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A FIRST COURSE IN COMPLEX DYNAMICAL SYSTEMS

PETRA ŠINDELÁŘOVÁ

ABSTRACT. The topic of this paper is to give some introductory lectures to complex dynamical systems. It was presented at the 6^{th} Czech-Slovak Workshop on Discrete Dynamical Systems organized by the Mathematical Institute of Silesian University at Opava, 9-16 June 2002, Praded, Czech Republic. A more systematic exposition can be found in the survey articles [Dou], [Bl], and [EL]; and books [Bea], [CG], [Mc1] or Milnor's lecture notes [Mil].

1. Introduction and History

Between the founders of the field of complex dynamics are:

Ernst Schröder 1841–1902
Hermann A. Schwarz 1843–1921
Henri Poincaré 1854–1912
Gabriel Kœnigs 1858–1931
Léopold Leau 1868–1940 (?)
Lucjan E. Böttcher 1872–?
Samuel Lattès 1873–1918
Constantin Carathéodory 1873–1950
Paul Montel 1876–1975
Pierre Fatou 1878–1929
Paul Koebe 1882–1945
Arnaud Denjoy 1884–1974
Gaston Julia 1893–1978
Carl L. Siegel 1896–1981
Hubert Cremer 1897–1983
Herbert Grötzsch 1902–1993
Charles Morrey 1907–1984

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Lars Ahlfors 1907–1996
Lipman Bers 1914–1993

The local study of iterated holomorphic mappings in a neighborhood of a fixed point, was quite well developed in the late 19^{th} century. However, except for one very simple case studied by Schröder in 1871 and Cayley in 1879 (Newton's method for $f(z) = z^2 + 1$), nothing was known about the global behavior of iterated holomorphic maps until 1906, when Pierre Fatou described the following starting example. For the map $z \mapsto z^2/(z^2+2)$, he showed that almost every orbit under iteration converges to zero (even though there is a Cantor set of exceptional points for which the orbit remains bounded away from zero). This aroused great interest. The subject was taken up in depth by Fatou and also Gaston Julia and others. The most fundamental contributions were those of Fatou himself. However, Julia was a determined competitor and tended to get more credit because of his status as a wondered war hero. (In 1918 Julia was awarded the "Grand Prix des Sciences Mathématiques" by the Paris Academy of Sciences for his work.)

Complex dynamics flourished in the 20's under mathematicians such as Fatou and Julia. This field then slept until the late seventies. Then, mainly due to the computer graphics of Mandelbrot and the work of many present day workers in the field, attention was once again turned to the rich dynamical behavior of elementary maps of complex plane. Let me mention a few of them:

John W. Milnor (1931)
I. Noel Baker (1932)
Adrien Douady (1935)
Denis P. Sullivan (1941)
Michael R. Herman (1942)
John H. Hubbard (1945)
William P. Thurston (1946)
Mary Rees (1953)
Jean-Christophe Yoccoz (1955)
Curtis T. McMullen (1958)
Mikhail Y. Lyubich (1959)
Mitsuhiro Shishikura (1960)

2. The Julia and Fatou sets

A complex analytic map always decomposes the plane into two disjoint subsets, the *stable set* called the *Fatou* set, on which the dynamics are relatively tame, and the *Julia set*, on which the map is chaotic. To simplify, we will concentrate mainly on rational or polynomial maps of the complex plane. Rational maps on the Riemann sphere are complex analytic, form a finite-dimensional manifold, and some of the conjectures once proposed for smooth dynamical systems and now known to be false seem to hold in the class of rational maps.

Definition 2.1. A *rational map* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a holomorphic dynamical system on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Any such map can be written as a quotient $f(z) = \frac{P(z)}{Q(z)}$ of two relatively prime polynomials P and Q. The *degree* of f can be defined topologically or algebraically; it is the number of preimages of a typical point z, as well as the maximum of degrees of P and Q.

The fundamental problem in the dynamics of rational maps is to understand the behavior of hight iterates $f^n(z) = f \circ f^{n-1}$.

Definition 2.2. The familly $\{F_n\}$ of complex analytic functions defined on an open set U is a *normal family* if every infinite sequence of maps from $\{F_n\}$ has a subsequence which converges uniformly on compact subsets of U, or converges uniformly to ∞ on U.

Theorem 2.3. Montel. *For any complex manifold the set of all holomorphic maps into $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is a normal family.*

Example 2.4.

$$F(z) = az, \ |a| < 1 \quad F_n(z) = F^n(z)$$

$\{F_n\}$ forms a normal family of functions on any domain in $\mathbb{C}$ since F_n converges to the constant function 0 on compact subsets.

Example 2.5.

$$F(z) = az, \ |a| > 1$$

The same family is normal on any domain which does not include 0, but fails to be normal if the domain includes 0. Indeed, in any neighborhood of 0, there is a point z such that $|F^n(z)|$ is arbitrary large for some n. Any such neighborhood U satisfies $\bigcup_{n=1}^{\infty} F^n(U) = \mathbb{C}$.

Definition 2.6. Let $f : \hat{\mathbb{C}} \leftarrow \hat{\mathbb{C}}$ be a rational map of degree > 1. The *multiplier* of a point z of period p is the derivative $(f^p)'(z)$ of the first return map. We say z is

– *repelling* if $|(f^p)'(z)| > 1$,
– *indifferent* if $|(f^p)'(z)| = 1$,
– *attracting* if $|(f^p)'(z)| < 1$, and
– *superattracting* if $|(f^p)'(z)| = 0$.

An indifferent point is *parabolic* if $(f^p)'(z)$ is a root of unity (the multiplier at z is equal to 1).

Definition 2.7. The *Fatou set* $\Omega(f) \subset \hat{\mathbb{C}}$ is the largest open set such that the iterates $\{f^n|_\Omega : n \geq 1\}$ form a normal family. The *Julia set* $J(f)$ is the complement of the Fatou set.

The Julia and Fatou sets are each *totally invariant* under f (i.e. $f^{-1}(J(f)) = J(f)$ and $f^{-1}(\Omega(f)) = \Omega(f)$); so the partition $\hat{\mathbb{C}} = J(f) \sqcup \Omega(f)$ is preserved by the dynamics. This definition has been standard since the time of Fatou and Julia.

Thus, by its very definition, the Julia set J is a closed subset of $\hat{\mathbb{C}}$, while the Fatou set $\hat{\mathbb{C}} \setminus J$ is the complementary open set.

The above examples show that the Julia set of $F(z) = az\,(+b)$ are quite simple.

Theorem 2.8. *The Julia set is equal to the closure of the set of repelling periodic points. It is also characterized as the minimal closed subset of the Riemann sphere satisfying $|J| > 2$ and $f^{-1}(J) = J$.*

Theorem 2.9. Classification of Fatou components. *Every component U of the Fatou set is preperiodic ($f^i(U) = f^j(U)$ for some $i > j > 0$). The number of periodic components is finite. A periodic component U with $f^p(U) = U$ is exactly one of the following types:*

1. *An* attracting basin: *there is an attracting periodic point $w \in U$ and $f^{np}(z) \to w$ for all $z \in U$ as $n \to \infty$.*
2. *A* parabolic basin: *there is a parabolic periodic point $w \in \partial U$ and $f^{np}(z) \to w$ for all $z \in U$.*
3. *A* Siegel disk: *the component U is a disk on which f^p acts by an irrational rotation $z \mapsto e^{2\pi i\alpha}z$, $\alpha \notin \mathbb{Q}$.*
4. *A* Herman ring: *the component U is an annulus and again f^p acts as an irrational rotation.*

Proposition 2.10. *Let p be a polynomial of degree ≥ 2. Then*

(i) *$J(p) = J(p^n) \neq \emptyset$;*
(ii) *$J(p)$ is a perfect set;*
(iii) *for some $z_0 \in J(p)$, $J(p) = \overline{\bigcup_{k=0}^{\infty} p^{-k}(z_0)}$;*
(vi) *p is chaotic in the sense of Devaney on $J(p)$.*

The fact (iii) yields a good algorithm for plotting Julia sets graphically. One simply finds a repelling fixed point for p and computes its preimages.

Corollary 2.11. *$J(p)$ has empty interior.*

This is the only result for polynomials not true for rational maps.

Example 2.12. Let $Q_2(z) = z^2 - 2$ and $Q_0(z) = z^2$. Then $J(Q_2)$ is the closed interval $[-2, 2]$ and $J(Q_0)$ is the unit circle. Such smooth Julia sets are rather exceptional. More complicated Julia sets can be found at http://abel.math. harvard.edu/˜ctm/gallery/ or http://aleph0.clarku.edu/ ˜djoyce/julia/julia.html

Definition 2.13. The *postcritical set* $P(f)$ is the closure of the strict forward orbits of the critical points $C(f)$:

$$P(f) = \overline{\bigcup_{\mathbb{C}\in C(f), n>0} f^n(c)}$$

Note that $f(P) \subset P$ and $P(f^n) = P(f)$. The postcritical set is also the smallest closed set containing the critical values of f^n for every $n > 0$. A rational map is *critically finite* if $P(f)$ is a finite set.

Theorem 2.14. *The postcritical set $P(f)$ contains attracting cycles of f, the indifferent cycles which lie in the Julia set, and the boundary of every Siegel disk and Herman ring.*

Theorem 2.15. Characterization of hyperbolicity. *Let f be a rational map of degree > 1. Then the following conditions are equivalent:*

1. *The postcritical set $P(f)$ is disjoint from the Julia set $J(f)$.*
2. *There are no critical point or parabolic cycles in $J(f)$.*
3. *Every critical point of f tends to an attracting cycle under forward iteration.*
4. *There is a smooth conformal metric p defined on a neighborhood of the Julia set such that $\|f'(z)\|_p > C > 1$ for all $z \in J(f)$.*

Definition 2.16. The map f is *hyperbolic* if any of the equivalent conditions above is satisfied. A hyperbolic rational map is also sometimes said to be *expanding*, or to satisfy Smale's *Axiom A*.

Lemma 2.17. *Every attracting periodic orbit is contained in the Fatou set. Every parabolic periodic point belongs to the Julia set.*

Theorem 2.18. Transitivity Theorem. *Let z be an arbitrary point of the Julia set $J(f) \in \hat{\mathbb{C}}$ and let N be an arbitrary neighborhood of z. Then the union U of the forward images $f^n(N)$ contains the entire Julia set, and contains all but at most two points of $\hat{\mathbb{C}} \setminus U$.*

Corollary 2.19. *If the Julia set contains an interior point, then it must be equal to the entire Riemann sphere.*

Corollary 2.20. *If $\mathcal{A} \subset \hat{\mathbb{C}}$ is the basin of attraction for some attracting periodic orbit, then the topological boundary $\partial\mathcal{A} = \overline{\mathcal{A}} \setminus \mathcal{A}$ is equal to the entire Julia set. Every connected component of the Fatou set $\hat{\mathbb{C}} \setminus J$ either coincides with some connected component of this basin $\mathcal{A}$ or else is disjoint from $\mathcal{A}$.*

Corollary 2.21. *For any rational map of degree ≥ 2 the Julia set is either connected or has uncountably many connected components.*

3. Local fixed point theory

In this lecture we will study the dynamics of a holomorphic map in some small neighborhood of a fixed point. This local theory is a fundamental tool in understanding more global dynamics. It has been studied for well over a hundred years by mathematicians such as Schröder, Kœnigs, Leau, Böttcher, Fatou, Siegel, Voronin, Cherry, Herman, Yoccoz and Perez-Marco. In most cases it is now well understood, but a few cases still present extremely difficult problems.

We start by expressing our map in terms of a local uniformizing parameter z, which can be chosen so that the fixed point corresponds to $z = 0$.

We can then describe the map by a power series of the form $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$, which converges for $|z|$ sufficiently small. Recall that the initial coefficient $\lambda = f'(0)$ is called the *multiplier* of the fixed point.

3.1. Attracting points.

Definition 3.1. A fixed point p of a map f is *topologically attracting* if it has a neighborhood U so that the successive iterates f^n are all defined throughout U, and so that the sequence $\{f^n|_U\}$ converges uniformly to the constant map $U \to P$.

Lemma 3.2. Topological characterization of attracting points. *A fixed point for a holomorphic map is topologically attracting if and only if its multiplier satisfies $|\lambda| < 1$.*

In either case we will show that f can be reduced to a simple normal form by a suitable change of coordinates. We assume that the origin is not a critical point.

Theorem 3.3. Kœnigs linearization. *If the multiplier λ satisfies $|\lambda| \neq 0, 1$, then there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(0) = 0$, so that $\phi \circ f \circ \phi^{-1}$ is the linear map $w \mapsto \lambda w$ for all w in some neighborhood of the origin. Furthermore, ϕ is unique up to multiplication by a non-zero constant.*

In other words, the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \\ \phi\downarrow & & \downarrow\phi \\ \mathbb{C} & \xrightarrow{\lambda\cdot} & \mathbb{C} \end{array}$$

where ϕ is univalent (conformal, one-to-one) on the neighborhood U of zero.

The usefulness of the functional equation

$$\phi \circ f \circ \phi^{-1}(w) = \lambda u$$

had been pointed out some years earlier by E. Schröder. However, Schröder had been able to find solutions in very special cases.

Suppose that $f : S \to S$ is a holomorphic map of Riemann surfaces with an attracting fixed point $\hat{p} = f(\hat{p})$ of multiplier $\lambda \neq 0$. Recall that the total *basin of attraction* $\mathcal{A} = \mathcal{A}(\hat{p}) \subset S$ consists of all $P \in S$ for which $\lim_{n\to\infty} f^n(p)$ exists and is $\hat{p}$. The *immediate basin* $\mathcal{A}_0$ is the connected component of the Fatou set $S \setminus J$ which contains $\hat{p}$.

Now let us specialize to the case of the Riemann sphere. Suppose that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is rational degree ≥ 2. Let $\hat{z} \in \hat{\mathbb{C}}$ be an attracting (not superattracting) fixed point with basin of attraction $\mathcal{A} \subset \hat{\mathbb{C}}$. In some small neighborhood $\mathbb{D}_\xi$ of $0 \in \mathbb{C}$, note that the map $\phi : \mathcal{A} \to \mathbb{C}$ has a well defined holomorphic inverse $\psi_\xi : \mathbb{D}_\xi \to \mathcal{A}_0$ with $\psi_\xi(0) = \hat{z}$.

Lemma 3.4. Finding a critical point. *The local inverse $\psi_\xi : \mathbb{D}_\xi \to \mathcal{A}_0$ extends, by analytic continuation, to some maximal open disk $\mathbb{D}_r$ about the origin in $\mathbb{C}$. This yields a uniquely defined holomorphic map $\psi : \mathbb{D}_r \to \mathcal{A}_0$ with $\psi(0) = \hat{z}$ and $\phi(\psi(w)) = w$. Furthermore, ψ extends homeomorphically over the boundary circle $\partial\mathbb{D}_r$, and the image $\psi(\partial\mathbb{D}_r) \subset \mathcal{A}_0$ neccessarilly contains a critical point of f.*

If we denote by U the image $\varphi(\partial\mathbb{D}_r) \subset \mathcal{A}_0$, the previous lemma gives a commutative diagram of conformal isomorphisms. Note that the closure $\overline{U} \subset \hat{\mathbb{C}}$.

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \\ \phi \big\uparrow\big\downarrow \psi & & \psi \big\uparrow\big\downarrow \phi \\ \mathbb{D}_r & \xrightarrow{\lambda\cdot} & \lambda\mathbb{D}_r \end{array}$$

The following fundamental result is due to Fatou and Julia.

Theorem 3.5. Finding periodic attractors. *If f is a rational map of degree ≥ 2, then the immediate basin of every attracting periodic orbit contains at least one critical point. Hence the number of attracting periodic orbits is finite, less than or equal to the number of critical points.*

Note that this theorem gives a constructive algorithm for locating the attracting periodic points, if they exist, for any non-linear rational map. Starting at each one of the critical points, simply iterate the map many times and then test for (approximate) periodicity. (Of course if the period is very large, then this becomes impractical. As an example, it is easy to check that the quadratic map $f(z) = z^2 - 1.5$ has no attracting orbits of reasonable period. However, we know no way of deciding whether it has an attracting orbit of some very high period.)

Theorem 3.6. Topology of $\mathcal{A}_0$. *Let $\mathcal{A}_0$ be the immediate basin of attracting fixed point. Then $\hat{\mathbb{C}} \setminus \mathcal{A}_0$ is either connected or has uncountable many connected components.*

3.2. **Repelling points.**

Definition 3.7. A fixed point $\hat{p} = f(\hat{p})$ of a continuous map will be called *topologically repelling* if there is a neighborhood U of $\hat{p}$ such that for every $p \neq \hat{p}$ in U there exists some $n \geq 1$ such that the n-th forward image $f^n(p)$ lies outside of U. In other words, the only infinite orbit which is completely contained in U must be the orbit of the fixed point itself. Such U is called a *forward isolating* neighborhood of $\hat{p}$.

Lemma 3.8. Characterization of topologically repelling points. *A fixed point of a holomorphic map is topologically repelling if and only if its multiplier satisfies $|\lambda| > 1$.*

Lemmas 3.2 and 3.8 work only over the complex numbers. Over the real numbers, examples such as $f(x) = x \pm x^3$ show that a fixed point with multiplier $\lambda = 1$ may perfectly well be topologically attracting or topologically repelling.

The Kœnigs linearization Theorem 3.3, in the repelling case, helps us to uderstand why the Julia set is so often a comlicated "fractal" set.

Corollary 3.9. *Suppose that the rational function f has a repelling periodic point $\hat{z}$ for which the multiplier λ is not a real number. Then $J(f)$ cannot be a smooth manifold, unless it is all of $\hat{\mathbb{C}}$.*

Corollary 3.10. Global extension of ϕ^{-1}. *If $\hat{p}$ is a repelling fixed point for the holomorphic map $f : S \to S$, then there is a holomorphic map $\psi : \mathbb{C} \to S$ with $\psi(0) = \hat{p}$, so that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{f} & S \\ \psi \big\downarrow & & \big\downarrow \psi \\ \mathbb{C} & \xrightarrow{\lambda\cdot} & \mathbb{C} \end{array}$$

is commutative and so that ψ maps a neighborhood of zero biholomorphically onto a neighborhood of $\hat{p}$. Here ψ is unique except that it may be replaced by $w \mapsto \psi(cw)$ for any constant $c \neq 0$.

3.3. **Superattracting case.** $f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$, $n \geq 2$, $a_n \neq 0$

Theorem 3.11. Böttcher. *There exists a local holomorphic change of coordinate $w = \phi(z)$ with $\phi(0) = 0$, which conjugates f to the n-th power map $w \mapsto w^n$ throughout some neighborhood of zero. Furthermore, ϕ is unique up to multiplication by a constant.*

Thus, near any critical fixed point, f is conjugate to a map of the form $\phi \circ f \circ \phi^{-1} : w \mapsto w^n$, with $n \geq 2$. This theorem is often applied in the case of a fixed point at infinity. For example, any polynomial of degree $n \geq 2$ has a superattracting point at infinity.

Corollary 3.12. Extension of $|\phi|$. *If $f : S \to S$ has a superattracting fixed point $\hat{p}$ with basin $\mathcal{A}$, then the function $p \mapsto |\phi(p)|$ of Theorem 3.11 extends uniquely to a continuous map $|\phi| : \mathcal{A} \to [0, 1)$, which satisfies the identity $|\phi(f(p))| = |\phi(p)|^n$.*

Theorem 3.13. Critical points in the basin. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function with superattracting fixed point $\hat{p}$ and let $\mathcal{A}_0$ be the immediate attracting basin of $\hat{p}$. Then there are two possibilities:*

1. *The Böttcher map extends to a conformal isomorphism from $\mathcal{A}_0$ onto the open unit disk $\mathbb{D}$, which necessarily conjugates $f|_{\mathcal{A}_0}$ to the n-th power map $w \mapsto w^n$ on $\mathbb{D}$. In this case f evidently has no critical points other than $\hat{p}$ in $\mathcal{A}_0$.*
2. *Otherwise, there exists a maximal number $0 < r < 1$ such that the local inverse $\psi_\xi : \mathbb{D}_\xi \to \mathcal{A}_0$ extends to a conformal isomorphism ψ from the open disk $\mathbb{D}_r$ of radius r onto an open subset $U = \psi(\mathbb{D}_r) \subset \mathcal{A}_0$. In this case, the closure $\bar{U}$ is a compact subset of $\mathcal{A}_0$ and the boundary $\partial U \subset \mathcal{A}_0$ contains at least one critical point of f.*

4. Open problems and Renormalization

4.1. **Hyperbolic rational maps.** We first state one of the central problems in the field.

Conjecture. HD. *Hyperbolic maps are open and dense among all rational maps.*

It is easy to see that hyperbolicity is an open condition, but the density of hyperbolic dynamics has so far eluded proof.

Definition 4.1. A pair of rational maps f and g are *topologically conjugate* if there is a homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\phi f \phi^{-1} = g$. A rational map f is *structurally stable* if f is topologically conjugate to all g in a neighborhood of f.

The following close relative of Conjecture HD is true. For details see [MSS], [McS].

Theorem 4.2. Mañé, Sad, Sullivan. *The set of structurally stable rational maps is open and dense.*

Given the density of structural stability, to settle HD it suffices to prove that a structurally stable rational map is a hyperbolic.

4.2. Quadratic polynomials $f_c(z) = z^2 + c,\ c \in \mathbb{C}$.

Conjecture. HD 2. *Hyperbolic maps are dense among quadratic polynomials.*

Note that f_c has only one critical point $z = 0$. Consequently,

Theorem 4.3. *The map $f_c(z) = z^2 + c$ is hyperbolic if and only if $f_c^n(0) \to \infty$ or f_c has an attracting periodic cycle in the finite plane $\mathbb{C}$.*

This theorem motivates the following definition:

Definition 4.4. The *Mandelbrot set* $M \subset \mathbb{C}$ is the set of c such that $f_c^n(0)$ stays bounded as $n \to \infty$.

The Mandelbrot set $M \subset \mathbb{C}$ is compact, connected and full (i.e., $\mathbb{C} \setminus M$ is also connected). The interior of M consists of countably many components.

Conjecture. HD 2'. *If c lies in the interior of the Mandelbrot set, then $f_c(z)$ has an attracting cycle.*

Conjecture. MLC. *The Mandelbrot set is locally connected.*

Known is:

$$\text{HD 2} \Longleftrightarrow \text{HD 2'}$$

$$\text{MLC} \Longrightarrow \text{HD 2} \qquad \text{(Douady–Hubbard)}$$

4.3. **Renormalization.** We next present some breakthroughs in the direction of the conjectures above. For this, we will need the notion of renormalization.

The local behavior of a rational map can sometimes be given by a linear model (e.g., near an attracting or a repelling fixed point p with $f'(p) = \lambda$, one can choose a complex coordinate z so that the dynamics take the form $f : z \mapsto \lambda z$; see the previous section).

Renormalization looks for a local model of the dynamics which is a polynomial of degree > 1.

Definition 4.5. Let $f(z) = z^2 + c$ with c in the Mandelbrot set. An iterate f^n is *renormalizable* if there exist disks U and V containing the origin, with $\bar{U}$ a compact subset of V such that

(a) $f^n : U \to V$ is a proper map of degree 2, and
(b) $f^{nk}(0) \in U$ for all $k > 0$.

This means that although f^n is a polynomial of degree 2^n, it behaves like a polynomial of degree two on a suitable neighborhood of the critical point $z = 0$. The restriction $f^n : U \to V$ is called a *quadratic-like map.* A fundamental theorem of Douady and Hubbard asserts that any quadratic-like map is topologically conjugate to a quadratic polynomial $g(z) = z^2 + c'$; condition (b) implies c' lies in the Mandelbrot set and, with a suitable normalization, is unique [DH].

A quadratic polynomial f is *infinitely renormalizable* if f^n is renormalizable for infinitely many $n > 1$.

Example 4.6. Feigenbaum polynomial $f(z) = z^2 - 1{,}401155\ldots$ A suitable restriction of f^2 is a quadratic-like map topologically conjugate to f. Thus, f^{2n} is renormalizable for every $n \geq 1$. Its attractor A_c is a Cantor set.

Theorem 4.7. Yoccoz. *If c belongs to the Mandelbrot set, then either $f_c(z) = z^2 + c$ is infinitely renormalizable or $J(f_c)$ admits no invariant line field and M locally connected at c.*

Definition 4.8. A *line field* on a subset E of a Riemann surface X is the choice of a real line through the origin in the tangent space $T_\xi X$ at each point of E. A line field is *invariant* if $f^{-1}(E) = E$ and if f' transforms the line at z to the line $f(z)$ (for a polynomial f).

Conjecture. NILF2. *A quadratic polynomial admits no invariant line field on its Julia set.*

We know:

$$\text{HD } 2 \Longleftrightarrow \text{NILF } 2$$

Sketch of the proof of Yoccoz theorem: The main case of the proof arises when all periodic cycles of f_c are repelling, let us assume this. The first step is to try to show that the Julia set $J(f_c)$ is locally connected. To this end, Yoccoz construct a sequence $\{\mathcal{P}_d\}$ of successively finer tilings of neighborhoods of $J(f_c)$ called *Puzzle pieces* of level d (connected neighborhoods of points in the Julia set). The pieces at level $d + 1$ are defined inductively as the components of the preimages of the pieces $\{\mathcal{P}_d\}$ at level d. The new pieces fit neatly inside those already defined. The image of a puzzle piece under f_c is again a puzzle piece. See e.g. [Hub].

To conclude, we mention two results in complex dynamics which are connected to the present discussion.

Lyubich proved the local connectivity of the Mandelbrot set at a large class of infinitely renormalizable points [Lyu]. Thus it seems that Conjecture

MLC itself is not far out of reach. It will complete the topological picture of the space of complex quadratic polynomials.

Shishikura solved that the boundary of the Mandelbrot set ∂M has Hausdorff dimension two [Shi].

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E-mail address: Petra.Sindelarova@math.slu.cz

Current address: Mathematical Institute, Silesian University in Opava, 746 01 Opava, Czech Republic