

6.5 GONIOMETRICKÉ FUNKCE

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{sinus}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{kosinus}$$

Věta 6.19

1. Funkce $\sin x$, $\cos x$ jsou spojité
2. funkce \sin lichá, \cos sudá.
3. $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

$$1 = \cos^2(x) + \sin^2(x)$$

Důkaz: 1. plyne z Věty 6.13

$$2. \sin(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$$

$$= -\sin(x)$$

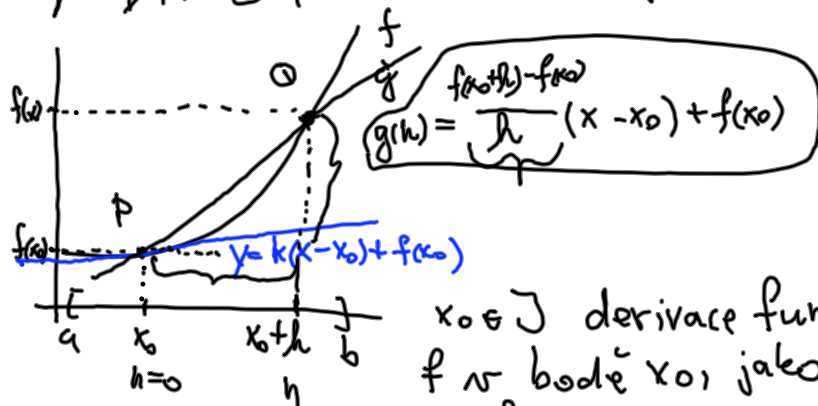
Tangens $\tan: \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi; k \in \mathbb{Z} \right\} \rightarrow \mathbb{R}$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Kotangens $\cotan: \mathbb{R} \setminus \{k\pi; k \in \mathbb{Z}\} \rightarrow \mathbb{R}$

$$\cotan(x) = \frac{\cos(x)}{\sin(x)}$$

7.7. DIFERENCIÁLNÍ POČET V \mathbb{R}



$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Příklad: $f(x) = c$.

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = 0$$

$$f(x) = x$$

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{x_0+h - x_0}{h} = 1$$

$$f(x) = x^2$$

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{(x_0+h)^2 - x_0^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x_0^2 + 2x_0h + h^2 - x_0^2}{h} = \lim_{h \rightarrow 0} \frac{2x_0 + h}{1} = 2x_0$$

$f: J \rightarrow \mathbb{R}$ pokud $f'(x)$ $x \in X \subset J$
existuje a je konečná
 $f': X \rightarrow \mathbb{R}$

$$(c)' = 0$$

$$(x)' = 1$$

$$(x^2)' = 2x$$

$$(x^n)' = n \cdot x^{n-1} \quad \forall n \in \mathbb{N}$$

$$(1/x)' = -\frac{1}{x^2}$$

$$f(x) = \frac{1}{x}$$

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x_0 - x_0 - h}{(x_0+h)x_0}}{h} = \lim_{h \rightarrow 0} \frac{\frac{-h}{(x_0+h)x_0}}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{-1}{x_0^2 + x_0 h} = -\frac{1}{x_0^2}$$

Tečna ke grafu funkce $f: J \rightarrow \mathbb{R}$ $x_0 \in J$
 $P = (x_0, f(x_0))$. Rovnice tečny ke grafu funkce f
bodem P je přímka určená rovnicí

$$y = f'(x_0) \cdot (x - x_0) + f(x_0).$$

Derivace zleva, zprava

$$f'_-(x_0) = \lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'_+(x_0) = \lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h}$$

Věta 7.1: Bud' $f: J \rightarrow \mathbb{R}$ a $x_0 \in J$, potom

$f'(x_0)$ existuje právě, když existují $f'_-(x_0)$ a $f'_+(x_0)$
a rovnají se.

$$f(x) = |x|$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$f'_-(0) = \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{|h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{|h|}{h} =$$
$$= \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1$$

$$f'_+(0) = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{|h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{|h|}{h} =$$
$$= \lim_{h \rightarrow 0^+} \frac{h}{h} = 1$$

Věta 7.2. Necht' $f: J \rightarrow \mathbb{R}$ $x_0 \in J$ a $f'(x_0)$ existuje
a je konečná, potom je f v x_0 spojitá.

$$\text{Důkaz: } \lim_{x \rightarrow x_0} f(x) = f(x_0) \Leftrightarrow \lim_{x \rightarrow x_0} f(x) - f(x_0) = 0$$

$$\lim_{x \rightarrow x_0} f(x) - f(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) =$$
$$= \underbrace{\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0) \in \mathbb{R}} \cdot \underbrace{\lim_{x \rightarrow x_0} x - x_0}_0 = f'(x_0) \cdot 0 = 0$$
$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$$f(x) = \text{sgn}(x) \quad x_0 = 0$$

$$f'_-(0) = \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{\text{sgn}(h)}{h} =$$
$$= \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

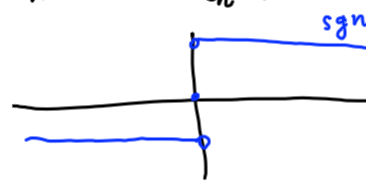
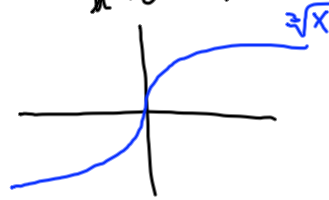
$$f'_+(0) = \dots \dots \dots$$

$$= \lim_{h \rightarrow 0^+} \frac{1}{h} = \infty$$

$$\text{sgn}'(0) = \infty$$

Príklad
 $f(x) = \sqrt[3]{x}$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt[3]{h} - 0}{h} = \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = \infty$$



$\sqrt[3]{x}$ inverze k x^3
 spojité funkcie

Veta 7.3. Necht (f_n) je posloupnost funkcií $f_n: J \rightarrow \mathbb{R}$
 f_n existujú na $J \forall n \in \mathbb{N}$, $\sum f_n \rightarrow f$ a $\sum f_n' \rightarrow g$
 na J . Potom patí existuje f' na J a
 $f' = g$ (nebo-li $f' = \sum f_n'$)

Dúsledok 7.4

1. $(e^x)' = e^x$;

2. $(\sin x)' = \cos x$;

3. $(\cos x)' = -\sin x$.

Dúkaz 1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum f_n$

$$(e^x)' = e^x \quad \begin{matrix} 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \dots \\ 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \end{matrix} = e^x$$

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \cos x$$

$\sin' x = \cos x$