. Potom, je-li f spojitá funkce na (a,b) platí

$$\int f(x)dx = G(\psi(x)) \quad \text{kde } G(t)=\int f(\varphi(t))\varphi'(t)dt$$

Příklad

$$\left(\frac{1}{x^2+1}\right)' = \arctan x \quad x\in\mathbb{R}$$

$$\int \underbrace{\frac{dx}{x^2+4}}_{f(x)} = \left|\begin{matrix} x=\overbrace{2t}^{\varphi(t)} \\ dx=2dt \\ \hline \psi=\frac{x}{2}\end{matrix}\right| = \int\frac{1}{(2t)^2+4}2\,dt = 2\int\frac{1}{4t^2+4}dt$$

$$=\frac{1}{2}\int\frac{1}{t^2+1}dt = \underbrace{\frac{1}{2}\arctan(t)}_{G(t)} = \frac{1}{2}\arctan\left(\frac{x}{2}\right)$$

Věta 8.18 (Newton-Leibnizova formule)
Buď $f:[a,b] \to \mathbb{R}$ ohraničená funkce, F její primitivní funkce
Je-li f integrovatelná na $[a,b]$ potom platí

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Důkaz: $\Delta = (x_0, x_1, \dots, x_n)$ – dělení $[a,b]$

$[x_i, x_{i+1}]$ $\quad x_i < \xi < x_{i+1}$

$$\frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i} = F'(\xi)$$

$$F(x_{i+1}) - F(x_i) = F'(\xi)(x_{i+1} - x_i)$$
$$= \underbrace{f(\xi)}_{m_i(f,\Delta) \le \quad \le M_i(f,\Delta)}(x_{i+1} - x_i)$$

$$\sum_{i=1}^{n} m_i(f,\Delta)(x_{i+1} - x_i) \le \sum f(\xi)(x_{i+1} - x_i) \le \sum_{i=1}^{n} M_i(f,\Delta)(x_{i+1} - x_i)$$

$$\underbrace{\sum m_i(f,\Delta)(x_{i+1} - x_i)}_{s(f,\Delta)} \le \sum f(\xi)(x_{i+1} - x_i) \le \sum M_i(f,\Delta)(x_{i+1} - x_i)$$

$$s(f,\Delta) \le \sum_{i=1}^{n} F(x_{i+1}) - F(x_i) \le S(f,\Delta)$$

$$F(\cancel{x_1}) - F(x_0) + F(\cancel{x_2}) - \cancel{F(x_1)} + \dots + F(x_n) - \cancel{F(x_{n-1})}$$
$$F(b) - F(a)$$

$$s(f,\Delta) \le F(b) - F(a) \le S(f,\Delta)$$

$$\int_a^b f(x)\,dx \le F(b) - F(a) \le \int_a^b f(x)\,dx$$

$$\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$