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TWO KINDS OF CHAOS AND RELATIONS BETWEEN THEM

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Abstract

In this paper we consider relations between chaos in the sense of Li and Yorke, and ω-chaos. The main aim is to show how important is the size of scrambled sets in definitions of chaos. We provide an example of an ω-chaotic map on a compact metric space which is chaotic in the sense of Li and Yorke, but any scrambled set contains only two points. Chaos in the sense of Li and Yorke cannot be excluded: We show that any continuous map of a compact metric space which is ω-chaotic, must be chaotic in the sense of Li and Yorke. Since it is known that, for continuous maps of the interval, Li and Yorke chaos does not imply ω-chaos, Li and Yorke chaos on compact metric spaces appears to be weaker. We also consider, among others, the relations of the two notions of chaos on countably infinite compact spaces.

The paper will be presented at the conference SVOČ 2002. The author's recent work "Scrambled sets for transitive maps", which was presented at SVOČ 2001, and which will appear in Real Analysis Exchange, also involves ω-chaos, but the results neither are related to, nor are used in the present paper.

Key words: ω-chaos, Li and Yorke chaos, scrambled sets.

1 Introduction

In this paper we study two different (but similar) definitions of chaos and relations between them.

Chaos in the sense of Li and Yorke, briefly LYC, was introduced in 1975 by T. Y. Li and J. A. Yorke [10]: A continuous map $f : I \to I$, where I is the unit interval, is LYC if there is an uncountable set $S \subset I$ such that trajectories of any two distinct points x, y in S are *proximal* and *not asymptotic*, i.e.,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

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The original definition contains an another condition which later appeared to be superfluous [10]. The condition on S in this definition (i.e., for continuous maps of the interval, but not in a general compact metric space) is equivalent to the condition that S contains two points [7], or that S is a perfect set (i.e., nonempty, compact and without isolated points) [12].

The second type of chaos is an ω-chaos, briefly ωC, introduced in 1993 by S. Li [9]: A continuous map $f : I \to I$ is ωC if there is an uncountable set S such that, for any distinct x and y in S,

$$\omega_f(x) \backslash \omega_f(y) \text{ is uncountable, } \omega_f(x) \cap \omega_f(y) \neq \emptyset, \text{ and } \omega_f(x) \setminus Per(f) \neq \emptyset.$$

If $f : I \to I$ is continuous, then ωC is equivalent to PTE (positive topological entropy) [9], and by [12] PTE implies LYC but not conversely. Moreover, f is ωC if and only if it has an ω-scrambled set containing two points [9], and this is if and only if it has a perfect ω-scrambled set [13]. However, in the general case, when X is a compact metric space, the size of S is essential.

By a *compactum* we mean an infinite compact metric space X (countable or uncountable), with a metric d, and all maps considered in this paper are continuous. The space of all continuous maps of X is denoted by $C(X, X)$. The set of ω-limit points of $x \in X$ under $f \in C(X, X)$ is denoted by $\omega_f(x)$. The set of strictly increasing sequences of positive integers is denoted by $\mathcal{A}$.

Definition 1 Let $f \in C(X, X)$, and let $S \subset X$ contain at least two points. We say that f is *chaotic in the sense of Li and Yorke* (briefly, f is LYC), and that if S is a *scrambled set* for f if, for any distinct $x, y \in S$,

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$,

(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

Stronger notions of Li and Yorke chaos are these with *infinite*, or with an *uncountable* scrambled set. To distinguish between these three types of chaos we use notation LY_2C, or $LY_\infty C$, or LY_uC, respectively. Also we say that f is *completely LYC* if $S = X$.

Now we introduce several modifications of the notion of ω-chaos.

Definition 2 Let $f \in C(X, X)$, and let $S \subset X$ contain at least two points. We say that f is *ω^u-chaotic* (briefly, f is ω^uC), and S is an *ω^u-scrambled set* for f if, for any distinct $x, y \in S$,

(1) $\omega_f(x) \backslash \omega_f(y)$ is uncountable,

(2) $\omega_f(x) \cap \omega_f(y)$ is nonempty,

(3) $\omega_f(x)$ is not contained in the set of periodic points.

In particular, f is *ω_2^u-chaotic*, or *ω_∞^u-chaotic*, or *ω_u^u-chaotic* (briefly, ω_2^uC, or ω_∞^uC, or ω_u^uC, respectively), if there is an ω^u-scrambled set containing two, or infinitely many, or uncountable many points, respectively).

The next definition modifies the notion of ω-chaos for countable compact spaces.

Definition 3 Let $f \in C(X, X)$, and let and $S \subset X$ contain at least two points. We say that f is ω^∞-*chaotic* (briefly, f is ω^∞C), and that S is an ω^∞ -*scrambled set* for f if, for any distinct $x, y \in S$,
(1) $\omega_f(x) \backslash \omega_f(y)$ is infinite,
(2) $\omega_f(x) \cap \omega_f(y)$ is nonempty,
(3) $\omega_f(x)$ is not contained in the set of periodic points.
In particular, the map f is ω_2^∞-*chaotic*, or ω_∞^∞-*chaotic* (briefly, $\omega_2^\infty C$, or $\omega_\infty^\infty C$, respectively), if f has an ω^∞-scrambled set possessing two, or infinitely many points, respectively.

Remark 1 It is obvious, that $\mathrm{LY}_u\mathrm{C} \Rightarrow \mathrm{LY}_\infty\mathrm{C} \Rightarrow \mathrm{LY}_2\mathrm{C}$; the converse implications are true for continuous maps on the interval [12] but, on general compact metric spaces they are no more valid [5], [4]. Also $\omega_u^u\mathrm{C} \Rightarrow \omega_\infty^u\mathrm{C} \Rightarrow \omega_2^u\mathrm{C} \Rightarrow \omega_2^\infty\mathrm{C}$ and $\omega_\infty^\infty\mathrm{C} \Rightarrow \omega_2^\infty\mathrm{C}$, and again it is possible to show that the converse implications are not true in the general case. Moreover, by [12] there is a $\mathrm{LY}_u\mathrm{C}$ map of the interval with zero topological entropy. This map has a unique infinite ω-limit set and consequently, by [9], it cannot be $\omega_2^\infty\mathrm{C}$. Thus, in the general case, no form of Li and Yorke chaos implies the weakest form of ω-chaos.

Thus it remains to answer the question: which forms of ω-chaos imply Li and Yorke chaos? In the present paper we show that any form of ωC implies $\mathrm{LY}_2\mathrm{C}$, cf. the next Theorem 1. But the implied LYC may be very small. In fact, we show that ωC map on a compact metric space may have only two-point LYC scrambled sets – cf. Theorem 5. On the other hand, we show that even completely LYC homeomorhisms may not be ωC (Theorems 2 and 4; compare with Theorem 3).

Theorem 1 *Let X be a compact metric space, and f a continuous map of X which is $\omega_2^\infty C$. Then f is LY_2C. In general, any point in an ω^∞-scrambled set of f forms a LYC pair with a suitable point in X.*

Proof. Let u, v be points in X forming an ωC scrambled set for f. By a results of Auslander [1] and Ellis [3], in a dynamical system on a compact metric space any point is proximal to a uniformly recurrent point in its orbit closure. Let x be such a uniformly recurrent point proximal to u. Then x belongs to a minimal set $M = \omega_f(x) \subset \omega_f(u)$. But M must be a proper subset of $\omega_f(u)$. For if $M = \omega_f(u)$ then $\omega_f(u) \cap \omega_f(v) \neq \emptyset$ and $\omega_f(v) \setminus \omega_f(u) \neq \emptyset$ would imply $\omega_f(u) \subset \omega_f(v)$ and consequently, $\omega_f(u) \setminus \omega_f(v) = \emptyset$ – a contradiction. Thus u and x are proximal points, which cannot be asymptotic since $\omega_f(x) \neq \omega_f(u)$.

□

2 Examples on countably infinite spaces

For a set $A \subset X$, and for any nonnegative integer n, define the n-th derivative A^n of A by $A^0 = A$, and A^{n+1} is the set of cluster points of A^n. Denote $X_0 = X \setminus X^1$, and $X_j = X^j \setminus X^{j+1}$ for each $j = 1, 2, \ldots$.

Proposition 1 ([5], Proposition 2.2.) *Let f be a completely LYC homeomorphism of a compactum X. Then, f has a unique fixed point.*

Remark 2 In [5] there is given a construction of a countably infinite compactum X' and completely LY$^\infty$C homeomorphism φ on X', with fixed point p.

The set X' is contained in the plane $\mathbb{R}^2$, $X' = \bigcup_{j=0}^{\infty} X_j \cup \{p\}$, and $\omega_f(x) = X^{(j+1)}$, for each $j = 0, 1, 2, \ldots$ and each $x \in X_j$ (cf. [5], Theorem 3.1). Note that $p \in \omega_f(x)$ and the compactum X' can be taken with arbitrarily small diameter.

Theorem 2 *There is a countable compactum X and completely LYC homeomorphism $f : X \to X$ such that f is not $\omega_2^\infty C$.*

Proof. Put $X = X'$, and $f = \varphi$ (see Remark 2). From the form of $\omega_f(x)$, $x \in X$, it is easy to see that, for each $x, y \in X$, $\omega_f(x) \subset \omega_f(y)$ or $\omega_f(y) \subset \omega_f(x)$. So, the map f cannot be ω_2^∞C. □

Proposition 2 ([5], Proposition 2.5.) *Let f be a completely LYC homeomorphism of a compactum X. Then for each $x \neq y \in X$, there is $\{n_i\} \in \mathcal{A}$ such that $f^{n_i}(x) \to p$ and $f^{n_i}(y) \to p$ where p is the unique fixed point of f (cf. Proposition 1).*

Theorem 3 *There is a countable compactum X and a completely LYC homeomorphism $\varphi : X \to X$ such that φ is $\omega_\infty^\infty C$.*

Proof. First, define a sequence of isosceles triangles Y_i. Let us denote the triangle Y_i by $A_i B_i C_i$ in the clockwise sense and in a such a way that C_i is the apex between the two sides of the same length.

Let the length of the side $C_i A_i$ is $1/(2^i)$ and the angle $A_i C_i B_i$ is $\pi/(2^{2i+1})$ for each $i = 0, 1, 2, \ldots$. Now, let us "glue" these triangles in such a way, that they have common apex C_i, $i = 0, 1, 2, \ldots$, and angle between $C_i B_i$ and $C_{i+1} A_{i+1}$ is $\pi/(2^{2(i+1)})$, $i = 0, 1, 2, \ldots$.

Finally, plug diminished copy X_i of X' into each Y_i, $i = 0, 1, 2, \ldots$, in such a way that each apex C_i is the fixed point p for the completely LY$^\infty$C homeomorphism $f_i : X_i \to X_i$ (it is possible by Remark 2). Then $X = \bigcup_{i=0}^{\infty} X_i$ with Euclid metric is countable compactum and a map $f : X \to X$ defined by $f|_{X_i} = f_i$, $i = 0, 1, 2, \ldots$ is completely LYC homeomorphism (with fixed point p).

Realy, for each $x \in X_i$ and $y \in X_j$, where $i \neq j$, $x \neq p \neq y$, we have $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$ and $\limsup_{n\to\infty} d(f^n(y), f^n(p)) > 0$ so, by triangular inequality, we obtain $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$. By Proposition 2, $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

Let $S = \bigcup_{i=0}^{\infty}\{x_i\}$, where x_i is an arbitrary point from $X_i \setminus \{p\}$. Then S is countably infinite, $\omega_f(y_i) \backslash \omega_f(y_j)$ is countably infinite and $\omega_f(y_i) \cap \omega_f(y_j) = \{p\}$ for each $x_i \neq x_j$ (see Remark 2). Finally, for each $x \in S$, $\omega_f(x)$ is not contained in the set of periodic points of f (note that it is singleton $\{p\}$), and hence the map f is ω_{∞}^{∞}C. $\square$

3 Examples on uncountable spaces

By a *Cantor set* we mean a compactum which is homeomorphic to the Cantor middle third set. Let C be a Cantor set. Collapsing $\{p\} \times C$ in $X' \times C$ we get a compactum (with induced Euclid metric) denoted by $X'C$.

Theorem 4 *There is a perfect compact set $X \subset \mathbb{R}^3$ possessing a completely LYC homeomorphism $\varphi : X \to X$, such that φ is not ω_2^uC.*

Proof. We can imagine the space $X'C$ as a union of slices S_i with one common point p. (So each S_i is countably infinite compactum, and $X'C$ is perfect since each point of $X'C$ is accumulation one.) There are $f_i : S_i \to S_i$ with the fixed point p. Define $F : X'C \to X'C$ by $F|_{S_i} = f_i$, for each i. It is easy to see, that F is a homeomorphism with the fixed point p.

It is clear that $\liminf_{n\to\infty} d(F^n(x), F^n(y)) = 0$ for each $x \neq y \in XC$ (by Proposition 2). Since, for any $i \neq j$ and any neighborhood U of p, the distance between $S_i \setminus U$ and $S_j \setminus U$ is positive, $\limsup_{n\to\infty} d(F^n(x), F^n(y)) > 0$ for each $x \neq y \in X'C$. Consequently, the map F is completely LYC.

On the other hand, the map F is not ω_2^uC, since, for each $x \in S$, $\omega_F(x)$ is countable. $\square$

To conclude this section we provide an example of a map which is not LY$_2$C but has a two point ω^u-scrambled set. The construction is based on symbolic dynamics. The standard notions and basic known results can be found, e.g., in [6].

Let Σ_2 denote the set of sequences $x = x_1x_2x_3\ldots$ where $x_n = 0$ or 1 for each n, equipped with the metric of pointwise convergence. Thus, for $y = y_1y_2y_3\ldots$, put $\rho(x, y) = 1/k$ if $x \neq y$, and $k = \min\{n = 1, 2, \ldots : x_n \neq y_n\}$, and let $\rho(x, y) = 0$ for $x = y$. Then Σ_2 is a compactum and the "shift" $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(x_1x_2x_3\ldots) = x_2x_3\ldots$ is continuous.

Recall that a subset M of X is *minimal* for a map f, if it is closed, invariant and no proper subset of M has the same property (or equivalently, M is minimal for a map f, if and only if $\omega_f(x) = M$, for each $x \in M$). A sequence $x =$

$x_1x_2x_3\ldots \in \Sigma_2$ is called *uniformly recurrent* if for each block $x_1x_2\ldots x_l$ there is k, such that for each i at least one of the sequences $\sigma^i(x), \sigma^{i+1}(x)\ldots\sigma^{i+k}(x)$ starts with the block $x_1x_2\ldots x_l$.

For the construction of our example we use the following special uniformly recurrent sequences which, among others, have all blocks periodic.

Denote by $\mathbb{N}_0$ the set of nonnegative integers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathcal{N} = \{N_n = 2^{n-1}(1+2\mathbb{N}_0), n = 1, 2, \ldots\}$. It is easy to verify that $\mathcal{N}$ is a decomposition of $\mathbb{N}$. Define a map $\Phi : \Sigma_2 \to \Sigma_2$ so that, for $x = x_1x_2x_3\ldots \in \Sigma_2$, $\Phi(x) = \tilde{x} = \tilde{x}_1\tilde{x}_2\tilde{x}_3\ldots$, where $\tilde{x}_k = x_s$ if $k \in N_s$, i.e. $\Phi(x) = x_1x_2x_1x_3x_1x_2x_1x_4x_1x_2x_1\ldots$. Then $\Phi(x)$ is not only uniformly recurrent but the blocks in $\Phi(x)$ are even periodic. This follows from the next lemma whose proof is obvious.

Lemma 1 *Let $B = \tilde{x}_i\tilde{x}_{i+1}\ldots\tilde{x}_j$ be a block of $\Phi(x)$, and let n be the maximal positive integer such that $i \leq 2^n \leq j$. Then the block B is periodic in the sequence $\Phi(x)$, with period 2^{n+1}.*

Let $\{r_i\}_{i=1}^{\infty} \in \Sigma_2$ be a sequence containing infinitely many zeros and infinitely many ones. Put $a = 11r_3r_4\ldots$, $b = 011r_4r_5\ldots$ and $c = 00r_3r_4\ldots$. Thus, keeping our notation, we have $\Phi(a) = \tilde{a} = \tilde{a}_1\tilde{a}_2\tilde{a}_3\ldots$, and similarly with $\Phi(b) = \tilde{b}$ and $\Phi(c) = \tilde{c}$. The sets $\omega_\sigma(\tilde{a}) = A$, $\omega_\sigma(\tilde{b}) = B$ and $\omega_\sigma(\tilde{c}) = C$ are minimal and uncountable, since the sequences $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ are uniformly recurent but not periodic.

For $x = x_1x_2x_3\ldots$ and $y = y_1y_2y_3\ldots$ in Σ_2, put

$$x \diamond y = x_1\ldots x_{m_1}y_1\ldots y_{n_1}x_1\ldots x_{m_2}y_1\ldots y_{n_2}x_1\ldots x_{m_3}y_1\ldots y_{n_3}\ldots,$$

where $\{m_i\}_{i=1}^{\infty}$, $\{n_i\}_{i=1}^{\infty}$ are given sequences in $\mathbb{N}$; we will specify them later. Finally, let $\alpha = \tilde{a} \diamond \tilde{b}$, $\beta = \tilde{c} \diamond \tilde{a}$, and let $X = \overline{\mathrm{Orb}(\alpha)} \cup \overline{\mathrm{Orb}(\beta)}$.

Lemma 2 *Let* $\lim_{k\to\infty} m_k = \lim_{k\to\infty} n_k = \infty$. *Then*

(i) $\overline{\mathrm{Orb}(\alpha)} = \mathrm{Orb}(\alpha) \cup \omega_\sigma(\alpha)$.

(ii) $\overline{\mathrm{Orb}(\tilde{a})} \cup \overline{\mathrm{Orb}(\tilde{b})} \subset \omega_\sigma(\alpha) = \overline{\mathrm{Orb}(\tilde{a})} \cup \overline{\mathrm{Orb}(\tilde{b})} \cup C_a \cup C_b$ *where C_a is a subset of the set* $\mathrm{Orb}^{-1}(\tilde{a})$ *of all σ-preimages of $\tilde{a}$ in Σ_2, and similarly for C_b. Consequently, both C_a and C_b are countable.*

(iii) *Similar formulas are valid with α replaced by β, b by c, and $\tilde{b}$ by $\tilde{c}$.*

Proof. (i) This equality is true, since $\omega_\sigma(\alpha)$ is the set of accumulation points of $\mathrm{Orb}(\alpha)$.

(ii) Obviously, $\tilde{a}, \tilde{b}$ belong to $\omega_\sigma(\alpha)$, and $\omega_\sigma(\alpha)$ is closed and invariant. Therefore, it contains $\overline{\mathrm{Orb}(\tilde{a})}$ and $\overline{\mathrm{Orb}(\tilde{b})}$. This proves the first inclusion.

To prove the second one, let $u \in \omega_\sigma(\alpha)$. There is a sequence $\{p_k\} \in \mathcal{A}$ such that $\sigma^{p_k}(\alpha) \to u$. Consider the four possible cases:

1. Infinitely many terms in the sequence $\sigma^{p_k}(\alpha)$ begin with a block of $\tilde{a}$ of the same length $\lambda \geq 0$, followed by a block $\tilde{b}_1\tilde{b}_2\ldots,\tilde{b}_{n_l}$ of $\tilde{b}$. Since $\lim_{l\to\infty} n_l = \infty$, $u = \tilde{a}_1\ldots\tilde{a}_j\tilde{b}$, where $j > i$. Thus, $u \in \mathrm{Orb}^{-1}(\tilde{b})$.

2. Similarly, if infinitely many terms in the sequence $\sigma^{p_k}(\alpha)$ begin with a block of $\tilde{b}$ of the same length, followed by a block $\tilde{a}_1\tilde{a}_2\ldots,\tilde{a}_{m_l}$ of $\tilde{a}$, $u \in \mathrm{Orb}^{-1}(\tilde{a})$.

3. If infinitely many terms in the sequence $\sigma^{p_k}(\alpha)$ begin with a block of $\tilde{a}$ whose length is unbounded as k tends to infinity then $u \in \overline{\mathrm{Orb}(\tilde{a})}$.

4. If infinitely many terms in the sequence $\sigma^{p_k}(\alpha)$ begin with a block of $\tilde{b}$ whose length is unbounded as k tends to infinity then $u \in \overline{\mathrm{Orb}(\tilde{b})}$. $\square$

Lemma 3 *Let* $\lim_{k\to\infty} m_k = \lim_{k\to\infty} n_k = \infty$. *Then* σ *restricted to* X *is* $\omega_2^u C$.

Proof. We show that $\{\alpha, \beta\}$ is an ω^u-scrambled set. By Lemma 2, $\omega_\sigma(\alpha) \setminus \omega_\sigma(\beta) \supset \omega_\sigma(\tilde{b})$ is uncountable, $\omega_\sigma(\beta) \cap \omega_\sigma(\alpha) \supset \omega_\sigma(\tilde{a}) \neq \emptyset$, and since $\omega_\sigma(\tilde{a})$ is infinite and minimal, it contains no periodic point. $\square$

Lemma 4 *Let* u, v *be distinct points in* $\{\tilde{a}, \tilde{b}, \tilde{c}\}$. *Then* $\rho(\sigma^i(u), \sigma^j(v)) \geq 1/8$, *for any* $i, j \in \mathbb{N}_0$.

Proof. The result follows from the following observation (cf. definition of the sequences $\tilde{a}, \tilde{b}, \tilde{c}$). Any block of $\tilde{a}$ of length 8 contains at least 6 ones, hence at most two zeros. Similarly, any block of $\tilde{c}$ of length 8 contains at most two ones, and the number of ones in any block of $\tilde{b}$ of length 8 is between 3 and 4. $\square$

For simplicity put $P_x = \overline{\mathrm{Orb}(\tilde{x})} \cup \mathrm{Orb}^{-1}(\tilde{x})$, for $x \in \Sigma_2$.

Lemma 5 *The sets* P_a, P_b, P_c *are distal (and hence, disjoint). Thus, for* u *and* v *belonging to distinct sets* P_a, P_b, P_c, $\liminf_{k\to\infty} \rho(\sigma^k(u), \sigma^k(v)) > 0$.

Proof. Apply Lemma 4. $\square$

Our next aim is to show that σ is LYC on P_x for no $x \in \Sigma_2$. For simplicity, we will consider only sequences x which contain infinitely many zeros and infinitely many ones. In this case, it is possible to reconstruct the original sequence x from $\tilde{y} = \sigma^k(\tilde{x})$ without knowing k. In fact, majority of the digits in $\tilde{y}$ must be equal to x_1: either the digits on odd places in $\tilde{y}$ are the same and equal to x_1, or the digits on the even places in $\tilde{y}$ are equal to x_1. Next, having fixed x_1 among the digits $\tilde{x}_j$, we remove from $\tilde{y}$ the digits corresponding x_1 (i.e., either all digits on the odd places, or all digits on the even places), and proceed by induction.

However, in a similar way we can reconstruct the first n digits in x from any block of a sequence in $\omega_\sigma(\tilde{x})$, with a sufficient length δ_n. Indeed, let $d = d_1d_2d_3\ldots \in \omega_\sigma(\tilde{x})$. Then for any $m = 1, 2, \ldots$ there is $\{n_k\} \in \mathcal{A}$ such that $d_1d_2\ldots d_m = \tilde{x}_{n_k+1}\tilde{x}_{n_k+2}\ldots\tilde{x}_{n_k+m}$, for any k. Let $r = \min\{i : x_i \neq x_1\}$. Then it suffice to take $m = 2^r$ to see, which members of $d_1d_2\ldots d_m$ are equal to x_1. (Thus, in our case, $\delta_1 = 2^r$.) Define a map $\mu : \omega_\sigma(\tilde{x}) \to \Sigma_2$ such that, for $d = d_1d_2d_3\ldots \in \omega_\sigma(\tilde{x})$, $\mu(d) = s = s_1s_2s_3\ldots$, where s_n is given inductively in the following way:

Stage 1: Let

$$s_1 = \begin{cases} 0, & \text{if } x_1 = d_1 = d_3 = d_5 \ldots, \\ 1, & \text{if } x_1 = d_2 = d_4 = d_6 \ldots \end{cases}$$

Let $d^1 = d^1_1 d^1_2 d^1_3 \ldots$ be a subsequence of d obtained by removing $d_1, d_3, d_5 \ldots$ from d if $s_1 = 0$, and by removing $d_2, d_4, d_6 \ldots$ otherwise.

Stage n: Sequence $d^{n-1} = \{d^{n-1}_i\}$ is available from *stage n-1*. Let

$$s_n = \begin{cases} 0, & \text{if } x_n = d^{n-1}_1 = d^{n-1}_3 = d^{n-1}_5 \ldots, \\ 1, & \text{if } x_n = d^{n-1}_2 = d^{n-1}_4 = d^{n-1}_6 \ldots, \end{cases}$$

and let d^n be obtained from d^{n-1} by removing the odd or even members, if $s_n = 0$ or $s_n = 1$, respectively. Obviously, we have the following.

Lemma 6 *Let x be a sequence in Σ_2 having infinitely many zeros and infinitely many ones. Then the map μ is a bijection from $\omega_\sigma(\tilde{x})$ to Σ_2.*

Lemma 7 *Let x be a sequence in Σ_2 having infinitely many zeros and infinitely many ones. Then, for any distinct d, h in P_x, $\liminf_{n\to\infty} \rho(\sigma^n(d), \sigma^n(h)) > 0$.*

Proof. Since any point in P_x is eventually in $\omega_\sigma(\tilde{x})$ we may assume without loss of generality that $d, h \in \omega_\sigma(\tilde{x})$. By Lemma 6, μ is bijective, hence $\mu(d) \neq \mu(h)$. Then, for some $m \in \mathbb{N}$, the sequences $\mu(d), \mu(h)$ differ the mth coordinate, $\mu(d)_m \neq \mu(h)_m$. But then $\liminf_{n\to\infty} \rho(\sigma^n(d), \sigma^n(h)) \geq 1/2^m$ (cf. the construction of μ). $\square$

Now we can return to our special sequences α and β.

Lemma 8 *Assume that $\lim_{k\to\infty} m_k = \lim_{k\to\infty} n_k = \infty$. Then neither $\sigma|\mathrm{Orb}(\alpha)$ nor $\sigma|\mathrm{Orb}(\beta)$ is LY_2C. Thus, $\liminf_{n\to\infty} \rho(\sigma^n(\alpha), \sigma^{n+k}(\alpha)) > 0$ whenever $k \in \mathbb{N}$, and similarly with β.*

Proof. Let us suppose that $\liminf_{n\to\infty} \rho(\sigma^n(\alpha), \sigma^{n+k}(\alpha)) = 0$. Then there is $\{n_i\}_{i=1}^\infty \in \mathcal{A}$ such that $\lim_{n\to\infty} \rho(\sigma^{n_i}(\alpha), \sigma^{n_i+k}(\alpha)) = 0$. Then $\sigma^{n_i}(\alpha) \to d$ and $\sigma^{n_i+k}(\alpha) \to d$ so $\sigma^k(d) = d$ and $d \in X$ is a periodic point. But σ restricted to X has no periodic point (cf. Lemma 2) – a contradiction. $\square$

Lemma 9 *Assume that $\lim_{i\to\infty} m_i = \lim_{i\to\infty} n_i = \infty$. Then σ restricted to $\mathrm{Orb}(\alpha) \cup \mathrm{Orb}(\beta)$ is not LY_2C.*

Proof. Because of the symmetry it suffices to show that, for any $k \in \mathbb{N}_0$, $\liminf_{n\to\infty} \rho(\sigma^{n+k}(\alpha), \sigma^n(\beta)) > 0$. Assume, contrary to what we wish to show, that this is not true. Then, by Lemma 4, in both sequences $\sigma^k(\alpha), \beta$, there must be arbitrarily large a-blocks at the same positions. However, if $k = 0$, then any a-blocks in α and β are at complementary positions (cf. the definition of α and β). If k is positive, then the blocks are shifted, and there is some overlapping of the

a-blocks. But since $\lim_{i\to\infty} m_i = \lim_{i\to\infty} n_i = \infty$, the parts of a-blocks in $\sigma^k(\alpha)$, β, respectively, that are overlapping, are small – their length is k. Consequently, by Lemma 4, we get $\liminf_{n\to\infty} \rho(\sigma^{n+k}(\alpha), \sigma^n(\beta)) \geq 1/(8+k) > 0$. □

Now we are able to prove our main result.

Theorem 5 *There is a compactum $X \subset \Sigma_2$ such that $\sigma(X) \subset X$, σ has no periodic points in X, σ restricted to X is $\omega_2^u C$ and any LY-scrambled set has only two points.*

PROOF. Let $\lim_{i\to\infty} m_i = \lim_{i\to\infty} n_i = \infty$, and let $X = \overline{\mathrm{Orb}(\alpha)} \cup \overline{\mathrm{Orb}(\beta)}$. By Lemma 3, σ is ω_2^uC on X. On the other hand, by Lemma 2,

$$X \subset \mathrm{Orb}(\alpha) \cup \mathrm{Orb}(\beta) \cup P_a \cup P_b \cup P_c.$$

The fact that σ on X has no LY-scrambled set now follows by Lemmas 5, 7 – 9. □

Concluding remarks. (i) R. Pikula [11] recently proved, that there is an ωC map f of a compact metric space with the property that any LY-scrambled set has not more than 8 points. He considers uncountable ω^u-scrambled sets. Our Theorem 5 gives a stronger result. On the other hand, our ω^u-scrambled set has only two points. However, the above approach is applicable so that one can obtain a continuous map f on a compactum which is ω_∞^uC, but not LY-scrambled set has three points. The construction is rather complicated.

(ii) The systems obtained in Theorems 2 - 5 can be inserted to the real line so that there is a continuous map f of the unit interval I which has as factors the systems from Theorems 2 – 5.

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