International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2011.**

1. Baron Munchausen has a set of 50 coins. The mass of each is a distinct positive integer not exceeding 100, and the total mass is even. The Baron claims that it is not possible to divide the coins into two piles with equal total mass. Can the Baron be right?

2. In the coordinate space, each of the eight vertices of a rectangular box has integer coordinates. If the volume of the solid is 2011, prove that the sides of the rectangular box are parallel to the coordinate axes.

3. Two cross-sections of an arbitrary right triangular prism are such that they do not intersect the base of the prism.

   (a) Can these cross-sections be similar but not congruent triangles?

   (b) Can these cross-sections be equilateral triangles of sides 1 and 2 respectively?

4. There are n red sticks and n blue sticks. The sticks of each colour have the same total length, and can be used to construct an n-gon. We wish to repaint one stick of each colour in the other colour so that the sticks of each colour can still be used to construct an n-gon. Is this always possible if

   (a) $n = 3$;

   (b) $n > 3$?

5. In the convex quadrilateral $ABCD$, BC is parallel to AD. Two circular arcs ω_1 and ω_3 pass through A and B and are on the same side of AB. Two circular arcs ω_2 and ω_4 pass through C and D and are on the same side of CD. The measures of ω_1, ω_2, ω_3 and ω_4 are α, β, β and α respectively. If ω_1 and ω_2 are tangent to each other externally, prove that so are ω_3 and ω_4.

6. In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.

7. Among a group of programmers, every two either know each other or do not know each other. Eleven of them are geniuses. Two companies hire them one at a time, alternately, and may not hire someone already hired by the other company. There are no conditions on which programmer a company may hire in the first round. Thereafter, a company may only hire a programmer who knows another programmer already hired by that company. Is it possible for the company which hires second to hire ten of the geniuses, no matter what the hiring strategy of the other company may be?

Note: The problems are worth 4, 6, 3+4, 4+4, 8, 8 and 11 points respectively.

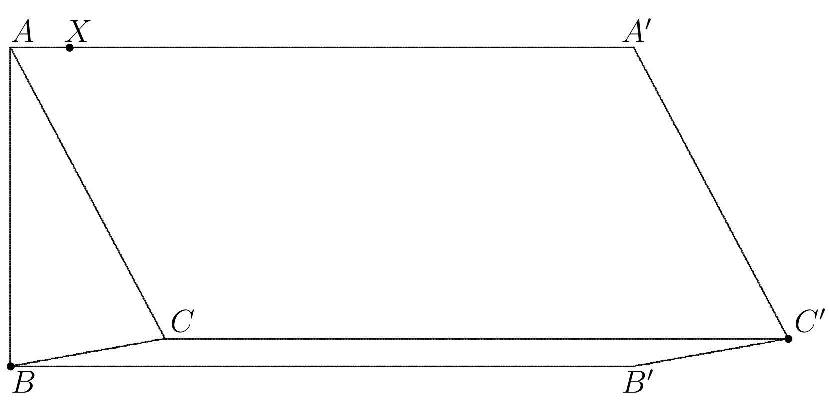
Solution to Junior A-Level Spring 2011

1. The Baron is right, as usual. The masses of the 50 coins in his collection may just be the 50 even numbers up to 100. Their total mass is 50 times 51, so that each of two piles would have total mass 25 times 51, which is odd. This is impossible since all the coins have even weights.

2. **Solution by Olga Ivrii:**
Let the side lengths of the rectangular box be $a \leq b \leq c$. Because the vertices are lattice points, each of a^2, b^2 and c^2 is an integer. Since 2011 is a prime, we must have either $a = b = 1$ and $c = 2011$, or $a = 1$ and $b = c = \sqrt{2011}$. In the former case, the sides of lengths 1 must be parallel to two of the coordinate axes. It follows that the side of length 2001 must also be parallel to the third coordinate axes. In the latter case, the side of length 1 is parallel to a coordinate axis, and the sides of length $\sqrt{2011}$ define a square in a plane parallel to a coordinate plane. Again, because the vertices are lattice points, we must be able to express 2011 as a sum of two squares. One of them must be the square of an even integer and the other the square of an odd integer, say $(2m)^2 + (2n+1)^2 = 2011$. This simplifies to $4(m^2 + n^2 + n) = 2010$, which is impossible since 2010 is not a multiple of 4.

3. **Solution by Yu Wu:**

   (a) Take a prism $ABCA'B'C'$ as shown in the diagram below, with $AB = AC = 2$ and $BC = 1$. Take the cross-section XBC' where X be a point on AA' to be chosen. Let $AX = x$ and $XA' = y$. Then $BC' = \sqrt{1 + (x+y)^2}$, $XC' = \sqrt{4 + y^2}$ and $BX = \sqrt{4 + x^2}$. We wish to choose x and y such that $BC' = XC' = 2BX$. From $XC' = 2BX$, we have $y = 2\sqrt{3 + x^2}$. From $BC' = XC'$, we have $0 = x^2 + 2xy - 3 = x^2 + 2x\sqrt{3 + x^2} - 3$. Denote the last expression by $P(x)$. Then $P(0) = -3$ and $P(1) = 6$. Since $P(x)$ is continuous, there exists a value for x, and consequently for y, which makes triangle XBC' similar to triangle ABC and yet not congruent to it.



   (b) We use here the same diagram above, but with $BC = a$, $CA = b$ and $AB = c$. If we can inscribe an equilateral triangle of side 1 in this prism, then each of a, b and c is less than 1. Now $BC' = \sqrt{a^2 + (x+y)^2}$, $BX = \sqrt{b^2 + x^2}$ and $XC' = \sqrt{c^2 + y^2}$. If XBC' is an equilateral triangle of side 2, then $x = \sqrt{4 - b^2} \geq \sqrt{3}$ and $y = \sqrt{4 - c^2} \geq \sqrt{3}$. Hence $4 \geq a^2 + (\sqrt{3} + \sqrt{3})^2 > 12$, which is a contradiction.

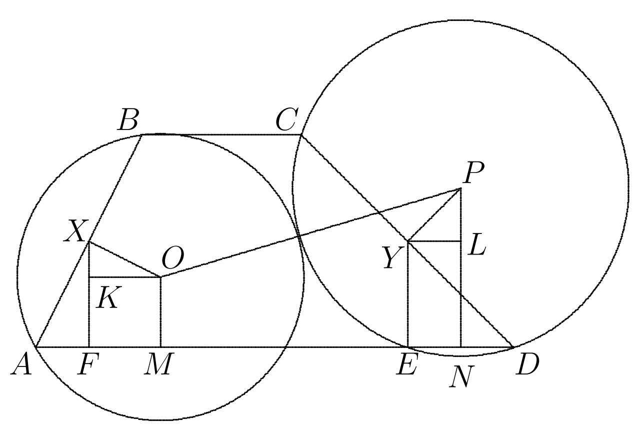
4. (a) The red sticks may be of lengths 16, 16 and 1, and the blue sticks may be of lengths 13,11 and 9. Since 16+16+1=13+11+9, $16 < 16+1$ and $13 < 11+9$, the conditions of the problem are satisfied. If we swap the red stick of length 1 with any blue stick, it will not form a triangle with the other two blue sticks. If we swap any blue stick with a red stick of length 16, it will not form a triangle with the other two red sticks.

(b) **Solution by Yu Wu:**
There is a red stick of length $5-\frac{1}{n(n+1)}$ and another of length $4+\frac{1}{n(n+1)}$. The other $n-2$ are of length $\frac{1}{n-2}$. There is a blue stick of length $5-\frac{1}{n^2(n+1)}$ and another of length $5-\frac{1}{n^3(n+1)}$. The other $n-2$ are of length $\frac{1}{n^3(n-2)}$. The total length of the sticks of each colour is 10. We consider four cases.
Case 1. A short red stick is swapped for a short blue stick.
We will have trouble on the predominantly red side, because

$$\begin{aligned} & 4-\frac{1}{n(n+1)}+\frac{n-3}{n-2}+\frac{1}{n^3(n-2)}-5+\frac{1}{n(n+1} \\ < & \frac{3}{n(n+1)}+\frac{n-3}{n-2}-1 \\ < & \frac{1}{n-2}+\frac{n-3}{n-2}-1 \\ = & 0. \end{aligned}$$

Case 2. A long red stick is swapped for a short blue stick.
We will have trouble on the predominantly red side because $4+\frac{1}{n(n+1)}>1+\frac{1}{n^3(n-2)}$.
Case 3. A long red stick is swapped for a long blue stick.
We will have trouble on the predominantly blue side, because $5-\frac{1}{n^2(n+1)}>5-\frac{1}{n(n+1)}+\frac{1}{n^3}$.
Case 4. A short red stick is swapped for a long blue stick.
We will have even greater trouble on the predominantly blue side.

5. **Solution by Charles Leytem:**
Let X and Y be the respective midpoints of AB and CD. Let O, P, Q' and P' be the respective centres of ω_1, ω_2, ω_3 and ω_4. Let E, F, M, N, M' and N' be points on AD such that YE, XF, OM, PN, $O'M'$ and $P'N'$ are perpendicular to AD. Let K be the point on XF such that OK is perpendicular to XF, L be the points on PN such that LY is perpendicular to PN, K' be the point on XF such that $O'K'$ is perpendicular to XF, and L' be the point on $P'N'$ such that $L'Y$ is perperndicular to $P'N'$. Let $AX=a$, $DY=c$, $DE=e$, $AF=f$ and $XF=YE=h$.



We first prove three preliminary results.

(1) $OB \cdot PC = O'B \cdot P'C$.

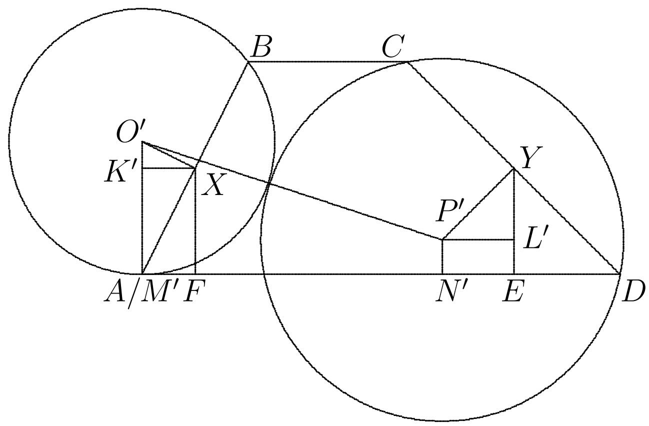
By the Law of Sines, $2OB\sin\alpha = 2a = 2O'B\sin\beta$ and $2PC\sin\beta = 2c = 2P'C\sin\alpha$. Hence $cOB = aP'C$ and $aPC = cO'B$, so that $OB \cdot PC = O'B \cdot P'C$.

(2) $OX \cdot PY = O'X \cdot P'Y$.

We have $c^2OX^2 = c^2(OB^2 - a^2) = a^2(P'C^2 - c^2) = a^2P'Y^2$ so that $cOX = aP'Y$. Similarly, $cO'X = aPY$. Hence $OX \cdot PY = O'X \cdot P'Y$.

(3) $KX \cdot LP = K'O' \cdot L'Y$.

We have $KX = \frac{f}{a}OX$, $LP = \frac{e}{c}PY$, $K'O' = \frac{f}{a}O'X$ and $L'Y = \frac{e}{c}P'Y$ It follows from (2) that $KX \cdot LP = K'O' \cdot L'Y$.



The horizontal distance between O and P is $MN = EF - MF + EN$. The vertical distance between O and P is $PN - OM = KX + LP$. We are given that ω_1 and ω_2 are tangent, so that $MN^2 + (KX + LP)^2 = (OB + PC)^2$. Similarly, the horizontal distance between O' and P' is $M'N' = EF - E'N + F'M$, and the vertical distance is $O'M' - P'N' = K'O' + L'Y$. To have ω_3 tangent to ω_4, we need to prove that $(M'N')^2 + (K'O' + L'Y)^2 = (O'B + P'C)^2$. Note that $FM = \frac{h}{a}OX = \frac{h}{c}P'Y = EN'$ and $FM' = \frac{h}{a}O'X = \frac{h}{c}PY = EN$. Hence $MN = M'N'$. In view of (1) and (3), the desired result now follows from

$$\begin{aligned}
& OB^2 - KX^2 + PC^2 - LP^2 \\
= \ & \frac{a^2}{c^2}P'C^2 - \frac{f^2}{a^2}OX^2 + \frac{c^2}{a^2}O'B^2 - \frac{e^2}{c^2}PY^2 \\
= \ & \frac{a^2}{c^2}P'C^2 - \frac{f^2}{c^2}P'Y^2 + \frac{c^2}{a^2}O'B^2 - \frac{e^2}{a^2}O'X^2 \\
= \ & \frac{h^2}{c^2}P'C^2 - f^2 + \frac{h^2}{a^2}O'B^2 - e^2 \\
= \ & \frac{h^2}{a^2}OB^2 - e^2 + \frac{h^2}{c^2}PC^2 - f^2 \\
= \ & \frac{a^2}{c^2}PC^2 - \frac{f^2}{c^2}PY^2 + \frac{c^2}{a^2}OB^2 - \frac{e^2}{a^2}OX^2 \\
= \ & \frac{a^2}{c^2}PC^2 - \frac{f^2}{a^2}O'X^2 + \frac{c^2}{a^2}OB^2 - \frac{e^2}{c^2}P'Y^2 \\
= \ & O'B^2 - (K'O')^2 + P'C^2 - L'Y^2.
\end{aligned}$$

6. **First Solution by Daniel Spivak.**
We only need to prove that $a \geq b$ since we will then have $b \geq a$ by symmetry. Circle the largest number x_j in column j, $1 \leq j \leq n$, where n is the number of rows and therefore of columns. By relabelling if necessary, we may assume that x_1 is the smallest of these n numbers. We consider two cases.
Case 1. Two circled numbers x_j and x_k are on the same row.
Then $a \geq x_j + x_k \geq 2x_1 \geq b$ since the sum of the largest two numbers in column 1 is b.
Case 2. Each circled number is in a different row.
Let the second largest number y_1 in column 1 be in row j, and let the circled number in row j be x_k. Then $b = x_1 + y_1 \leq x_k + y_1 \leq a$ since the sum of the largest two numbers in row k is a.

Second Solution:
Suppose to the contrary that $a \neq b$. We may assume by symmetry that $a > b$. Circle in each row the largest two numbers. Let the number of circled numbers in column i be c_i, $1 \leq i \leq n$, where n is the number of rows and therefore of columns. We have $c_1 + c_2 + \cdots + c_n = 2n$. Now $\binom{c_1}{2} + \binom{c_2}{2} + \cdots + \binom{c_n}{2} = \frac{1}{2}(c_1^2 + c_2^2 + \cdots + c_n^2) - \frac{1}{2}(c_1 + c_2 + \cdots + c_n)$. By the Root-Mean-Square Inequality, $\sqrt{\frac{c_1^2+c_2^2+\cdots+c_n^2}{n}} \geq \frac{c_1+c_2+\cdots+c_n}{n}$. It follows that $\binom{c_1}{2} + \binom{c_2}{2} + \cdots + \binom{c_n}{2} \geq n$. We now construct a graph with n vertices representing the n rows. Two vertices are joined by an edge if and only if the corresponding rows have circled numbers in the same column. The number of edges of this graph is given by $\binom{c_1}{2} + \binom{c_2}{2} + \cdots + \binom{c_n}{2} \geq n$, so that the graph has a cycle, say of length k. By relabelling if necessary, the k vertices on this cycle represent rows 1 to k, with the circled numbers on row i in columns i and $i+1$ for $1 \leq i \leq k-1$, and the circled numbers on row k in columns k and 1. Now the circled numbers in a column may or may not be the largest two of its numbers, but the sum of these $2k$ numbers is ka. This means that the sum of the largest two numbers of some column is at least $a > b$, which is a contradiction.

7. **Solution by Central Jury:**
Let there be eleven attributes on which the companies rank the candidates. The ranking of each attribute for each candidate is a non-negative integer. It turns out that for each candidate, the sum of the eleven rankings is exactly 100. Moreover, no two candidates have exactly the same set of rankings, and for each possible set of rankings, there is such a candidate. The eleven genius are those with a ranking of 100 in one attribute and a ranking of 1 in every other atrribute. Two candidates know each other if their sets of rankings differ only in two attributes, and those two rankings differ by 1. Consider candidate A who is the first hired by the first company. By the Pigeonhole Principle, the ranking of at least one attribute for A is at least 10, and we may assume that this is the first attribute. The second company hires the candidate whose ranking in the first attribute is exactly 10 lower than that of A, but exactly 1 higher in each of the other ten attributes. At this point, the first company has a big edge in hiring the genius of the first attribute, but the second company has a small edge in hiring the genius of each of the other ten attributes. The second company concedes the genius of the first attribute to the first company, but aims to hire the other ten geniuses by maintaining these small advantages. Note that among the candidates hired by each company, the highest ranking in any attribute can only increase by 1 with each new hiring. Whenever the first company makes a hiring, the second comapny will respond by hiring a candidate whose rankings change in the same attributes and in the same directions.