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OMEGA LIMIT SETS AND DISTRIBUTIONAL CHAOS ON GRAPHS

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Abstract. We prove the following results for general continuous maps on graphs. We give a full topological characterization of ω-limit sets. We show that basic sets have similar properties as in the case of the compact interval. Finally, we prove that the presence of distributional chaos, the existence of basic sets, and positive topological entropy (among other properties) are mutually equivalent.

Omega limit sets give fundamental information about asymptotic behavior of a dynamical system. One of the basic tasks is to provide a topological characterization of them. The task is more complicated than it could be thought and the first full characterization in the simplest onedimensional case — the compact interval — was given only in the end of 1980's in [ABCP]. A slightly stronger version was proved in a completely different and simpler way in [BS]. This was extended to the circle maps in [P]. One of the main aims of this paper is to give a full characterization of ω-limit sets on graphs. This is done in Section 1.

The notion of distributional chaos was introduced in [ScS]. In this paper it is shown that the presence of distributional chaos is equivalent to the existence of basic sets and positive topological entropy in the case of continuous interval maps. Natural question arises on what spaces these equivalences hold. The positive answer in the circle case was given in [M1]. Equivalence of positive sequence entropy and the presence of distributional chaos for a subclass of tree maps was proved in [C] and [CH]. This is not anymore true in higher dimension as was shown for skew-product maps of the square [Ba].

In this paper we show that the presence of distributional chaos, the existence of basic sets, and positive topological entropy are mutually equivalent in the case of general continuous graph maps.

The interest in studying graph maps is, besides their own attractivity, due to the fact that for maps on manifolds with an invariant foliation of codimension one, the corresponding quotient map turns out to be defined in general on a graph. Furthermore, the dynamics of pseudo-Anosov homeomorphisms on a surface can be essentially reduced to the analysis of some special graph maps (see eg. [FM]).

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Finally, a graph map sometimes imitates the behavior of a flow in a neighbourhood of a hyperbolic attractor (see eg. [W]).

Let (X, f) be a dynamical system given by a compact topological space and a continuous map $f : X \to X$ (we also write $f \in C(X)$). For $x \in X$ we define its *orbit*, $\mathrm{Orb}_f(x)$, to be the set $\{f^n(x) \mid n \geq 0\}$ (analogously we define the orbit of a set) and its *ω-limit set*, $\omega_f(x)$, as the set of all limit points of $\mathrm{Orb}_f(x)$. It is well known that each ω-limit set is non-empty, compact and strongly invariant (ie. $f(S) = S$).

An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$. A *graph* is a nonempty compact connected metric space which can be written as a union of finitely many arcs any two of which can intersect only in their endpoints (ie. it is a one-dimensional compact connected polyhedron). A *subgraph* is a subset of a graph which is itself a graph (note that this is more general than a subgraph in the common combinatorial sense). Let G be a graph and $f \in C(G)$ then by a *periodic subgraph* (of f) we mean a subgraph $H \subseteq G$ such that there is an $n \geq 1$ for which $H, f(H), \dots, f^{n-1}(H)$ are pairwise disjoint and $f^n(H) = H$; in this case we also speak about a *periodic orbit* of subgraphs.

1. Topological characterization of ω-limit sets

The ideas in this section are inspired by those in [BS]. In the following theorem we give a full topological characterization of ω-limit sets of continuous maps on graphs.

Theorem 1. *Let G be a graph, $f \in C(G)$, and ω an ω-limit set of f. Then ω is*

(i) *a finite set (in fact, a periodic orbit), or*
(ii) *an infinite compact nowhere dense set, or*
(iii) *a finite union of connected subgraphs (which forms a periodic orbit).*

Conversely, whenever $\omega \subseteq G$ is of one of the above forms then there is a map $f \in C(G)$ such that ω is an ω-limit set of f.

Proof. Let $\omega = \omega_f(x)$ be an ω-limit set of f. It is straightforward to see that if ω is finite then it is a periodic orbit (cf. also [BC]). So, suppose that ω is infinite and it is dense in an open subset of G. Then it contains an arc. Take S to be a connected component of ω containing an arc. It is easy to see that $f^n(S)$ is not a singleton for any $n \geq 0$ otherwise ω would reduce to a periodic orbit. From this and from the fact that there is the smallest $m \geq 1$ such that $f^m(x) \in S$ follows that $\overline{\mathrm{Orb}_f(S)}$ is an orbit of a subgraph with at most $m - 1$ components. The fact that $\omega = \overline{\mathrm{Orb}_f(S)}$ follows from $\mathrm{Orb}_f(x) \subseteq \mathrm{Orb}_f(S)$. Periodicity of S is obvious.

Now, conversely, let $\omega \subseteq G$ be of one of the forms (i), (ii) or (iii). Case (i) is trivial — we can extend continuously any map defined on a closed subset to the whole graph (see [BHS]).

Case (ii). This is a consequence of Lemmas 1 and 2 below.

Case (iii). We are going to define f first as a transitive map on ω. Then the proof will be finished by taking a continuous extension of this to the whole G. Let $\omega = \omega_1 \cup \dots \cup \omega_k$ is the decomposition of ω to its connected components. Any of these subgraphs can be written as a union of finitely many arcs with pairwise disjoint interiors $\omega_i = A_i^1 \cup \dots \cup A_i^{n_i}$. We define f to satisfy $f(\omega_i) = \omega_{i+1 \pmod k}$, in fact such that the image of any arc A_i^j will be the whole component $\omega_{i+1 \pmod k}$. It is easy to see that this can be done "piecewise linearly". Then the image of any subarc of ω will be always an entire component of ω after finitely many steps. Clearly f is transitive on ω. $\square$

Definition 1. Let E be an infinite compact, totally disconnected metric space, $P = \{p_1, \dots, p_k\}$ a set of its limit points, and $f : E \to E$ a continuous map such that the following two properties are satisfied

(i) there is a system $\{E_i^n \mid n \in \mathbb{N},\ i = 1, \dots, k\}$ of nonempty, compact, pairwise disjoint subsets of E such that $E \setminus \bigcup_{i,n} E_i^n = P$ and $\lim_{n\to\infty} E_i^n = p_i$ (in the sense that $\operatorname{diam}(E_i^n \cup \{p_i\}) \to 0$);

(ii) the set P forms a periodic orbit of f ($f(p_i) = p_{i+1 \pmod k}$) and $f(E_i^n) = E_{i+1}^n$ for $i = 1, \dots, k-1$, $f(E_k^n) = E_1^{n-1}$ and $f(E_k^0) = p_1$.

Then we say that E is *finitely forward homoclinic* with respect to f.

Lemma 1. *Let $E \subseteq G$ be a nowhere dense closed set, $f : E \to E$ be continuous with E finitely forward homoclinic with respect to f. Then there is a continuous extension $F : G \to G$ of f such that $E = \omega_F(x)$ for some $x \in G$.*

Proof. We construct F as the limit of a uniformly convergent series of maps. First, fix a system $\{A_n \mid n \in \mathbb{N}\}$ of open arcs such that $\operatorname{diam} A_n \to 0$ and any open set intersecting E contains some A_n. Let F_0 be any continuous extension of f to G (it always exists, see [BHS]) and put $B_0 = \emptyset$ and define the rest of the series inductively. Let F_n and B_n be defined then choose B_{n+1}, a compact subarc of $G \setminus (E \cup B_0 \cup \dots \cup B_n)$. In order to define F_{n+1} we change F_n on B_{n+1} in such a way that the resulting map is continuous, $F_{n+1}(B_{n+1})$ contains a neighborhood of $F_n(A_n \cap E) = f_n(A_n \cap E)$ and its diameter is less than $2 \operatorname{diam} f_n(A_n \cap E)$. Clearly, $\{F_n \mid n \in \mathbb{N}\}$ is uniformly convergent and we denote its limit by F. Notice, that F has the following property important for our purposes: for any $u \in E$ and any neighborhood U of u, $F(U)$ is a neighborhood of $F(u) = f(u)$.

Using this property we can show another one: for any $u, v \in E$, any neighborhood U of u and any $\varepsilon > 0$ there is a closed set $K \subseteq U$ and $n \in \mathbb{N}$ such that $F^n(K)$ is a neighborhood of v and $F^i(K) \subseteq B(E, \varepsilon)$ for $i = 0, 1, \dots, n$. To see this, realize that there are l and m such that $F^l(u) = f^l(u) = p_m$. Take a closed neighborhood $L \subseteq U$ of u such that $\operatorname{diam} F^i(L) < \varepsilon$ for $i = 0, 1, \dots, l$. We know that $F^l(L)$ is a neighborhood of p_m hence it contains sets E_j^m for arbitrarily large j so there is a point $w \in F^l(L) \cap E$ such that $F^p(w) = f^p(w) = v$ for some p (since v belongs either to P or to some E_r^s). Now choose a closed neighborhood $M \subseteq F^l(L)$ of w with $\operatorname{diam} F^i(M) < \varepsilon$ for $i = 0, 1, \dots, p$ and put $K = F^{-l}(M) \cap L$.

We finally show that E is an ω-limit set of F. Using the last property of F we can define inductively a nested sequence of compact sets $\{K_i \mid i \in \mathbb{N}\}$ and an increasing sequence of nonnegative integers $\{n_i \mid i \in \mathbb{N}\}$ such that for each i we have $F^{n_i}(K_i) \subseteq A_i$, and $F^j(K_i) \subseteq B(E, 1/i)$ for $j = n_i, \dots, n_{i+1}$. We claim that E is the ω-limit set of any $x \in \bigcap_{n\in\mathbb{N}} K_i \neq \emptyset$. Since $\{F^j(x) \mid j \in \mathbb{N}\}$ visits each of the arcs $\{A_n \mid n \in \mathbb{N}\}$ and thus each open set intersecting E, we have $\omega_F(x) \supseteq E$. The opposite inclusion we get from the fact that $F^j(x) \in B(E, 1/i)$ for $j = n_i, n_{i+1}, \dots$. $\square$

Lemma 2. *Any compact, totally disconnected metric space E is finitely forward homoclinic with respect to some continuous map.*

Proof. Let us begin with the uncountable case. Let $x \in E$ be a condensation point of E (ie. any neighborhood of x contains uncountably many points) then we can decompose $E\setminus\{x\}$ to countably many uncountable pairwise disjoint clopen sets E_n, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} E_n = x$. Each of these sets contains a Cantor set (which is its retract) thus there is a continuous map from E_{n+1} onto E_n for $n = 0, 1, \dots$. We finish the construction of our map by mapping the set E_0 as well as the point x to x. Continuity of the map can be seen easily.

For the rest of the proof we suppose E being countable. First, define a transfinite sequence $\{E(\alpha) \mid \alpha < \Omega\}$ putting $E(0) = E$, $E(\alpha+1)$ to be the set of limit points of $E(\alpha)$, and $E(\alpha) = \bigcap_{\beta<\alpha} E(\beta)$ if α is a limit ordinal. Then there is the unique ordinal $\tau = \tau(E) < \Omega$ such that $E(\tau)$ is nonempty and finite. Let $E(\tau) = \{x_1, \dots, x_k\}$. Choose a clopen cover $\{U_1^0, \dots, U_k^0\}$ of the space E by pairwise disjoint sets such that U_i^0 is a neighborhood of x_i, $i = 1, \dots, k$. Clearly $\tau(U_i^0) = \tau$. There is a clopen neighborhood $U_k^1 \subset U_k^0$ of x_k such that $E_k^0 := U_k^0 \setminus U_k^1 \neq \emptyset$. Obviously $\tau(E_k^0) < \tau$ since it does not contain any x_i. Thus there is a clopen neighborhood $U_{k-1}^1 \subset U_{k-1}^0$ of x_{k-1} such that for $E_{k-1}^0 := U_{k-1}^1 \setminus U_{k-1}^0$ we have $\tau(E_{k-1}^0) \geq \tau(E_k^0)$. We can also easily fulfil the cardinality of $(E_{k-1}^0)_{\tau(E_{k-1}^0)}$ to be at least the same as the cardinality of $(E_k^0)_{\tau(E_k^0)}$. In this way we construct clopen sets $E_k^0, E_{k-1}^0, \dots, E_1^0$.

Now, using the same argument, we can choose a clopen set E_k^1 satisfying $\tau(E_k^1) \geq \tau(E_1^0)$ and following in this direction we finally construct a sequence of clopen sets

$$E_k^0, \dots, E_1^0, E_k^1, \dots, E_1^1, E_k^2, \dots, E_1^2, \dots$$

such that $\tau(E_i^n)$ is nondecreasing and $(E_i^n)_{\tau(E_i^n)}$ is nondecreasing in cardinality. Moreover, we can choose these sets to satisfy $\lim_{n\to\infty} E_i^n = x_i$ for $i = 1, \dots, k$. For any n there always exists a continuous map from E_i^n onto E_{i+1}^n $(i = 1, \dots, k-1)$ and from E_k^{n+1} onto E_1^n (cf. [BS, Lemma 6]). These maps define the unique continuous map f on E being its restrictions. The space E has the property H (with respect to f) by the construction. □

2. Properties of basic sets

Let ω be an infinite ω-limit set. Put

$$P_\omega = \bigcap_U \overline{\operatorname{Orb} U},$$

where U is taken over all open connected subgraphs intersecting ω. If P_ω is a nowhere dense set then ω is a *solenoid*. If P_ω consists of finite number of connected components and ω contains a periodic point then ω is a *basic set*. If P_ω consists of finite number of connected components and ω contains no periodic points then we call ω a *singular set*. It is easy to see that other cases for infinite ω-limit sets are not possible. We say that f has a *horseshoe* if there are disjoint subgraphs U V such that $f(U) \cap f(V) \supset U \cup V$.

Lemma 3. *Let $\tilde\omega$ be a basic set for $f \in C(G)$. Every open subgraph U with $U \cap \tilde\omega \neq \emptyset$ contains a periodic point.*

Proof. By Theorem 2(b) in [B2] f has the specification property on $\tilde\omega$. Particularly this means that arbitrary large part of every trajectory from $\tilde\omega$ can be approximated by a periodic orbit with sufficiently large period. □

Lemma 4. *Let $\tilde\omega$ be a basic set for $f \in C(G)$, periodic points are dense in $\tilde\omega$.*

Proof. If $\tilde{\omega}$ is a union of periodic subgraphs then Lemma 3 gives the result. If $\tilde{\omega}$ is a nowhere dense set then by Theorem 2 in [B1] there is a subgraph $H \subset G$ containing $\tilde{\omega}$ such that if $\{J_i\}_{i=1}^{\infty}$ is an enumeration of graphs contiguous to $\tilde{\omega}$ in H then $n \in \mathbb{N}$ $f(J_n) \subset \overline{J_k}$ for some $k \in \mathbb{N}$. This shows that any J_n is either periodic or wandering set. Using this and Lemma 3 we finish the proof. □

Lemma 5. *Let $\tilde{\omega}$ be a basic set for $f \in C(G)$ such that P_ω is connected, let U be a connected subgraph such that $U \subset \operatorname{int} P_{\tilde{\omega}}$ and J be an open connected subgraph with $J \cap \tilde{\omega} \neq \emptyset$. Then there is an $n \in \mathbb{N}$ such that $U \subset f^n(J)$.*

Proof. By Lemma 4 there are periodic points $p, q \in J$ from the same edge and such that for the closed arc $[p, q] \subset J$ connecting p and q holds $[p, q] \cap \tilde{\omega}$ is infinite. (This is possible since $\tilde{\omega}$ is perfect cf. (i) of Theorem 2.) Let k be the common multiple of periods of p and q and take $H = \bigcup_{i=1}^{\infty} f^{ik}([p, q])$. Now, $\overline{H}$ is an invariant connected subgraph and such that $P_{\tilde{\omega}} \subset \overline{H}$ this implies that $f^n([p, q]) \supset U$, for all sufficiently large n. □

The above situation we describe saying that f is *strongly transitive on $P_{\tilde{\omega}}$*. We summarize some known as well as knew results in the next

Theorem 2. *Let $\tilde{\omega}$ be a basic set for $f \in C(G)$. Then*

(i) *$\tilde{\omega}$ is perfect;*
(ii) *system of all basic sets is countable;*
(iii) *periodic points are dense in $\tilde{\omega}$;*
(iv) *if P_ω is connected then f is strongly transitive on $P_{\tilde{\omega}}$;*
(v) *f^n has a horseshoe, for some $n \in \mathbb{N}$.*

Proof. Statements (i) and (ii) are proved in [B1]. Statement (iii) follows from Lemma 4. Statement (iv) follows from Lemma 5. Statement (v) follows from (iv). □

3. Distributional chaos

Let f be a continuous self map of a graph G. For any two point $x, y \in G$ any positive integer n and any real t define

$$\xi(x, y, n, t) = \sum_{i=0}^{n} \chi_{[0,t)}(\delta_{xy}(i)) = \#\{i \mid 0 \leq i < n \text{ and } \delta_{xy}(i) < t\},$$

where $\delta_{xy}(i) = \varrho(f^i(x), f^i(y))$. Now define the *upper distributional* and *lower distributional functions* of points x and y by formulas

$$F^{xy}(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x, y, n, t)$$

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, n, t)$$

respectively. We say that f is *distributionally chaotic* if there are two different points x and y from G such that $F^{xy}(t) = 1$ for all $t > 0$ and there is a point $s > 0$ of continuity of F^{xy} and F_{xy} such that $F^{xy}(s) > F_{xy}(s)$.

Lemma 6. *If $f \in C(G)$ has a horseshoe then it is distributionally chaotic.*

Proof. If f has a horseshoe then there is a set $M \subset G$ such that $f|_M$ is semiconjugated to the full shift σ on the space $\{0,1\}^{\mathbb{N}}$. But σ is distributionally chaotic. To see this, define a sequence $\{p_i \mid n \in \mathbb{N}\}$ of nonnegative integers as follows, put $p_1 = 1$ and $p_n = n\sum_{i=1}^{n-1} p_i$ for $n > 1$. Consider a sequence of blocks A_i of zeros and ones as follows. For every $n \in \mathbb{N}$ put $A_{2n} = 0^{p_{2n}}$ and $A_{2n+1} = 1^{p_{2n+1}}$, where 0^k means block of length k containing only symbol 0 analogously for 1^k. Take $u, v \in \{0,1\}^{\mathbb{N}}$ such that $u = 000\ldots$ and $v = A_1A_2A_3\ldots$. Now for every $t > 0$ we have

$$\frac{1}{p_{2k+1}}\xi(u,v,p_{2k+1},t) \geq \frac{1}{p_n}(p_n - p_{n-1}) = \frac{n-1}{n^2p_{n-1}}\left(\frac{n^2p_{n-1}}{n-1} - p_{n-1}\right) = 1 - \frac{n-1}{n^2}.$$

This shows $F^{uv}(t) = 1$ for every $t > 0$. Similarly for every $t < 1$ we have

$$\frac{1}{p_{2k}}\xi(u,v,p_{2k},t) \leq \frac{1}{p_{2k}}p_{n-1} = \frac{n-1}{n^2p_{n-1}}p_{n-1} = \frac{n-1}{n^2}.$$

This gives $F_{uv}(t) = 0$ for every $t < 1$. □

Lemma 7. *Let $f \in C(G)$. If $\omega_f(u)$ and $\omega_f(v)$ are solenoids then $F_{uv} = F^{uv}$ or $F^{uv}(t) \neq 1$ for some $t > 0$.*

Proof. By Theorem 1 in [B1] either $\omega_f(u)$ and $\omega_f(v)$ are disjoint or both are contained in the same solenoid. If $\omega_f(u) \cap \omega_f(v) = \emptyset$ then $\liminf_{i\to\infty} \delta_{uv}(i) = d > 0$ and consequently $F^{uv}(t) \neq 1$ for $t < d$. So suppose that $\omega_f(u), \omega_f(v) \subset \omega$, where ω is a solenoid. Consider two cases. Case 1. If there is a periodic sequence $J_1, \ldots, J_n$ of disjoint connected closed subgraphs covering ω and there is an $i \in \mathbb{N}$ such that $f^i(u) \in J_k \neq J_l \ni f^i(v)$ then again $\liminf_{i\to\infty} \delta_{uv}(i) = d > 0$ and consequently $F^{uv}(t) \neq 1$ for $t < d$. Case 2. Let for every periodic sequence $J_1, \ldots, J_n$ of disjoint connected closed subgraphs covering ω and all sufficiently large $i \in \mathbb{N}$ both $f^i(u)$ and $f^i(v)$ belongs to the same $J_{k(i)}$. Since there are no more than D/t sets from $\{J_i\}$ with diam $> t$ (where D is the length of G) $\xi(u,v,n,t) = \#\{i \mid 0 \leq i < n \text{ and } \delta_{uv}(i) < t\} \geq n - D/t$. From fact that solenoid has periodic covering with arbitrary large period we get $F_{uv}(t) = F^{uv}(t) = \lim_{n\to\infty} 1/n \cdot \xi(u,v,n,t) = 1$ for all $t > 0$. □

Lemma 8. *Let $f \in C(G)$. If f has no basic set then f is not distributionally chaotic.*

Proof. Take $x, y \in G$ two different points. Let $\omega_f(x)$ and $\omega_f(y)$ be their ω-limit sets. If these sets are of different kind (possible periodic orbit, singular set or solenoid) then $\liminf_{i\to\infty} \delta_{xy}(i) = d > 0$ and consequently $F^{xy}(t) \neq 1$ for $t < d$. Similar result we get when both $\omega_f(x)$ and $\omega_f(y)$ are different singular sets (whose distance must be positive). If $\omega_f(x) = \omega_f(y)$ is a singular set then by [B1] $f|_{\omega_f(x)}$ is conjugated to an irrational rotation. If both $\omega_f(x)$ and $\omega_f(y)$ are periodic obits then we conclude that either $F_{xy} = F^{xy}$ (if $\omega_f(x) = \omega_f(y)$ and points x and y are "synchronous") or $F^{xy}(t) \neq 0$ for some $t > 0$ elsewhere. In remaining case when both $\omega_f(x)$ and $\omega_f(y)$ are solenoids, apply Lemma 7. □

Lemma 9. *Let f be a continuous selfmap of a compact metric space X. If f^k is distributionally chaotic for some k then f is distributionally chaotic.*

Lemma was proved in [M2] in the circle case but the proof equally works for any compact metric space. For completeness we include the proof.

Proof. Let G_{uv}, G^{uv} be the lower and upper distribution functions for f^k respectively. Then $G_{uv}(t) < 1$ implies $F_{uv}(t) < 1$, where F_{uv} is lower distributional function for f. Let F^{uv} be the upper distributional function for f. Since f is continuous, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\varrho(u, v) < \delta$ implies $\delta_{uv}(i) < \varepsilon$, for $i = 0, 1, \dots, n-1$. Consequently, $G^{uv}(\delta) = 1$ implies $F^{uv}(\varepsilon) = 1$. □

The reverse implication (which is not needed here) also holds in the case of continuous graph maps. To see this, from the fact that f is distributionally chaotic we get by Theorem 3 that f^n has a horseshoe for some n. Then also f^{nk} has a horseshoe and again by Theorem 3 we get that f^{nk} is distributionally chaotic as well. Now just use Lemma 9 again.

Theorem 3. *Let $f \in C(G)$, then the following conditions are equivalent*

(i) $h(f) > 0$;
(ii) f^n *has a horseshoe, for some* $n \in \mathbb{N}$;
(iii) f *has a basic set;*
(iv) f *is distributionally chaotic.*

Proof. Equivalence of (i) and (ii) is proved in [LM]. Theorem 2(iv) yields that (iii) implies (ii). By Lemma 8, (iv) implies (iii). To prove the implication from (ii) to (iv), combine Lemmas 6 and 9. □

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