Silesian University at Opava
Mathematical Institute

Jiří Kupka

A classification of the triangular mappings with closed set of periodic points

Abstract of the Ph.D. Thesis
September 2004

Mathematical Analysis

Slezská univerzita v Opavě
Matematický ústav

Jiří Kupka

Klasifikace trojúhelníkových zobrazení s uzavřenou množinou periodických bodů

Autoreferát dizertační práce
Září 2004

Matematická analýza

Výsledky tvořící dizertační práci byly získány během doktorského studia oboru Matematická analýza na Matematickém ústavu Slezské univerzity v Opavě v letech 2000 – 2004.

Výzkum byl podporován granty GA201/00/0859, GA201/03/1153, GD201/03/H152 a MSM192400002.

Dizertant: Mgr. Jiří Kupka
Matematický ústav
Slezská univerzita v Opavě

Školitel: Prof. RNDr. Jaroslav Smítal, DrSc.
Matematický ústav
Slezská univerzita v Opavě

Školicí pracoviště: Matematický ústav
Slezská univerzita v Opavě

Oponenti: Host. prof. Vladimír Averbuch, DrSc.
Matematický ústav
Slezská univerzita v Opavě

Doc. RNDr. Katarína Janková, CSc.
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského v Bratislavě

Autoreferát byl rozeslán 6. září 2004.

Státní doktorská zkouška a obhajoba dizertační práce se konají dne 29. září 2004 v 11 hodin, před oborovou radou doktorského studia matematické analýzy v zasedací místnosti rektorátu Slezské univerzity. S dizertací je možno se seznámit v knihovně Matematického ústavu Slezské univerzity (Na Rybníčku 1, Opava).

Předseda oborové rady: Prof. RNDr. Jaroslav Smítal, DrSc.
Matematický ústav
Slezská univerzita v Opavě
Na Rybníčku 1
746 01 Opava

The results included in this Thesis have been obtained during author's doctoral study at Mathematical Institute of Silesian University at Opava (2000 – 2004). The research was supported by the grants GA201/00/0859, GA201/03/1153, GD201/03/H152 and MSM192400002.

Supervisor: Prof. RNDr. Jaroslav Smítal, DrSc.
Mathematical Institute
Silesian University at Opava
Na Rybníčku 1
746 01 Opava
Czech Republic

Contents

Introduction

In 1989 A. N. Sharkovsky [KS] formulated the following problem: The class of continuous maps of the interval can decomposed into several subclasses, each being characterized by a certain degree of complexity of their dynamics. For example, the class $[h(\varphi)=0]$ of the maps with zero topological entropy can be characterized by more than 60 other mutually equivalent conditions [Sh]. Other sublass, $[P(\varphi)=\overline{P(\varphi)}]$, containing the maps with closed set of periodic points, is characterized by other 18 mutually equivalent conditions [Sh]. On the other hand, it is well known that for triangular (i.e., skew-product) maps $(x,y)\mapsto(f(x),g_x(y))$ of the square many of these conditions fail to be equivalent. E.g., there is a triangular map of the square of type 2^∞ with positive topological entropy [K].

Sharkovsky's problem is to find a decomposition of the above quoted classes $[h(\varphi)=0]$, $[P(\varphi)=\overline{P(\varphi)}]$, etc., with respect to the conditions which are in the one-dimensional case equivalent. It should be mentioned that the triangular maps of the square have many important properties which are common for the continuous maps of the interval. For example, the Sharkovsky's theorem on coexistence of periodic orbits for the triangular maps is the same as for the continuous maps of the intervals. Contrary to this, for general continuous maps of the square no theorem on coexistence of periodic orbits is possible. The reason is that the projection of some standard sets (like the set of periodic points, recurrent points, ω-limit points, nonwandering points, etc.) of a triangular map F to the x-axis is the corresponding standard set for the base map f [K].

For the class $[h(\varphi)=0]$ the problem is almost solved, in a series of papers, see, e.g., [FPS1], [FPS2], [FPS3], [Ko1], [Ko2] and [GCh]. The main aim of this Thesis is to provide similar classification of the class $[P(\varphi)=\overline{P(\varphi)}]$.

The principal part of the Thesis is formed by three papers dealing with the above mentioned question. In the first part (Section 2) we present a short history and an alternative solution of a problem concerning triangular maps for which any nonwandering point is periodic. In the next part (Section 3) the characterization of triangular maps possessing a nonperiodic chain recurrent point is given. Section 4 is the only one where generalized triangular maps are studied. The question whether some regularly recurrent point of the triangular map lies in the fibres over minimal set of the base map possessing a regularly recurrent point is stated and partially solved. Finally, an almost complete classification of triangular maps with closed set of periodic points is presented in the last Section 5; in fact, the problem is completely solved for triangular maps monotone on the fibres, but not yet completed for general maps.

1 Basic terminology and notation

Denote by I the closed interval $[0,1] \subseteq \mathbb{R}$ with the induced topology, by I^2 the Cartesian product $I \times I$, by X an arbitrary compact metric space with metric d_X. If $A \subseteq X$ then $\overline{A}$ is the closure of A. Let $C(X)$ denote the set of continuous maps of X into itself.

Take $\varphi \in C(X)$ and $x \in X$. Define inductively the *nth iteration* of φ by $\varphi^0(x) = x$, and $\varphi^n(x) = \varphi(\varphi^{n-1}(x))$. The *trajectory* of x is the sequence $\{\varphi^n(x)\}_{n=0}^{\infty}$, and the set of limit points of the trajectory of x is the *ω-limit set* $\omega_\varphi(x)$ of x. Let $\omega(\varphi) = \bigcup_{x \in X} \omega_\varphi(x)$. A point x is *a periodic* point of φ, if $\varphi^p(x) = x$ for some positive integer p. The minimal set containing a periodic point x of φ and closed with respect to φ, is a *periodic orbit* of φ, and its cardinality is the *period* of x. A point x is *nonwandering* if for every open neighborhood U of x there is an $n \in \mathbb{N}$ such that $\varphi^n(U) \cap U \neq \emptyset$. Denote by $P(\varphi)$ and $\Omega(\varphi)$ the set of periodic and nonwandering points of φ, respectively. In general, if we put $\Omega_1(\varphi) = \Omega(\varphi)$ and $\Omega_{n+1}(\varphi) = \Omega(\varphi|_{\Omega_n(\varphi)})$, $n \in \mathbb{N}$, we have an infinite decreasing sequence of closed sets. By using transfinite induction, we get a **centre** $C(\varphi)$.

Let $\varepsilon > 0$, and let x, y be points of X. An *ε-chain* from x to y is a finite sequence $\{x_0, x_1, \ldots, x_n\}$ of points in X with $x = x_0$, $y = x_n$ and $d_X(\varphi(x_{k-1}), x_k) < \varepsilon$ for $k = 1, 2, \ldots, n$. A point x is *chain recurrent* if and only if, for every $\varepsilon > 0$, there is an ε-chain from x to itself. Let $CR(\varphi)$ denote the set of all chain recurrent points of φ.

A point $x \in X$ is called *recurrent* if $x \in \omega_\varphi(x)$, i.e., for any neighborhood U of x there exists an integer $m > 0$ such that $\varphi^m(x) \in U$. Consequently, one can find an infinite increasing sequence of return times $\{m_i\}$ such that $\varphi^{m_i}(x) \in U$ for $i = 1, 2, \ldots$. Recurrent points can be classified depending on the properties of the sequence $\{m_i\}$. For instance if, for any neighborhood U, $\{m_i\}$ is relatively dense in $\mathbb{N}$, then x is a *uniformly recurrent* point. If, for any U, $m_i = mi$

for some m, then x is a *regularly recurrent* point. We denote by $R(\varphi)$, $UR(\varphi)$ and $RR(\varphi)$ the set of recurrent, uniformly recurrent (sometimes called *almost periodic* or *regularly recurrent*) and regularly recurrent (sometimes called *isochronously recurrent* or *almost periodic*) points, respectively. The mentioned sets can be ordered by inclusions in the following way, see, e.g., [BC].

$$P(\varphi) \subseteq RR(\varphi) \subseteq UR(\varphi) \subseteq R(\varphi) \subseteq \omega(\varphi) \subseteq \Omega(\varphi) \subseteq CR(\varphi) \quad (1)$$

$$R(\varphi) \subseteq C(\varphi) \subseteq \Omega(\varphi)$$

For a given X and $\varphi \in C(X)$, a set $A \subseteq X$ is *φ-invariant* (or simply *invariant*) if $\varphi(A) = A$. A set A is *φ-minimal* (or simply *minimal*) if it is nonempty, closed and φ-invariant and no proper subset of A has these three properties. Consider the set $\mathcal{M}(X)$ of Borel probabilistic measures endowed with the usual weak* topology. Then $\mu \in \mathcal{M}(X)$ is *φ-invariant* if $\mu(A) = \mu(\varphi^{-1}(A))$, for any measurable $A \subseteq X$. If, for any φ^{-1}-invariant set A, either $\mu(A) = 0$ or $\mu(A) = 1$, then the invariant measure μ is *ergodic*.

By a *triangular map* we mean a continuous map $F : X \times I \to X \times I$ of the form $F(x, y) = (f(x), g_x(y))$ where $f \in C(X)$ and $g_x \in C(I)$ for all $x \in X$. The maps f, g_x and the set $I_x := \{x\} \times I$ are called the *base* map, the *fibre* map and the *layer* (*fibre*) over $x \in X$. Denote by $T(X)$ and $T_m(X)$ the set of triangular maps on $X \times I$ and the set of triangular maps on $X \times I$ with nondecreasing fiber maps, respectively.

A periodic orbit P of an $f \in C(I)$ of period $m = 2^k$, where $k \geq 1$, is *simple* [Bl] if for any points $x_1 < x_2 < \ldots < x_n$ in P where $m = nr$, $r \geq 1$ and $n \geq 2$, such that $\{x_1, x_2, \ldots, x_n\}$ is periodic orbit of f^r, we have

$$f^r\left(\{x_1, x_2, \ldots, x_{\frac{n}{2}}\}\right) = \{x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n\}.$$

A cycle α of a triangular map $F \in T(I)$ is *simple* [Ko1] if the first canonical projection of α, $\pi(\alpha)$, forms a simple cycle of the base map f and, for any $x \in \pi(\alpha)$, the set $\alpha \cap I_x$ is a simple cycle of the map $F^m|_{I_x}$, where m denotes the period of x.

Later we will need the following example, which is a slight modification of an example from [FP].

Example 1. Let F be a triangular map of the square I^2 defined by $F = (x, g_x(y))$ where all g_x are continuous increasing interval maps whose graphs lie over or on the diagonal of the square I^2. Besides, let the "horseshoe" $I_1 \cup (\{0,1\} \times I)$ be the set of fixed (and also periodic) points of F. Then each point of the square I^2 is chain recurrent.

2 Nonwandering points

In the nineties several mathematicians tried to solve the following problem concerning the triangular maps of the square with the closed set of periodic points. In 1996 Forti and Paganoni [FP] considered relatively simple system admitting the condition $(\star)$ that every ω-limit set is a cycle. Under this assumption, they proved that the periodic set of F is closed and equal to $\omega(F)$. The converse statement was formerly disproved by S. Kolyada [K]. Forti and Paganoni also constructed a triangular map having some nonperiodic chain recurrent point, see Example 1. In view of (1), the following question arose:

Is there a triangular map F satisfying $(\star)$ and possessing a nonperiodic nonwandering point?

Or even, what about the validity of the following condition:

$$\text{If } P(F) \text{ is closed then } P(F) = \Omega(F)? \tag{2}$$

This question was answered in positive by C. Arteaga in 1995 [Ar] under strong regularity conditions, involving differentiability of f and all g_x and hyperbolicity of the periodic points of f. The problem for the general continuous triangular map without any restrictions was solved in [1]. However, during refereeing process it appeared that the complete positive answer was already given by L. S. Efremova ([Ef]) in eighties. It should be noted that the Efremova's paper is not generally accessible.

In her paper, L. S. Efremova found a suitable description of the nonwandering set of a triangular map whose base map has closed set of periodic points, and the implication (2) is a direct consequence. Our paper [1] provides different approach. Its advantage is a shorter and more transparent proof of (2). It is decomposed in a sequence of technical lemmas concerning one-dimensional continuous maps, some of them quoting and other extending known results given, e.g., in [BC]. In particular, Lemma A, [1, Lemma 4], is essential for our purpose. This lemma refers to the interval maps with the closed periodic set and describes the stability or unstability of points lying within a small neighborhood of each fixed point. It reads as follows:

Lemma A. *Let $f \in C(I)$, let $P(f)$ be closed and let $x \in I$ be fixed. Then there is an arbitrarily small open neighborhood U of x such that*

$$n \in \mathbb{N},\ y \in U \text{ and } f(y) \notin U \text{ imply } f^n(y) \notin U.$$

3 Chain recurrent points

Example 1 in Section 2 exhibits a triangular map F whose set $P(F)$ of periodic points is closed and properly contained in the set $CR(F)$ of chain recurrent points. In this section we give a characterization of such maps [2]. It is shown that inside the set of periodic points there are possible all phenomena, causing the existence of nonperiodic chain recurrent points. Example 1, and the following Example 2 present two typical situations ensuring $CR(F) \setminus P(F) \neq \emptyset$.

Example 2. Define a triangular map F of the square I^2 in the same way as in Example 1, with the exception that we need another set of fixed points of F and no periodic point of period greather than 1. Denote by $|a, b| \subseteq I^2$ the abscissa connecting points a and b and choose, for $i \in \mathbb{N}$, $w_i := (2^{-(i-1)}, 2^{-i})$ and $u_i := (2^{-i}, 2^{-(i+2)})$. The set of fixed points consist of $\{0, 1\} \times I$ and $\bigcup_{i \geq 2} |w_i, u_i|$. Then, the point $(1/2, 1/16)$ is chain recurrent but not periodic.

The main result of this section, Theorem D, uses the following notions.

Definition B. Let $z_1 = (x, y_1)$, $z_2 = (x, y_2)$ be periodic points of $F(x, y) = (f(x), g_x(y))$. Denote by A the periodic orbit of x. The point z_1 is *accessible* from z_2 if there is an ε-chain from z_2 to z_1 with respect to the map F restricted to $I_A = \bigcup_{y \in A} I_y$, for any $\varepsilon > 0$. The point z_1 is *nontrivially accessible* from z_2 if z_1 is accessible from z_2 and, for any sufficiently small $\varepsilon > 0$, any ε-chain from z_2 to z_1 with respect to the map F restricted to I_A contains a nonperiodic point.

Let $K_1,\ K_2$ be subsets of $P(F)$. Then K_1 is *accessible* from K_2 if there are points $z_1 \in K_1$ and $z_2 \in K_2$ so that z_1 is accessible from z_2.

Definition C. Points $z_1,\ z_2 \in P(F)$ form a *t-pair* if

(i) z_1 is nontrivially accessible from z_2,

and, for any $\delta > 0$,

(ii) there exists a finite system $\{K_i\}_{i=1}^m$ of connected components of $P(F)$ such that $z_1 \in K_1,\ z_2 \in K_m$ and, for any $1 \le i < m$, either K_{i+1} is accessible from K_i or $\inf\{d_{I^2}(x,y) | x \in K_i, y \in K_{i+1}\} < \delta$.

Theorem D. *Let $F \in T(I)$ be a triangular map with closed set $P(F)$ of periodic points. Then $CR(F) \setminus P(F) \neq \emptyset$ if and only if there exists a t-pair.*

It should be noted that, in general, it is not easy to check the conditions given in Definition C. However, a specific information about nontrivial ε-chains is provided by this characterization.

4 Minimal sets

This section apparently is not closely related with triangular maps possessing closed set of periodic points. However, we will be able to show, in Section 5, that there are relations. It is well known, that any minimal set is characterized by the property that each its point is uniformly recurrent, cf., e.g., [BC]. Since any regularly recurrent point is uniformly recurrent (1), in general, we have three types of minimal sets.

(i) Minimal sets containing no regularly recurrent point (e.g., an irrational rotation of a circle).

(ii) So-called *odometers*, minimal sets consisting only of regularly recurrent points (e.g., a periodic orbit or a solenoid of an interval map). Detailed study of the odometers can be found in the recent paper [BK].

(iii) Minimal sets containing both regularly and not regularly (only uniformly) recurrent points. An example of such minimal set was given by Forti and Paganoni in [FP]. They initially wanted to show that no regularly recurrent point of a triangular map F has to lie in a fibre over some regularly recurrent point of the base map of F; compare this with the Kolyada's projection theorem [K]. Moreover, a strange behavior (impossible for one-dimensional maps) of triangular maps with infinite minimal sets of the last type is described in [FPS1] and [FPS2].

It is known that there is a two-dimensional system possessing an infinite minimal set with no regularly recurrent point, e.g., see [P]. On the other hand, any continuous map of the interval with an infinite ω-limit set contains an infinite odometer [BC]. Now, there is a question what about the triangular maps of the square or even, of $X \times I$? Clearly, there is a triangular map having an odometer, since $F(x, y) = (f(x), y)$ has the same types of minimal sets as f. As we mentioned above, Forti and Paganoni found a triangular map with monotone fiber maps possessing an infinite minimal set of the type (iii). We show that their result is the best possible one [4]. More precisely, we have the following.

Theorem E. *Let $F \in T_m(X)$ and $M \subseteq X \times I$ be an infinite F-minimal set whose projection is an infinite minimal set $Q \subseteq X$ of the base map f. Then $Q \cap RR(f) \neq \emptyset$ implies $M \cap RR(F) \neq \emptyset$.*

Unfortunately, we are not able to give a similar proposition without the assumption of monotonicity. Only partial results are known, see [4]. One can find additional information on minimal sets of generalized triangular maps [KST] and on almost 1-1 extensions of odometers, i.e., minimal sets possessing a regularly recurrent point. A survey paper concerning odometers and their almost 1-1 extensions is prepared by T. Downarowicz at present.

5 Classification of triangular maps

The main aim of this section is to provide a classification of triangular maps of the square with closed sets of periodic points. For interval maps, similar classification is given et al., by A. Sharkovsky, see, e.g., [Sh]. In this book, the following list of mutually equivalent conditions is given for interval maps (here φ and X are chosen in accordance with context):

P1 $P(\varphi) = CR(\varphi)$
P2 $CR(\varphi) = \{x \in X \mid \exists n(x) : \varphi^{2^n}(x) = x\}$
P3 $CR(\varphi)$ is a union of all simple cycles of the map φ
P4 $\forall x \in X : \omega_\varphi(x)$ is a cycle
P5 $\forall x \in X : \omega_\varphi(x)$ is a simple cycle
P6 $P(\varphi) = \Omega(\varphi)$

P7 $P(\varphi) = \omega(\varphi)$
P8 $P(\varphi) = C(\varphi)$
P9 $P(\varphi) = R(\varphi)$
P10 $P(\varphi) = UR(\varphi)$
P11 $P(\varphi) = \overline{P(\varphi)}$
P12 $\Omega(\varphi) = \{x \in X \mid \exists n(x) : \varphi^{2^n}(x) = x\}$
P13 $\omega(\varphi) = \{x \in X \mid \exists n(x) : \varphi^{2^n}(x) = x\}$
P14 $C(\varphi) = \{x \in X \mid \exists n(x) : \varphi^{2^n}(x) = x\}$
P15 $R(\varphi) = \{x \in X \mid \exists n(x) : \varphi^{2^n}(x) = x\}$
P16 $UR(\varphi) = \{x \in X \mid \exists n(x) : \varphi^{2^n}(x) = x\}$
P17 any invariant ergodic measure is concentrated on a cycle
P18 $RR(\varphi) = \{x \in X \mid \exists n(x) : \varphi^{2^n}(x) = x\}$
P19 $P(\varphi) = RR(\varphi)$.

As we could see above (e.g. Example 1), these conditions are not mutually equivalent for a triangular map F. Almost complete account of relations among them is stated in the next result.

Theorem F. *Let $F \in T(I)$. Then*
(i) $P1 \Rightarrow P11$ and $P4 \Rightarrow P11$,
(ii) no other implication among $P1, P4$ and $P11$ is true,
(iii) $P1 - P3$ are mutually equivalent,
(iv) $P4$ and $P5$ are equivalent,
(v) $P6 - P17$ are mutually equivalent.
Thus, all implications among the conditions $P1 - P17$ can be displayd by the following diagram:

$$(P1 - P3) \Rightarrow (P6 - P17) \Leftarrow (P4, P5)$$

As regards the remaining conditions, only two implications are known:

$$(P6 - P17) \Rightarrow P18 \Rightarrow P19$$

Thus there is an **open problem** whether, in the general case, either of the implications

$$P18 \Rightarrow (P6 - P17) \text{ or } P19 \Rightarrow P18$$

is true. Let us note that, from Theorem E in Section 4, it can be deduced that either of these implications is true in the class $T_m(I)$

of triangular maps nondecreasing on the fibres. Thus, we have the following result.

Theorem G. *Let $F \in T_m(I)$. Then*
(i) $P1 \Rightarrow P11$ *and* $P4 \Rightarrow P11$*,*
(ii) no other implication among $P1, P4$ *and* $P11$ *is true,*
(iii) $P1 - P3$ *are mutually equivalent,*
(iv) $P4$ *and* $P5$ *are equivalent,*
(v) $P6 - P19$ *are mutually equivalent.*
Thus, the all implications connecting conditions $P1 - P19$ *can be displayed by the following diagram:*

$$(P1 - P3) \Rightarrow (P6 - P19) \Leftarrow (P4, P5)$$

6 Publications

[1] J. Kupka, Triangular maps which have the nonwandering points periodic, *Preprint Series in Mathematical Analysis*, Mathematical Institute, Silesian University at Opava, MA 30/2001.

[2] J. Kupka, Triangular maps with the chain recurrent points periodic, *Acta Math. Univ. Comenianae*, **72** (2003), 245 – 251. ISSN 0862-9544

[3] J. Kupka, Triangular maps with the chain recurrent points periodic, *Real Analysis Exchange*, Summer Symposium 2003, 39 – 40. ISSN 0147-1937

[4] J. Kupka, Triangular maps with closed set of periodic points, *Preprint Series in Mathematical Analysis*, Mathematical Institute, Silesian University at Opava, MA 43/2004.

7 Presentations

[5] 5th Czech - Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, June 2001. Talk on [1].

[6] 30th Winter School in Abstract Analysis, Lhota nad Rohanovem, January 2002. Talk on [1].

[7] Summer Symposium in Real Analysis XXVII, Opava, Czech Republic, June 2003. Talk on [3].

[8] 7th Czech - Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, September 2003. Talk on [2].

[9] 32th Winter School in Abstract Analysis, Lhota nad Rohanovem, January 2004. Talk on [2].

[10] Universidad de Murcia, Spain, April 2004. Lecture on [2].

[11] European Conference on Iteration Theory (ECIT' 04), Batschuns, Austria. Talk on [4].

References

[Ar] C. Arteaga, Smooth triangular maps of the square with closed set of periodic points, *J. Math. Anal. Appl.* **196** (1995), 987–997.

[Bl] L. Block, Simple periodic orbits of mappings of the interval, *Trans. Am. Math. Soc.* **254**, 391-398 (1979).

[BC] L. S. Block, W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math. 1513, Springer, Berlin – Heidelberg – New York, 1992.

[BK] L. S. Block, J. Keesling, *A characterization of adding machine maps*, Topology Appl. **140** (2004), 151–161.

[Ef] L. S. Efremova, On the nonwandering set and the center of triangular maps with closed set of periodic points in the base (in Russian), *Dynam. Systems and Nonlinear Phenomena*, Inst. of Mathem. of NAS of Ukraine, Kiev (1990), 15–25.

[GCh] J. L. Garcia Guirao, J. Chudziak, *A characterization of zero topological entropy for a class of triangular mappings*, J. Math. Anal. Appl. **287** (2003), 516–521.

[FP] G. L. Forti, L. Paganoni, On some properties of triangular maps, *Iteration theory (ECIT '96) (Urbino)*, Grazer Math. Ber. **339** (1999), 125–140.

[FPS1] G. L. Forti, L. Paganoni, J. Smítal, *Strange triangular maps of the square*, Bull. Austral. Math. Soc. **51** (1995), 395–415.

[FPS2] G. L. Forti, L. Paganoni, J. Smítal, *Dynamics of homeomorphisms on minimal sets generated by triangular mappings*, Bull. Austral. Math. Soc. **59** (1999), 1–20.

[FPS3] G. L. Forti, L. Paganoni, J. Smítal, *Triangular maps with all periods and no infinite omega-limit set containing periodic points*, Topology Appl. (to appear)

[Ko1] Z. Kočan, *The problem of classification of triangular maps with zero topological entropy*, Ann. Math. Sil. **13** (1999), 181–192.

[Ko2] Z. Kočan, *Triangular maps non-decreasing on the fibres*, Real Analysis Exchange **29** (2003/04).

[K] S. F. Kolyada, On dynamics of triangular maps of the square, *Ergodic Theory Dynam. Systems* **12** (1992), 749–768.

[KS] S. F. Kolyada, A. N. Sharkovsky, *On topological dynamics of triangular maps of the plane*, in Proceedings of ECIT-89, Batschuns, Austria, World Scientific, Singapore (1991), 177–183.

[KST] S. Kolyada, L. Snoha, S. Trofimchuk, *Minimal sets in fibred systems*, Preprint (2004).

[P] K. Petersen, Ergodic Theory, *Cambridge University Press*, 1983.

[Sh] V. V. Fedorenko, S. F. Kolyada, A. N. Sharkovsky, A. G. Sivak, *Dynamics in One-Dimensional Maps*, Kluwer Academic Publishers, 1997.