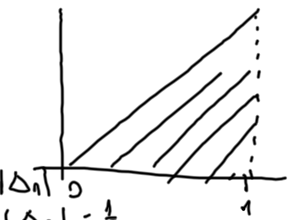
Príklad $f:[0,1]\longrightarrow \mathbb{R}$ $f(x)=x$

$$\int_0^1 x\,dx = \frac{1}{2}$$



Δ_1 0 … 1 $|\Delta_1|$

Δ_2 … $|\Delta_2| = \frac{1}{2}$

Δ_3 … $= \frac{1}{3}$

$\vdots$

$$\Delta_n = (x_0, x_1, \dots, x_n) = \left(0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-1}{n}, 1\right)$$

$[x_{i-1}, x_i]$ $\quad m_i(f,\Delta_n) = f(x_{i-1}) = x_{i-1} = \frac{i-1}{n}$

$M_i(f,\Delta) = f(x_i) = x_i = \frac{i}{n}$

$$s(f,\Delta_n) = \sum_{i=1}^{n} m_i(f,\Delta_n)(x_i - x_{i-1}) =$$
$$= \sum_{i=1}^{n} \frac{i-1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i-1$$
$$= \frac{1}{n^2}\,\frac{(n-1+0)}{2}\,n = \frac{1}{2}\,\frac{n-1}{n}$$

$$\lim_{n\to\infty} \frac{1}{2}\,\frac{n-1}{n} = \frac{1}{2}$$

$$S(f,\Delta_n) = \sum_{i=1}^{n} M_i(f,\Delta_n)(x_i - x_{i-1}) =$$
$$= \sum_{i=1}^{n} \frac{i}{n}\left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum_{i=1}^{n} \frac{i}{n}\,\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i =$$
$$= \frac{1}{n^2}\,\frac{n+1}{2}\,n = \frac{1}{2}\,\frac{n+1}{n}$$

$$\lim_{n\to\infty} S(f,\Delta_n) = \frac{1}{2}\,\frac{n+1}{n} = \frac{1}{2}$$

$$\lim S(f,\Delta_n) = \lim_{n\to\infty} s(f,\Delta_n) = \frac{1}{2}$$

$$\int_0^1 x\,dx = \frac{1}{2}$$

Věta 8.7 Je-li f spojitá na $[a,b]$, potom je na $[a,b]$ integrovatelná.

Důkaz: Spojitá funkce je na $[a,b]$ stejnoměrně spojitá!

Ukážeme $\forall \varepsilon > 0\ \exists \delta > 0$ tak, že $|\Delta| < \delta$ $\boxed{S(f,\Delta) - s(f,\Delta) < \varepsilon}$

Ze stejnoměrné spojitosti k $\varepsilon > 0\ \exists \delta > 0$ takové, že

$|x-y| < \delta$ potom $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$

$[x_{i-1}, x_i]$ $\quad x_i - x_{i-1} < \delta$

$x, y \in [x_{i-1}, x_i]$ $\quad |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$

$$M_i(f,\Delta) - m_i(f,\Delta) \leq \frac{\varepsilon}{2(b-a)}$$

$$\underline{S(f,\Delta) - s(f,\Delta)} = \sum_{i=1}^{n} \left(M_i(f,\Delta) - m_i(f,\Delta)\right)(x_i - x_{i-1})$$

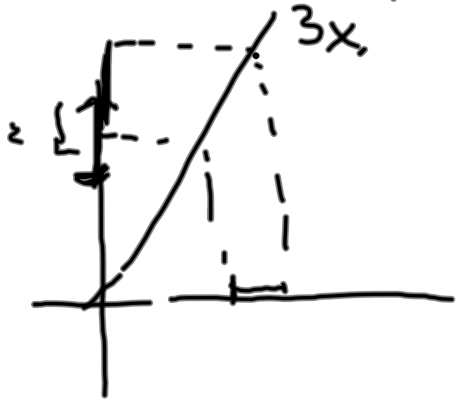
$$\leq \sum_{i=1}^{n} \frac{\varepsilon}{2(b-a)} \cdot (x_i - x_{i-1}) = \frac{\varepsilon}{2(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= \frac{\varepsilon}{2(b-a)} \left(\cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \cancel{x_3} - \cancel{x_2} + \cancel{\dots} + x_n - \cancel{x_{n-1}} \right)$$

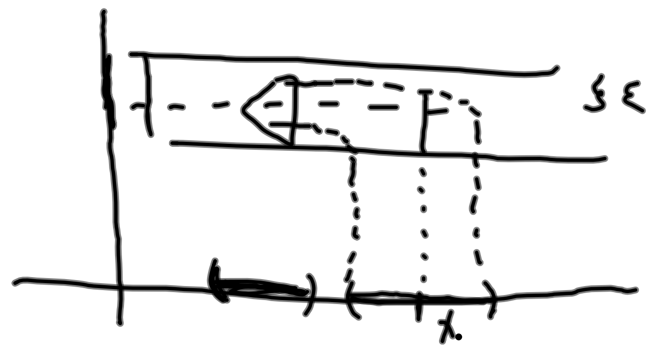
$$= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2} \underline{< \varepsilon}$$

Title: 4. 10-3:11 odp. (2 of 6)

Funkce $f: X \subset \mathbb{R} \to \mathbb{R}$ stejnoměrně spojitá na $Y \subset X$ jestliže $\forall \varepsilon > 0\ \exists \delta > 0$ takové, že $\forall x, y \in Y$ $|x-y| < \delta$ platí $|f(x)-f(y)| < \varepsilon$

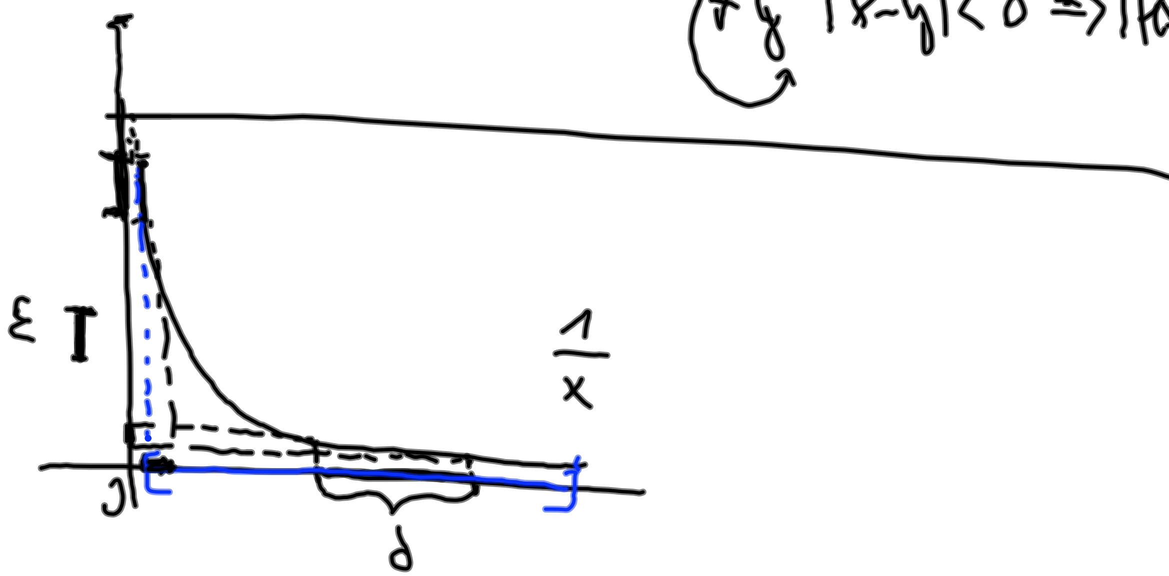


$\varepsilon > 0$



$\forall x \in Y,\ \forall \varepsilon > 0\ \exists \delta > 0$

$\forall y\ |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$



Veta 8.8 Buďte f, g - omezené funkce $[a,b]$ f, g integrovatelné na $[a,b]$, $c \in \mathbb{R}$. Potom

1. $f+g$ je integrovatelná a $\int_a^b (f+g)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
2. $c \cdot f$ je integrovatelná a $\int_a^b (cf)(x)\,dx = c\int_a^b f(x)\,dx$.

Důkaz: $\Delta = (x_0, x_1, \dots, x_n)$ $\quad M_i(f,\Delta)$

$[x_{i-1}, x_i]$

$$\sup_{x \in [x_{i-1}, x_i]} f+g \leq \sup_{x \in [x_{i-1}, x_i]} f + \sup_{x \in [x_{i-1}, x_i]} g$$

$M_i(f+g,\Delta)$ $\qquad\qquad$ $M_i(g,\Delta)$

$$\inf f+g \geq \inf f + \inf g$$

$$M_i(f+g,\Delta) \leq M_i(f,\Delta) + M_i(g,\Delta)$$

$$\sum_{i=1}^{n} M_i(f+g)(x_i - x_{i-1}) \leq \sum M_i(f,\Delta)(x_i - x_{i-1}) + \sum M_i(g,\Delta)(x_i - x_{i-1})$$

$$\rightarrow \quad S(f+g,\Delta) \leq S(f,\Delta) + S(g,\Delta)$$

$$s(f+g) \geq s(f,\Delta) + s(g,\Delta)$$

$$\underbrace{S(f,\Delta_n)}_{\downarrow} + \underbrace{S(g,\Delta_n)}_{\downarrow} \geq \underbrace{S(f+g,\Delta_n)}_{\downarrow} \geq \underset{\downarrow}{s(f+g,\Delta_n)} \geq \underbrace{s(f,\Delta_n)} + \underbrace{s(g,\Delta_n)}$$

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int \quad + \int \quad = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Title: 4. 10-3:38 odp. (4 of 6)

Věta 8.9 Buďiž f omezená na $[a,b]$, $c \in (a,b)$. Potom $\int_a^b f(x)dx$ existuje právě když existují $\int_a^c f(x)dx$ a $\int_c^b f(x)dx$ platí

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Důkaz: $\Delta = (x_0, x_1, \dots, x_k, \dots, x_n)$ $\quad x_k = c$

$$\underbrace{(x_0, x_1, \dots, c)}_{\Delta^L} \underbrace{(c, x_{k+1}, \dots, x_n)}_{\Delta^P}$$

$$S(f,\Delta) = \sum_{i=1}^{n} M_i(f,\Delta)(x_i - x_{i-1}) =$$

$$= \underbrace{\sum_{i=1}^{k} M_i(f,\Delta)(x_i - x_{i-1})}_{S(f,\Delta^L)} + \underbrace{\sum_{i=k+1}^{n} M_i(f,\Delta)(x_i - x_{i-1})}_{S(f,\Delta^P)}$$

$$S(f,\Delta) = S(f,\Delta^L) + S(f,\Delta^P)$$

$$\boxed{s(f,\Delta) = s(f,\Delta^L) + s(f,\Delta^P)}$$

Předpokládejme, že f je integrovatelná na $[a,b]$

$\exists$ dělení Δ intervalu $[a,b]$ $\quad S(f,\Delta) - s(f,\Delta) < \varepsilon$

Můžeme předpokládat, že c je dělicí bod Δ

$$S(f,\Delta) - s(f,\Delta) = S(f,\Delta^L) + S(f,\Delta^P) - s(f,\Delta^L) - s(f,\Delta^P)$$

$$= \underbrace{S(f,\Delta^L) - s(f,\Delta^L)}_{<\varepsilon} + \underbrace{S(f,\Delta^P) - s(f,\Delta^P)}_{<\varepsilon} < \varepsilon$$

Předpokládejme f je integrovatelná na $[a,c]$ i $[c,b]$

$\Delta^L = (a, x_1, x_2, \dots, x_{k-1}, c)$ $\quad S(f,\Delta^L) - s(f,\Delta^L) < \frac{\varepsilon}{2}$

$\Delta^P = (c, y_1, y_2, \dots, y_{e-1}, b)$ $\quad S(f,\Delta^P) - s(f,\Delta^P) < \frac{\varepsilon}{2}$

$\Delta = (a, x_1, x_2, \dots, x_{k-1}, c, y_1, y_2, \dots, y_{e-1}, b)$

$$S(f,\Delta) - s(f,\Delta) = S(f,\Delta^L) + S(f,\Delta^P) - s(f,\Delta^L) - s(f,\Delta^P)$$

$$= \underbrace{S(f,\Delta^L) - s(f,\Delta^L)}_{<\frac{\varepsilon}{2}} + \underbrace{S(f,\Delta^P) - s(f,\Delta^P)}_{<\frac{\varepsilon}{2}}$$

$$< \varepsilon$$

8.2 Integrál jako funkce horní meze

$f:[a,b]\to\mathbb{R}$ řekneme, že funkce $F:(a,b)\to\mathbb{R}$ je primitivní k f jestliže $\underline{F'(x)=f(x)}$ $\forall x\in(a,b)$

Př: $f(x)=x$ $\quad F(x)=\underline{\frac{x^2}{2}+50}$

<u>Věta 8.11</u> Buďte $f:[a,b]\to\mathbb{R}$ spojitá a F, G dvě primitivní funkce k f. Potom $F-G$ – konstantní

Důkaz: $H(x)=F(x)-G(x)$ $\qquad F'(x)=f(x)$, $G'(x)=f(x)$

$x,y\in(a,b)$

$$\frac{H(y)-H(x)}{y-x}=H'(c)=\underline{F'(c)}-\underline{G'(c)}=\underline{f(c)}-\underline{f(c)}=0$$

$$H(y)-H(x)=0$$

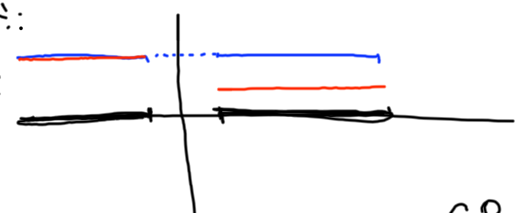
<u>Neurčitý integrál z f na $[a,b]$</u>

množina všech primitivních funkcí k f

$$\int f(x)\,dx=F(x)+c$$

Př:

f



Příklad: $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=\begin{cases}0 & x\neq 0\\ 1 & x=0\end{cases}$

kdyby existovala F – primitivní k f.

$(0,\infty)$ – $F(x)=a$ $\quad F(0)$

$(-\infty,0)$ – $F(x)=b$

F' $+\infty$ $-\infty$ neexistuje