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ON A PROBLEM CONCERNING ω-LIMIT SETS OF TRIANGULAR MAPS IN I^3

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ABSTRACT. We show that there is a continuous triangular map $I^3 \to I^3$ with $\omega(F) = \{0\} \times I^2 = \omega_F(x, y, z)$ for any $(x, y, z) \in I^3$ such that $x \neq 0$. This map is of the form $F(x, y, z) = (f(x), g(x, y), h(x, z))$, where the maps $g(x, .)$ and $h(x, .)$ are non-decreasing. This solves a problem by F. Balibrea, L. Reich, and J. Smítal.

1. Main result

In the sequel, $I = [0, 1]$ is the unit compact interval and I^n the n-dimensional cube. For a compact metric space X, $C(X, X)$ is the set of continuous maps of X into itself. For $\varphi \in C(I, I)$, let φ^n denote the n-th iterate of φ. The set of accumulation points of the sequence $\{\varphi^n(x)\}_{n=0}^{\infty}$ is the *ω-limit set of x with respect to φ*, and is denoted by $\omega_\varphi(x)$. By $\omega(\varphi)$ we denote the set of ω-limit points.

A map $F \in C(I^3, I^3)$ such that $F(x, y, z) = (f(x), g(x, y), h(x, y, z))$ is a *triangular map*, f is the *base* of F, and the set $I_x := \{x\} \times I^2$ is the *layer over x*.

Theorem. *There is a triangular map $F \in C(I^3, I^3)$ with $\omega(F) = \{0\} \times I^2 = \omega_F(x, y, z)$, for any $(x, y, z) \in I^3$ such that $x \neq 0$. This map has a special form $F(x, y, z) = (f(x), g_x(y), h_x(z))$.*

Remark. The above theorem is true in more general settings: For any positive integer n there is a triangular map $F \in C(I^{n+1}, I^{n+1})$ of the form $F(x, x_1, \dots, x_n) = (f(x), g_1(x, x_1), g_2(x, x_1, x_2), \dots, g_n(x, x_1, x_n))$, which has as an ω-limit set the n-dimensional cube $\{0\} \times I^n$. The proof is similar and we omit it.

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2. Proof of Theorem

For the base of F take $f(x) = \lambda x$, where $\lambda \in (0,1)$ is fixed. Let $g_0 = h_0$ be the identity. Let $\{m_k\}_{k=0}^{\infty}$ be an increasing sequence with $m_0 = 0$ and such that $m_{k+1} - m_k \geq k+2$, for any k. For each $k \geq 0$ let φ_k be a nondecreasing continuous map of I such that

$$||\varphi_k^{j+1} - \varphi_k^j|| \leq \frac{1}{k+1}, \; m_k \leq j < m_{k+1} \tag{1}$$

and, for any $y \in I$,

$$\varphi_k(y) \leq y \;\text{ and }\; \varphi_k^{m_{k+1}-m_k-1}(y) = \{0\}, \;\text{ if }\; k \;\text{ is odd,} \tag{2}$$

and

$$\varphi_k(y) \geq y \;\text{ and }\; \varphi_k^{m_{k+1}-m_k-1}(y) = \{1\}, \;\text{ if }\; k \;\text{ is even.} \tag{3}$$

Let

$$g_x(y) = \varphi_k(y) \;\text{ if }\; x \in [\lambda^{m_{k+1}-1}, \lambda^{m_k}] \text{ and } y \in I, \tag{4}$$

and for $x \in [\lambda^{m_{k+1}}, \lambda^{m_{k+1}-1}]$ let g_x be the convex combination of the maps $g_{\lambda^{m_{k+1}}}$ and $g_{\lambda^{m_{k+1}-1}}$, i.e.,

$$g_x = t g_{\lambda^{m_{k+1}}} + (1-t) g_{\lambda^{m_{k+1}-1}} \text{ if } x = t\lambda^{m_{k+1}} + (1-t)\lambda^{m_{k+1}-1}. \tag{5}$$

Denote by $G(x,y)$ the two-dimensional triangular map $(f(x), g_x(y))$. Then $\omega_G((x,y)) = \{0\} \times I$ whenever $x > 0$. This follows since, by the contractivity of f, the set of ω-limit points of G is contained in $\{0\} \times I$ and since, by (1) – (5), the second coordinates of the trajectory of any point (x,y) with $x > 0$ form a dense subset of I. The above example is a modification of an example from [2].

Now we have to extend G to a three-dimensional triangular map F. The map h_x will be defined similarly as g_x, but with the sequence $\{m_k\}$ replaced by a subsequence $\{n_k\}$, and $\{\varphi_k\}$ replaced by a sequence $\{\psi_k\}$ satisfying conditions similar to (4) and (5), but with (1) – (3) replaced by the next three ones. For any k and any s with $n_k \leq m_s < n_{k+1}$,

$$||\psi_k^j - \psi_k^i|| \leq \frac{1}{k+1} \;\text{ if }\; m_s \leq i \leq j < m_{s+1} \tag{6}$$

and, for any $y \in I$,

$$\psi_k(y) \leq y \;\text{ and }\; \psi_k^{n_{k+1}-n_k-1}(y) = \{0\}, \;\text{ if }\; k \;\text{ is odd,} \tag{7}$$

and

$$\psi_k(y) \geq y \;\text{ and }\; \psi_k^{n_{k+1}-n_k-1}(y) = \{1\}, \;\text{ if }\; k \;\text{ is even.} \tag{8}$$

Conditions (1) – (3) and (6) – (8) are consistent if the sequence $\{n_k\}$ increases faster than $\{m_k\}$ such that, for any k,

$$m_{k+1} - m_k \geq k+2, \tag{9}$$

and

$$\#\{s;\; n_k \leq m_s < n_{k+1}\} \geq k+1. \tag{10}$$

These two conditions can be satisfied if

$$n_{k+1} - n_k \geq (k+1)(k+2), \tag{11}$$

or, in particular, if $n_0 = m_0 = 0$ and $n_k = (k+3)!$ for $k > 0$. Then it is possible to choose the numbers m_k satisfying (9) such that $\#\{s;\ n_k \leq m_s < n_{k+1}\} = k+1$, or equivalently, $n_k = m_{k(k+1)/2}$. Thus, we may assume

$$n_0 = m_0 = 0, \quad \text{and} \quad n_k = m_{k(k+1)/2} = (k+3)! \ \text{ for } k > 0. \tag{12}$$

Now we have to specify φ_k and ψ_k. First, for any nonnegative integer k, define auxillary maps ν_k, μ_k by $\mu_k(y) = y + \frac{1}{k}$ if $y \leq 1 - \frac{1}{k}$, and $\mu_k(y) = 1$ otherwise. Similarly, $\nu_k(y) = y - \frac{1}{k}$ for $y \geq \frac{1}{k}$, and $\nu_k(y) = 0$ otherwise. Obviously,

$$\mu_k^k(I) = \{1\}, \text{ and } \nu_k^k(I) = \{0\}. \tag{13}$$

Now we can let

$$\varphi_k = \mu_{m_{k+1}-m_k-1}, \text{ if } k \text{ is even},$$

$$\varphi_k = \nu_{m_{k+1}-m_k-1}, \text{ if } k \text{ is odd}.$$

Similarly, for s and k such that $n_k \leq m_s < n_{k+1}$ let

$$\psi_s = \mu_{(m_{s+1}-m_s-1)(k+1)}, \text{ if } s \text{ is even},$$

$$\psi_s = \mu_{(m_{s+1}-m_s-1)(k+1)}, \text{ if } s \text{ is odd}.$$

This choice satisfies the conditions (1) – (3) and (6) – (8). Indeed, (1) and (4) follow by the definition of φ_k and ψ_k, the remaining conditions by (12).

To prove that $\omega(F) = \{0\} \times I^2 = \omega_F(x,y,z)$ whenever $x \neq 0$, it suffices to show the following. For any integers p, q, r such that $0 < p, q < r$ and for any $\alpha = (x,y,z) \in I^3$ with $x \neq 0$,

$$F^l(\alpha) \in I \times \left[\frac{p}{r}, \frac{p+1}{r}\right] \times \left[\frac{q}{r}, \frac{q+1}{r}\right] = K, \text{ for some } l. \tag{14}$$

To do this note that if k is an integer such that, for some $j > 0$, $f^j(x) = \lambda^j x \in (\lambda^{n_k+1}, \lambda^{n_k}]$ then

$$\lambda^{j+n_{k+1}-n_k} x \in (\lambda^{n_{k+1}+1}, \lambda^{n_{k+1}}] \tag{15}$$

and, by (2), (3), (7), and (8),

$$F^{j+n_{k+1}-n_k}(\alpha) = (\lambda^{j+n_{k+1}-n_k} x, 0, 0) \ \text{ if } \ k, t(k) \ \text{ are odd}, \tag{16}$$

where $t(k)$ stands for $\frac{1}{2}(k+1)(k+2) - 1$, cf. (12). Thus, (16) is satisfied if k is a multiple of 4.

Now the argument follows from the following two easily observable facts:

(i) Any point $\alpha = (x, 0, 0)$, where $x \in (\lambda^{n_k+1}, \lambda^{n_k}]$ and $k+1 \geq 2r$ satisfies (14). Indeed, since $\frac{1}{k+1} \leq \frac{1}{2r}$, by (6) – (8) there is an s such that $n_k \leq m_s < n_{k+1}$, and

$$F^{m_s - n_k}(\alpha) \in (\lambda^{m_s+1}, \lambda^{m_s}] \times \{0, 1\} \times \left[\frac{q}{r}, \frac{q+\frac{1}{2}}{r}\right].$$

By (1) – (3) there is an $l \in [m_s, m_{s+1} - 1]$ such that $\varphi_s^{l-m_s}(0) \in [p/r, (p+1)/r]$. By (4) and (5), l satisfies (13).

(ii) By (15) and (16), any point $\alpha = (x, y, z), x \neq 0$ can be mapped by an iterate of F to a point satisfying (i). □

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