International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Spring 2011.**

1. The numbers from 1 to 2010 inclusive are placed along a circle so that if we move along the circle in clockwise order, they increase and decrease alternately. Prove that the difference between some two adjacent integers is even.

2. A rectangle is divided by 10 horizontal and 10 vertical lines into 121 rectangular cells. If 111 of them have integer perimeters, prove that they all have integer perimeters.

3. Worms grow at the rate of 1 metre per hour. When they reach their maximum length of 1 metre, they stop growing. A full-grown worm may be dissected into two new worms of arbitrary lengths totalling 1 metre. Starting with 1 full-grown worm, can one obtain 10 full-grown worms in less than 1 hour?

4. Each diagonal of a convex quadrilateral divides it into two isosceles triangles. The two diagonals of the same quadrilateral divide it into four isosceles triangles. Must this quadrilateral be a square?

5. A dragon gave a captured knight 100 coins. Half of them were magical, but only the dragon knew which were. Each day, the knight divided the coins into two piles which were not necessarily equal in size. If each pile contained the same number of magic coins, or the same number of non-magic coins, the knight would be set free. Could the knight guarantee himself freedom in at most

   (a) 50 days;

   (b) 25 days?

Note: The problems are worth 3, 4, 5, 5 and 2+3 points respectively.

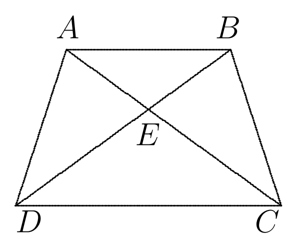
Solution to Junior O-Level Spring 2011

1. If all differences between two adjacent numbers are odd, then the numbers must be alternately odd and even. Also, the numbers are given to be alternately higher than both neighbours and lower than both neighbours. Now 2010 must be a high number. Hence all the even numbers are high numbers. However, 2 clearly cannot be a high number. This is a contradiction.

2. Let the widths of the columns be x_i and the heights of the rows be y_i, $1 \le i \le 11$. A cell is said to be good if its perimeter is an integer. Since there are at least 111 good cells, there are at most 10 bad cells, so that an entire row and an entire column is free of bad cells. Let these be the first row and the first column. In particular, $2x_1 + 2y_1$ is an integer. Consider the cell on the i-th row and the j-th column, $2 \le i, j \le 11$. Its perimeter is $2x_j + 2y_i$. Both the cells on the i-th row and the first column and on the j-th column and the first row are good. Hence $2x_1 + 2y_i$ and $2x_j + 2y_1$ are integers. It follows that $2x_j + 2y_i$ is also an integer, meaning that every cell is good.

3. We divide 1 metre into 1024 sillimetres and 1 hour into 1024 sillihours. Then an under-sized worm grows at the rate of 1 sillimetre per sillihour. At the start, we cut the full-grown worm into two, one of length 1 sillimetre and the other 1023 sillimetres. After 1 sillihour, the shorter worm has length 2 sillimetres and the longer worm is full-grown. It is then dissected into two, so that the shorter one is also of length 2 sillimetres. After another 2 sillihours, the shorter worms have length 4 sillimetres and the longer worm is full-grown. It is then dissected into two so that the shorter one is also of length 4 sillimetres. Continuing in this manner, we will have 10 full-grown worms after $1+2+4+\cdots+512 = 1023$ sillihours, just beating the deadline of 1 hour.

4. The answer is no. In the convex quadrilateral $ABCD$ in the diagram below, where the diagonals intersect at E, we have $\angle ADB = \angle BDC = \angle DCA = \angle ACB = \angle BAC = \angle ABD = 36°$. Then $\angle ADC = \angle DAE = \angle DEA = \angle CEB = \angle CBE = \angle BCD = 72°$. It follows that all of the triangles ABC, BAD, ACD, BDC, ABE, BCE, CDE and DAE are isosceles, and yet $ABCD$ is not a square.



5. **Solution by Victor Ivrii.**
We just solve (b), and (a) follows. On the first day, the knight divides the 100 coins into a small pile containing 25 coins and a large pile containing 75 coins. Each day for which he is still in confinement, he transfers 1 coin from the large pile to the small pile. If he escapes, the dragon continues the transfer. By the 25th day, there will be 49 coins in the small pile and 51 coins in the large pile. On the 1st day, if the small pile contains 25 magic coins or 25 non-magic coins, the knight will be free. If not, the pile contains less than 25 magic coins and less than 25 non-magic coins. On the 25th day, the small pile contains either at least 25 magic coins or at least 25 non-magic coins. By symmetry, we may assume it contains at least 25 magic coins. Since the coins are transferred one at a time to the small pile, the number of magic coins must be exactly 25 at some point, on or before the 25th day. Hence the knight can guarantee freedom by the 25th day at the latest.