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DISTRIBUTIONAL CHAOS AND SPECTRAL DECOMPOSITION OF DYNAMICAL SYSTEMS ON THE CIRCLE

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Abstract. Schweizer and Smítal [Tran. Amer. Math. Soc. **344** (1994), 737–754] introduced the notion of distributional chaos for continuous maps of the interval. In this paper we show that for the continuous mappings of the circle the results are very similar, up to natural modifications. Thus any such mapping has a finite spectrum, which is generated by the map restricted to a finite collection of basic sets, and any scrambled set in the sense of Li and Yorke has a decomposition into three subsets (on the interval into two subsets) such that the distribution function generated on any such subset is lower bounded by a distribution function from the spectrum. While the results are similar, the original argument is not applicable directly and needs essential modifications. Thus, e.g., we had first to develop the theory of basic sets on the circle.

1. Introduction and Main results

Let (X, ϱ) be a compact metric space. For f in the space $C(X,X)$ of continuous mappings from X into itself, $x, y \in X$, real t, and any positive integer n define

$$(1)\quad \xi(x,y,t,n) = \sum_{i=0}^{n-1} \chi_{[0,t)}(\delta_{xy}(i)) = \#\{i; 0 \le i < n \text{ and } \delta_{xy}(i) < t\},$$

$$(2)\qquad F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x,y,t,n),$$

and

$$(3)\qquad F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n}\xi(x,y,t,n),$$

where $\delta_{xy}(i) = \varrho(f^i(x), f^i(y))$, and χ_A is the characteristic function of the set A.

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Clearly both F^*_{xy}, F_{xy} are nondecreasing functions such that $F^*_{xy}(t) = F_{xy}(t) = 0$ for $t < 0$, and $F^*_{xy}(t) = F_{xy}(t) = 1$ for $t > \operatorname{diam} X$. We identify any two nondecreasing functions that coincide everywhere except at a countable set, and adopt the convention to chose functions F^*_{xy}, F_{xy} as left-continuous. Functions F^*_{xy}, F_{xy} are called the *upper* and *lower distribution function* of x and y, respectively. A function f *exhibits distributional chaos* if there are points $x, y \in \mathbb{S}$ such that $F^*_{xy}(t) = 1$ for all $t > 0$ and there is a point $s \in (0,1)$ such that $F^*_{xy}(s) > F_{xy}(s)$.

It is well known that in the case $X = I = [0,1]$ distributional chaos is equivalent to positive topological entropy ($h(f) > 0$). For details see [ScSm] or [ScSkSm]. Note that this relation does not hold in general; e.g. a counterexample in the case $X = I^2$ can be found in [Ba]. For $X = \mathbb{S} = \mathbb{R}/\mathbb{Z}$ (the circle) distributional chaos apears if and only if $h(f) > 0$, or equivalently if f has a basic set. See [M1].

The main aim of this paper is to extend results concerning the spectral decomposition of a dynamical system on the interval, as given in [ScSm], to the case $X = \mathbb{S}$. To do this we need to understand properties of basic sets since only these sets (both on the interval and on the circle) support distributional chaos. A *basic set* is a maximal infinite ω-limit set which contains a periodic point. This definition fits both the interval and circle. Recall that a basic set has decompositions into finite number of periodic portions which form a single orbit. The supremum of numbers of such portions is finite; if it is one the basic set is *indecomposable.* A *solenoid* is a (maximal) infinite ω-limit set which has decompositions of arbirarily high order; thus a solenoid cannot contain a periodic point. Properties of basic sets on the interval are well-known due to Sharkovsky [S] and Blokh [Bl], for example. The case $X = \mathbb{S}$ cannot be deduced simply from the previous one. It is considered in [M2]. We summarize the main results in the next Theorem 2.1.

We use the following terminology. For $f \in C(\mathbb{S}, \mathbb{S})$, let $\omega_f(x)$ denote the ω-limit set of x. Points $x, y \in \mathbb{S}$ are *synchronous* if the sets $\omega_f(x)$ and $\omega_f(y)$ are contained in the same maximal ω-limit set ω and if, for any periodic interval J such that its orbit $\mathrm{Orb}J$ contains ω, there is a $j \geq 0$ such that $f^j(x), f^j(y) \in J$. The *spectrum* $\Sigma(f)$ of f is the set of minimal elements of the set $D(f) = \{F_{xy};\ x \text{ and } y \text{ are synchonous}\}$. And the *weak spectrum* $\Sigma_w(f)$ of f is the set of minimal elements of the set $D_w = \{F_{xy};\ \liminf_{i\to\infty} \delta_{xy}(i) = 0\}$. We will see that, for a continuous map f of the circle, both the spectrum and the week spectrum are nonempty and finite (similarly as on the interval). A *scrambled set* (in the sense of Li and Yorke) is any set $S \subset \mathbb{S}$ such that, for any distinct points x and y in $\mathbb{S}$,

$$\liminf_{i\to\infty} \delta_{xy}(i) = 0, \quad \limsup_{i\to\infty} \delta_{xy}(i) > 0.$$

Other notions are explained in the text, or can be found in standard references like, e.g., [Bl]. To summarize the main results, we first describe the dynamics on a single basic set (Theorem A) and then the general case (Theorem B). Recall that part (A) of Theorem B was already proved in [M1].

Theorem A. *Let $f \in C(\mathbb{S}, \mathbb{S})$and let $\tilde{\omega}$ be its basic set. Then there are a nondecreasing function $F : \mathbb{R} \to [0, 1]$, a nonempty perfect set $P \subset \tilde{\omega}$, and a positive ε with the following properties:*

(i) *$F(\varepsilon) = 0$ and $\Sigma(f|_{\tilde{\omega}}) = \{F\}$;*
(ii) *$F = F_{xy} < F^*_{xy} = \chi_{(0,\infty)}$ for any $x \neq y$ in P;*
(iii) *if S is a scrambled set for f such that $\omega_f(x) \subset \tilde{\omega}$ for any $x \in \mathbb{S}$ then there are sets S_0, S_1, S_2, $(S_0 \neq \emptyset)$, such that $S = S_0 \cup S_1 \cup S_2$ and $F_{xy} \geq F$ whenever $x, y \in S_k$.*

Theorem B. *Let $f \in C(\mathbb{S}, \mathbb{S})$.*

(A) *If the topological entropy of f is zero, then $\Sigma(f) = \Sigma_w(f) = \{\chi_{(0,\infty)}\}$.*
(B) *If the topological entropy of f is positive, then:*
  (B1) *Both the spectrum $\Sigma(f)$ and the weak spectrum $\Sigma_w(f)$ are finite and nonempty. Specifically $\Sigma(f) = \{F_1, \dots , F_m\}$ for some $m \geq 1$, and $\Sigma \setminus \Sigma_w(f) = \{F_{m+1}, \dots , F_n\}$ where $n \geq m$. Furthermore, for each i there is an $\varepsilon_i > 0$ such that $F_i(\varepsilon_i) = 0$.*
  For any positive integer $k \leq n$, let π_k be the system of sets P such that $\#P \geq 2$ and for any distinct u, v in P, $F_k = F_{uv} < F^*_{uv} = \chi_{(0,\infty)}$.
  (B2) *If $k \leq m$, then π_k contains a nonempty perfect set P_k.*
  (B3) *If, on the other hand, $m < k \leq n$ then π_k is nonempty and any P in π_k contains two or three points.*
  (B4) *If S is a scrambled set for f (or more generally if, for any u, v in S, $\liminf_{i\to\infty} \delta_{uv}(i) = 0$), then there are integers i, j, $k \leq m$ and a decomposition $S = S_i \cup S_j \cup S_k$ such that $F_{uv} \geq F_l$ if $u, v \in S_l$, for $l \in \{i, j, k\}$.*

2. Properties of basic sets

The following Theorem summarizes the properties of basic sets for continous maps of the circle which have been proved in [M2]. It is easy to see (cf. also [M2]) that, similarly as for mappings in $C(I, I)$, any indecomposable basic set $\tilde{\omega} \subset \mathbb{S}$ is contained in a minimal compact invariant interval, possibly equal to $\mathbb{S}$; we call this interval the *envelope* of $\tilde{\omega}$ and denote it by $\mathrm{Env}(\tilde{\omega})$. If $\tilde{\omega}$ is not indecomposable then it has a maximal decomposition $\tilde{\omega}_1 \cup \dots \cup \tilde{\omega}_k$, $k > 1$, into periodic portions that form a single orbit of period k. Thus, every $\tilde{\omega}_i$ is an indecomposable basic set for f^k, and has the envelope $\mathrm{Env}(\tilde{\omega}_i)$ with respect to f^k. In this case we define $\mathrm{Env}(\tilde{\omega}) = \mathrm{Env}(\tilde{\omega}_1) \cup \dots \cup \mathrm{Env}(\tilde{\omega}_k)$. Clearly, $\mathrm{Env}(\tilde{\omega})$ consits of k periodic intervals forming a single orbit.

Theorem 2.1. *Let $f \in C(\mathbb{S}, \mathbb{S})$, $x \in \mathbb{S}$ and let $\tilde{\omega}$ be a basic set.*

(i) *$\tilde{\omega}$ is perfect;*
(ii) *if $\omega_f(x) \subset \tilde{\omega}$, then $\{y \in \tilde{\omega};\ \omega_f(y) = \omega_f(x)\}$ is dense in $\tilde{\omega}$;*

(iii) *if J is an interval such that $J \cap \tilde{\omega}$ is infinite then $\tilde{\omega} \cap J$ contains a periodic point;*
(iv) *the system of basic sets of f is countable;*
(v) *if $\tilde{\omega}_1 \neq \tilde{\omega}_2$ are indecomposable basic sets and $U = \mathrm{Env}(\tilde{\omega}_1)$, $V = \mathrm{Env}(\tilde{\omega}_2)$, then $U \cap V = \emptyset$, or U and V have at most two points in common, or $U \subset \mathrm{int}(V)$, or $V \subset \mathrm{int}(U)$; in particular, $U \neq V$;*
(vi) *if $\tilde{\omega}$ is indecomposable then, for every compact interval K contained in the interior of $\mathrm{Env}(\tilde{\omega})$, and every compact interval J such that $J \cap \tilde{\omega}$ is infinite, there is a $k \in \mathbb{N}$ such that $f^k(J) \supset K$.*

Definition 2.2. Let $f \in C(\mathbb{S}, \mathbb{S})$, and let $\tilde{\omega}$ be a basic set. Then $f|_{\tilde{\omega}}$ is *strongly transitive* in $\mathrm{Env}(\tilde{\omega})$, in the following sense. For every compact interval K contained in the interior of $\mathrm{Env}(\tilde{\omega})$ and every compact interval J such that $J \cap \tilde{\omega}$ is infinite, there is a $k \in \mathbb{N}$ such that $f^k(J) \supset K$. This follows by induction from Theorem 2.1 (vi).

We finish this section by couple of lemmas which are necessary in the remainder of this paper.

Lemma 2.3.(Cf. Theorem 3.4 in [M2].) *Let $f \in C(\mathbb{S}, \mathbb{S})$ and $u, v \in \mathbb{S}$. Let $\{U_i\}_{i=0}^{\infty}$, $\{V_i\}_{i=0}^{\infty}$, U, and V be compact intervals, possibly degenerate, with $\lim_{i\to\infty} U_i = U$ and $\lim_{i\to\infty} V_i = V$. Let, for any i and j, there exist positive integers $u(i,j)$ and $v(i,j)$ such that $f^{u(i,j)}(U_i) \supset V_j$ and $f^{v(i,j)}(V_i) \supset U_j$. Then there are $u \in U$ and $v \in V$ such that $\{u, v\} \subset \omega_f(y)$, for some $y \in \mathbb{S}$.*

Lemma 2.4. *Let $\tilde{\omega}$ be an indecomposable basic set for $f \in C(\mathbb{S}, \mathbb{S})$ and let $U = \mathrm{Env}(\tilde{\omega})$. Let $\{J_n\}_{n\in K}$, $J_n \subset U$, be an enumeration of intervals complementary to $\tilde{\omega}$. Then, for any $n \in K$, either $f(J_n)$ is a singleton or there is a $k \in K$ such that $f(J_n) \subset \overline{J_k}$.*

Proof. It is very similar to the proof of Lemma 3.5 in [M2] and we omit it. $\square$

Proof of the main result in [ScSm] is based on the following result stated by Sharkovsky [S] in 1966. *If $\tilde{\omega}$ is a basic set for a map $f \in C(I, I)$, and $\omega_f(x) \subset \tilde{\omega}$ for some $x \in \mathbb{S}$, then there is a $k > 0$ with $f^k(x) \in \tilde{\omega}$.* Unfortunately, this result is wrong. To see this let $f \in C(I, I)$ be transitive on $[0, 1/2]$, with $f[0, 1/2] = [0, 1/2]$, $f(1/2) = 1/2$, and let $f(x) = x/2 + 1/4$ for $x \in [1/2, 1]$. Then $\tilde{\omega} = [0, 1/2]$ is a unique basic set for f, $\omega_f(1) = \{1/2\} \subset \tilde{\omega}$ but $f^k(1) \notin \tilde{\omega}$ for all $k > 0$. However, we can replace the wrong result by its modification which works also in the case $X = I$.

Lemma 2.5. *Let $\tilde{\omega}$ be a basic set for $f \in C(\mathbb{S}, \mathbb{S})$, and let $\omega_f(x) \subset \tilde{\omega}$. Then there is a $y \in \tilde{\omega}$ such that $\lim_{i\to\infty} \delta_{xy}(i) = 0$ and hence, $\omega_f(x) = \omega_f(y)$.*

Proof. Let $f^n(x) \notin \tilde{\omega}$ for all n. We may assume that $\omega_f(x)$ is infinite since otherwise it is a cycle, and that $\tilde{\omega}$ is nowhere dense since otherwise $\tilde{\omega}$ consists of a finite number of compact intervals. In both cases the result would follow immediatelly.

Since $\omega_f(x)$ is infinite $f^n(x)$ is in an interval $B \subset \mathrm{Env}(\tilde{\omega})$ complementary to $\tilde{\omega}$, for some $n \geq 0$; we may assume $x \in B$. By Lemma 2.4 the closure $\overline{B}$ of B is wandering or periodic. In the first case $\lim_{i\to\infty} \mathrm{diam}\,(f^i(B)) = 0$ and as y we take an endpoint of B. If $\overline{B}$ is periodic then $\omega_f(z)$ would be finite. $\square$

Lemma 2.6. *Let $f \in C(\mathbb{S},\mathbb{S})$ and let $\{\omega_i\}_{i=1}^{\infty}$ be a sequence of distinct minimal (i.e., indecomposable) periodic portions of basic sets of f. If the periods of ω_i are bounded then $\lim_{i\to\infty} \mathrm{diam}\,\omega_i = 0$.*

Proof. Let m be a common multiple of the periods of all ω_i and $K_i = \mathrm{Env}(\omega_i)$. Assume that the lemma is not true. Replacing f by f^m we can assume that $m = 1$ and that, for any i, $\mathrm{diam}\,K_i > \varepsilon$, where ε is positive. By (v) of Theorem 2.1 it suffices to consider the case when $K_1, K_2, \dots$ is a monotone sequence. We may assume that it is a decreasing sequence, since in the other case the argument is similar.

Choose $\delta > 0$ such that $\mathrm{diam}\,f(A) < \varepsilon$ for any set A with $\mathrm{diam}\,A < \delta$. Again by (v) of Theorem 2.1 assume that $K_i = [a_i, b_i] \neq \mathbb{S}$ and $0 \notin K_i$, for every i. Then $a_i < a_{i+1}$ and $b_{i+1} < b_i$. Let $\omega^0 = [a_1, a_2] \cap \omega_1$ and $\omega^1 = [b_2, b_1] \cap \omega_1$. Since $\omega_1 = \omega^0 \cup \omega^1$ is indecomposable, one of the sets, $[a_1, a_2]$, $[b_1, b_2]$ say $[a_1, a_2]$ is mapped by f over $[a_2, b_2]$. Thus $|a_2 - a_1| > \delta$ and $\mathrm{diam}\,K_2 < \mathrm{diam}\,K_1 - \delta$. By induction we get $\mathrm{diam}\,K_{i+1} < \mathrm{diam}\,K_1 - i\delta$, for any i, which is impossible. $\square$

3. Distributional chaos on basic sets

The next three lemmas are slight modifications of results proved in [ScSm] for the interval mappings. For the reader's convenience we insert the arguments which are very simple, and are almost the same as in the original paper.

Lemma 3.1. *Let $f \in C(\mathbb{S},\mathbb{S})$. Then for any t, λ in $(0,1)$ there is an integer $n(t,\lambda)$ with the following property: If A is a periodic set of period $m \geq n(t,\lambda)$, and convex hulls of sets $f^s(A)$ for $s < m$, are nonoverlapping, then for any u, v in A, $F_{uv}(t) > \lambda$.*

Proof. Fix t and λ. Let $n(t,\lambda)$ be such that $(n(t,\lambda) - 1/t)/n(t,\lambda) > \lambda$. Since there are at most $1/t$ distinct sets $f^s(A)$ with $\mathrm{diam}\,f^s(A) \geq t$ we have $F_{uv}(t) \geq 1/m \cdot \#\{s < m;\ \mathrm{diam}\,f^s(A) < t\} \geq (m - 1/t)/m > \lambda$ $\square$

Lemma 3.2. *Let $f \in C(\mathbb{S},\mathbb{S})$ and both F^*_{xy} and F_{xy} be continuous at $t \in (0,1)$. Then, for any $\varepsilon > 0$, there are arbitrarily large positive integers k, q, and $\delta > 0$ such that*

$$\frac{1}{k}\xi(u,v,k,t) < F_{xy}(t) + \varepsilon$$

and

$$\frac{1}{q}\xi(u,v,q,t) > F^*_{xy}(t) - \varepsilon$$

whenever $\varrho(u,x) < \delta$ *and* $\varrho(v,y) < \delta$.

Proof. Choose $\varepsilon_1 > 0$ such that $F_{xy}(t+2\varepsilon_1) < F_{xy}(t) + \varepsilon/2$ and $F^*_{xy}(t - 2\varepsilon_1) > F^*_{xy}(t) - \varepsilon/2$. Then choose $k \in \mathbb{N}$ such that $1/k \cdot \xi(x,y,k,t+2\varepsilon_1) < F_{xy}(t+2\varepsilon_1) + \varepsilon/2$. The first equality follows from the fact that $\xi(u,v,k,t) \leq \xi(x,y,k,t+2\varepsilon_1)$ whenever $\delta > 0$ is sufficiently small. The argument for the second inequality is similar. □

Lemma 3.3. *Let* $f \in C(\mathbb{S},\mathbb{S})$*, let* $\tilde\omega_1, \tilde\omega_2$ *be basic sets, and let* U *and* V *be the minimal compact periodic intervals with* $\mathrm{Orb}(U) \supset \tilde\omega_1$ *and* $\mathrm{Orb}(V) \supset \tilde\omega_2$*. Then, for any* $u \in U \cap \tilde\omega_1$ *and* $v \in V \cap \tilde\omega_2$*, there are* $u^* \in \mathrm{int}(U) \cap \tilde\omega_1$ *and* $v^* \in \mathrm{int}(V) \cap \tilde\omega_2$ *such that* $F_{uv} = F_{u^*v^*}$.

Proof. First we show that $F_{uv} = F_{u(0)v(0)}$ where $u(0) \in \tilde\omega_1 \cap U$ and $v(0) \in \tilde\omega_2 \cap V$ are suitable nonperiodic points. To do this take $u(0) = u$ if u is not periodic; otherwise by (ii) of Theorem 2.1 there is a nonperiodic point $u(0)$ in $U \cap \tilde\omega_1$ such that $\omega_f(u) = \omega_f(u(0))$, and one can easily verify that $u(0)$ can even be chosen such that $\liminf_{i\to\infty} \delta_{u(0)u}(i) = 0$. Then clearly $F_{uv} = F_{u(0)v}$. The point $v(0)$ is defined similarly.

Now let $m > 0$ be a common multiple of the periods of U and V. Since $f^m(U \cap \tilde\omega_1) = U \cap \tilde\omega_1$, there is a sequence $\{u(i)\}_{i=0}^\infty$ of points in $U \cap \tilde\omega_1$ such that $f^m(u(i+1)) = u(i)$ for any $i > 0$. Choose $\{v(i)\}_{i=0}^\infty$ in $V \cap \tilde\omega_2$ similarly. Now the point $u(i), v(i)$ are not periodic, hence for some j, $u(j) \in \mathrm{int}(U) \cap \tilde\omega_1$ and $v(j) \in \mathrm{int}(V) \cap \tilde\omega_2$. Take $u^* = u(j)$ and $v^* = v(j)$. □

4. Spectral decomposition

Also the results in this section are modifications of these found in [ScSm]. Before stating the first one recall Definition 2.2 for the notion of strong transitivity.

Lemma 4.1. *Let* $\tilde\omega_1$*,* $\tilde\omega_2$ *be basic sets for* $f \in C(\mathbb{S},\mathbb{S})$*. Assume that there are periodic intervals* U, V *and countable set* $Q \subset \mathbb{S}^2$ *of pairs* (u,v) *such that*

$$f|_{\tilde\omega_1} \text{ is strongly transitive in } \mathrm{int}(U) \text{ and } f|_{\tilde\omega_2} \text{ in } \mathrm{int}(V) \tag{1}$$

and furthermore, that

$$u \in \tilde\omega_1 \cap \mathrm{int}(U) \text{ and } v \in \tilde\omega_2 \cap \mathrm{int}(V) \text{ if } (u,v) \in Q. \tag{2}$$

Then there are points $x \in \tilde\omega_1 \cap U$ *and* $y \in \tilde\omega_1 \cap V$ *such that, for any* $t > 0$,

$$F_{xy}(t) \leq \inf\{F_{uv}(t);\ (u,v) \in Q\} \tag{3}$$

and

$$F^*_{xy}(t) \geq \sup\{F^*_{uv}(t);\ (u,v) \in Q\}. \tag{4}$$

Proof. Let T be a countable set, dense in $[0,1]$, and such that, for any $(u,v) \in Q$ and any $t \in T$, both F_{uv} and F^*_{uv} are countinuous at t. Let $\{t_j\}_{j=1}^\infty$ and $\{u(j), v(j)\}_{j=1}^\infty$ be sequences of points from T and Q, respectively, such that for any $t \in T$ and any $(u, v) \in Q$, $t = t_j$, $u = u(j)$ and $v = v(j)$ for infinitely many j.

Next, using induction, we define positive integers

$$k(1) < q(1) < k(2) < q(2) < \dots < k(i) < q(i) < \dots$$

and decreasing sequences $\{U_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ of compact intervals with

$$\lim_{i\to\infty} \operatorname{diam}(U_i) = \lim_{i\to\infty} \operatorname{diam}(V_i) = 0,$$

and such that for any $u \in U_n$ and $v \in V_n$ and any $j \le n$,

$$\frac{1}{k(j)}\xi(u, v, k(j), t_j) \le F_{u(j)v(j)}(t_j) + \frac{1}{j} \tag{5}$$

and

$$\frac{1}{q(j)}\xi(u, v, q(j), t_j) \ge F^*_{u(j)v(j)}(t_j) - \frac{1}{j}. \tag{6}$$

To do this, we take $U_1 = U$, $V_1 = V$, $k(1) = 1$, $q(1) = 2$, and assume that U_n, V_n, $k(n)$ and $q(n)$ have been defined such that $f^j(U_n) \cap \omega_1$ and $f^j(V_n) \cap \omega_2$ are infinite whenever j is sufficiently large. Since U and V are periodic, by (1) and (2) there is some $s > q(n)$ such that $u(n + 1) \in f^s(U_n)$ and $v(n+1) \in f^s(V_n)$. Let $a \in U_n$ and $b \in V_n$ be such that $f^s(a) = u(n+1)$ and $f^s(b) = v(n + 1)$. Then clearly $F_{ab} = F_{u(n+1)v(n+1)}$ and $F^*_{ab} = F^*_{u(n+1)v(n+1)}$. Now the existence of $U_{n+1} \subset U_n$, $V_{n+1} \subset V_n$, $k(n + 1)$ and $q(n + 1)$ follows easily by Lemma 3.2. (We take as U_{n+1} and V_{n+1} compact neighborhoods of a and b, respectively, with $\operatorname{diam}(U_n) > 2\operatorname{diam}(U_{n+1})$ and $\operatorname{diam}(V_n) > 2\operatorname{diam}(V_{n+1})$. By (i) of Theorem 2.1, a, b, U_{n+1} and V_{n+1} can be chosen such that both $f^s(U_{n+1}) \cap \tilde\omega_1$ and $f^s(V_{n+1}) \cap \tilde\omega_2$ are infinte.)

Take $x' \in \bigcap_{i=1}^{\infty} U_j$ and $y' \in \bigcap_{i=1}^{\infty} V_j$. For any $t \in T$ and any $(u, v) \in Q$, take j such that $t = t_j$, $u = u(j)$ and $v = v(j)$. Since $x' \in U_j$ and $y' \in V_j$ (5) applies with $u = x'$ and $v = y'$. Since j can be arbitrarily large we have $F_{x'y'}(t) \le F_{uv}(t)$. This implies (3) for $x = x'$, $y = y'$, any $t \in T$, and since T is dense in $[0, 1]$, also for any t. The argument for (4) is similar.

Finally, let $w \in \tilde\omega_1 \cap U$ be such that $\omega_f(w) = \tilde\omega_1$ (see (ii) of Theorem 2.1) and let $\{W_i\}_{i=1}^{\infty}$ be a decreasing sequence of compact neighborhoods of w with $\lim_{i\to\infty} W_i = w$. Since $f|_{\tilde\omega_1}$ is strongly transitive we can apply Lemma 2.3 and obtain $\omega_f(x') \subset \tilde\omega_1$. Similary we get $\omega_f(y') \subset \tilde\omega_2$. Now by Lemma 2.5 we get $x \in \tilde\omega_1$ and $y \in \tilde\omega_2$ such that $\lim_{i\to\infty} \delta_{x'x}(i) = 0$ and $\lim_{i\to\infty} \delta_{y'y}(i) = 0$ which implies (3) and (4). □

Lemma 4.2.(Cf. Lemma 5.4 in [ScSm].) *Let $\{N_i\}_{i=0}^{\infty}$ be decomposition of the set $\mathbb{N}$ of positive integers into infinite sets. Then there is an uncountable Borel set $B \subset \{0, 1\}^{\mathbb{N}}$ such that, for any distinct $\alpha = \{\alpha(i)\}_{i=0}^{\infty}$ and $\beta = \{\beta(i)\}_{i=0}^{\infty}$ in B and any n,*

$$\{j \in N_n; \alpha(j) \ne \beta(j)\} \text{ is infinite.} \tag{7}$$

Lemma 4.3. *Let $f \in C(\mathbb{S}, \mathbb{S})$, and let $\tilde{\omega} = \omega_f(z)$ be a basic set. Let U be a minimal compact periodic interval with* $\mathrm{Orb}(U) \supset \tilde{\omega}$, *and let x_0, x_1 be in $U \cap \tilde{\omega}$. Then there is nonempty perfect set $P \subset \tilde{\omega}$ such that, for any distinct $u, v \in P$,*

$$F_{uv} \leq F_{x_0 x_1} \text{ and } F^*_{uv} \geq F^*_{x_0 x_1}. \tag{8}$$

Proof. It is similar to that one for Lemma 5.5 in [ScSm]. We use from Lemma 4.1, and methods of symbolic dynamics. Let T be a countable subset of $\mathbb{S}$, dense in $\mathbb{S}$ and such that both $F_{x_0 x_1}$ and $F^*_{x_0 x_1}$ are continuous at each $t \in T$, and let $\{t_j\}_{j=1}^{\infty}$ be a sequence of points from T that contains every t from T infinitely many times. Let $X_n = \{0, 1\}^n$, for $n = 1, 2, \dots$, and define a system of compact intervals $\{I_\alpha;\ \alpha \in X_j\}_{j=1}^{\infty}$ and positive integers

$$k(1) < q(1) < k(2) < q(2) < \dots < k(j) < q(j) < \dots$$

such that, for every $\alpha = \alpha(1)\alpha(2)\dots\alpha(n)$ and $\beta = \beta(1)\beta(2)\dots\beta(n)$ in X_n, the following is true

$$f^j(I_\alpha) \cap \tilde{\omega} \text{ is infinite if } j > k(n+1); \tag{9}$$

$$\text{if } \alpha \neq \beta \text{ then } I_\alpha \cap I_\beta = \emptyset; \tag{10}$$

$$\text{if } \gamma \text{ is in } X_k \text{ for some } k \text{ then } I_{\alpha\gamma} \subset I_\alpha \subset \mathrm{int}(U); \tag{11}$$

for any $u \in I_\alpha$ and $v \in I_\beta$, and any $j \leq n$,

$$\frac{1}{k(j)} \xi(u, v, k(j), t_j) \leq F_{x_{\alpha(j)} x_{\beta(j)}}(t_j) + \frac{1}{j} \tag{12}$$

and

$$\frac{1}{q(j)} \xi(u, v, q(j), t_j) \geq F^*_{x_{\alpha_j} x_{\beta_j}}(t_j) - \frac{1}{j}. \tag{13}$$

To do this, let I_0 and I_1 be disjoint compact subintervals of $\mathrm{int}(U)$, such that both $I_0 \cap \tilde{\omega}$ and $I_1 \cap \tilde{\omega}$ are infinite. Put $k(1) = 1$ and $q(1) = 2$ and assume by induction that $\{I_\alpha;\ \alpha \in X_n\}$, $k(n)$ and $q(n)$ have been defined. Assume that $f^j(I_\alpha) \cap \tilde{\omega}$ is infinite whenever $j > r$ and $\alpha \in X_n$. Let m be the period of U. By Lemma 3.3 we may assume that $x_0, x_1 \in \mathrm{int}(U)$ and since $f|_{\tilde{\omega}}$ is strongly transitive in $\mathrm{int}(U)$ ((vi) of Theorem 2.1), there is an $s > \max\{r, q(n)\}$ such that $x_0, x_1 \in \mathrm{int}(f^{mj}(I_\alpha))$ whenever $\alpha \in X_n$ and $j \geq s$ Since $\tilde{\omega}$ is perfect ((i) of Theorem 2.1), it is easy to see that, for $i = 0, 1$ and any $\alpha \in X_n$, there is a point $a(\alpha, i) \in \mathrm{int}(I_\alpha)$ such that $f^{ms}(a(\alpha, i)) = x_i$ and such that for any neighborhood V of $a(\alpha, i)$, $f^{ms}(V) \cap \tilde{\omega}$ is infinite.

Applying Lemma 3.2 we can find $q(n+1) > k(n+1) > ms$ and pairwise disjoint compact neighborhoods I_β of the points $a(\alpha, i)$ for all $\beta \in X_{n+1}$, where $\beta = \alpha i$ (we use αi for concatenation of α and i) such that (9)–(13) are satisfied when n is replaced by $n + 1$.

Let $A = \bigcap_{n=1}^{\infty} \bigcup \{I_\alpha;\ \alpha \in X_n\}$. Define a map, code: $A \to X$, by $\mathrm{code}(x) = \alpha(1)\alpha(2)\dots\alpha(n)\dots$ if $x \in I_{\alpha(1)\alpha(2)\dots\alpha(n)}$, for any n. Clearly code is a continuous map of A onto X. Moreover, code is constant on each connected component $J(\alpha) = \bigcap_{n=1}^{\infty} I_{\alpha(1)\alpha(2)\dots\alpha(n)}$ of A; we have $\text{code}(x) = \alpha$ for any $x \in J(\alpha)$. Thus if $A^* \subset A$ is a set that contains just one point from any connected component of A, then A^* is a Borel set and code is a continuous one-to-one map from A^* onto X. For $t \in T$, let $N_t = \{i \in \mathbb{N};\ t_i = t\}$. Apply Lemma 4.2 to the decomposition $\{N_t\}_{t\in T}$ of $\mathbb{N}$; let B be corresponding set. Then $\text{code}^{-1}(B) \cap A^*$ is an uncountable Borel set, hence it contains a nonempty perfect subset Q (cf. e.g., [K]).

Let $u, v \in Q$, $u \neq v$, let $\text{code}(u) = \{\alpha(i)\}_{i=1}^{\infty}$ and let $\text{code}(v) = \{\beta(i)\}_{i=1}^{\infty}$. By Lemma 4.2 there is an arbitrarily large j such that $\alpha(j) \neq \beta(j)$ and $t = t_j$. Hence (12) gives $F_{uv}(t) \leq F_{x_0x_1}(t)$, and since t is arbitrary in T, $F_{uv} \leq F_{x_0x_1}$. The argument for the second inequality in (8) is similar. Similarly, as at the end of proof of Lemma 4.1, we see that $\omega_f(u) \subset \tilde{\omega}$ for any $u \in Q$. By Lemma 2.5 there are $u^*, v^* \in \tilde{\omega}$ such that $\lim_{i\to\infty} \delta_{uu^*}(i) = 0$ and $\lim_{i\to\infty} \delta_{vv^*}(i) = 0$. Clearly equation (8) remains valid also for u^* and v^* instead of u and v. Since $\text{code}(u^*) = \{\alpha^*(i)\}_{i=1}^{\infty}$ and $\text{code}(v^*) = \{\beta^*(i)\}_{i=1}^{\infty}$ differ at infinitely many places Q and consequently, $\tilde{\omega}$ contains a nonempty perfect set with the requred properties. $\square$

Lemma 4.4. *Let $f \in C(\mathbb{S}, \mathbb{S})$ and let $\{\omega_i\}_{i=1}^{\infty}$ be the minimal periodic portions of basic sets of f. For any i, j, set $G_{ij} = \inf\{F_{uv};\ u \in \omega_i, v \in \omega_j\}$. Then*

- (i) *Each G_{ij} is zero on an interval $[0, \varepsilon(i,j)]$, where $\varepsilon(i,j)$ is a positive number.*
- (ii) *The set $\{G_{ij};\ \omega_i \cap \omega_j \neq \emptyset\}$ has a finite number of minimal elements.*
- (iii) *The set $\{G_{ii}\}_{i=1}^{\infty}$ has a finite number of minimal elements.*

Proof. (i) By (iii) of Theorem 2.1, there are distinct periodic points p in ω_i and q in ω_j. Since $\min_s \delta_{pq}(s) = \varepsilon > 0$ we have $F_{pq}(t) = 0$, and $F_{pq} \geq G_{ij}$ implies $G_{ij}(t) = 0$ for $t \leq \varepsilon$. Take $\varepsilon(i,j) = \varepsilon$.

(ii) We may assume that $\omega_i \neq \omega_j$, for $i \neq j$. Denote $K_i = \text{Env}(\omega_i)$, and $\varepsilon = \varepsilon(1,1)$ (from assertion (i)). We say that an ω_i is *extremal,* if $\text{diam}\,(\omega_i) > \varepsilon/2$ and if K_i is properly contained in no K_j. Note that there are only finitely many extremal ω_i's ((v) of Theorem 2.1). Let $\omega_1, \dots, \omega_{n(1)}$ be all extremal ω_i. Note that $n(1) \geq 1$ since f has positive topological entropy (cf. [M1]).

Let $m > 0$ be an integer. We say that an ω_i is *significant* if $\text{diam}\,(\omega_i) > \varepsilon/2$ and the period of ω_i is less than m. By Lemma 2.6 there are only finitely many significant ω_i's. Without loss of generality we may assume that there are integers $n(3) \geq n(2) \geq n(1) > 0$ such that the system $\{\omega_1, \dots, \omega_{n(2)}\}$ is invariant with respect to f, contains all extremal and all significant, and the system $\{\omega_{n(2)+1}, \dots, \omega_{n(3)}\}$ consists of all ω_i such that it has a common point with some ω_j, for $j \leq n(2)$.

From definitions of significant and extremal ω_i it follows that $n(2)$ and $n(3)$ but not $n(1)$ depend on the parameter m. We show that the minimal elements of $\{G_{ij};\ \omega_i \cap \omega_j \neq \emptyset\}$ are in the set $M = \{G_{ij};\ i, j \leq n(3)\}$, for sufficiently large m. Let $i > n(3)$ and $\omega_i \cap \omega_j \neq \emptyset$. Then ω_j cannot be significant and consequently $j > n(2)$. Take $u \in \omega_i$ and $v \in \omega_j$ and show that $F_{uv} \geq G$, for some $G \in M$. If $\operatorname{diam} f^s(\omega_i \cup \omega_j) \leq \varepsilon$ for any s then $F_{uv}(t) = 1$ for $t > \varepsilon$, and hence $F_{uv} \geq G_{11}$. Assume now that $\operatorname{diam} f^s(\omega_i \cup \omega_j) > \varepsilon$ for some s. Since the set $\{\omega_1, \ldots, \omega_{n(2)}\}$ is invariant, without loss of generality we can assume that $\operatorname{diam}(\omega_i \cup \omega_j) > \varepsilon$ and that

$$\operatorname{diam} f^s(\omega_i \cup \omega_j) \leq \operatorname{diam}(\omega_i \cup \omega_j) \text{ for any } s. \tag{14}$$

Since one of the sets ω_i, ω_j, say ω_i, has diameter $> \varepsilon/2$, there is an extremal ω_r such that $\omega_i \subset \operatorname{int}(K_r)$ (note that ω_i cannot be extremal since $i > n(1)$), and consequently with $\omega_i \cup \omega_j \subset \operatorname{int}(K_r)$, since $\omega_i \cap \omega_j \neq \emptyset$ (see (v) of Theorem 2.1). By (i) and (iii) of Theorem 2.1 there are periodic points $a, b \in \omega_r$ (sufficiently close to end points of K_r) with the following property: If J is an interval such that

$$J \cap \omega_r \text{ is infinite, } \operatorname{diam} J > \varepsilon, \text{ and } J \subset K_r \tag{15}$$

then $J \subset (a, b)$.

If $K_i \cap \omega_r$ would be infinite then the minimal compact periodical interval containing ω_i contains K_r and by Lemma 2.3 and (vi) of Theorem 2.1 $\omega_i = \omega_r$, which is impossible. Similarly $K_j \cap \omega_r$ is finite. Take $J = K_i \cup K_j$ satisfies (15) and get periodic points a, b such that $\omega_i \cup \omega_j \subset (a, b)$. Let t_0 be such that $\operatorname{diam}(\omega_i \cup \omega_j) < t_0 < \operatorname{diam}(\{a, b\})$. Take $\lambda_r = F_{a,b}(t_0)$. Clearly $\lambda_r < 1$. Let $\varepsilon(r, r)$ be as in (i). Now we have $G_{rr}(t) = 0$ for $t \leq \varepsilon(r, r)$, and by (14), $G_{rr}(t) \leq 1 = F_{uv}$ for $t > t_0$. And if $\varepsilon(r, r) < t_0$ then by Lemma 3.1, $G_{rr}(t) \leq F_{ab}(t) \leq \lambda_r < F_{uv}(t)$ whenever $\varepsilon(r, r) < t < t_0$ and $m \geq n(\varepsilon(r, r), \lambda_r)$. Thus $G_{rr} \leq F_{uv}$ if $m \geq n(\varepsilon(r, r), \lambda_r)$.

When the parameter $m = \max\{n(\varepsilon(r, r), \lambda_r);\ 1 \leq r \leq n(1)\}$ it follows that $\{G_{ij};\ i, j \leq n(3)\}$ contains the minimal elements of $\{G_{ij};\ \omega_i \cap \omega_j \neq \emptyset\}$.

(iii) It suffices to show that the minimal elements of $\{G_{ii}\}_{i=1}^{\infty}$ are contained in $\{G_{ij};\ i \leq n(2)\}$. But it follows from the argument given above. □

5. Proof of the main Theorem

We give the proof of Theorem B. Theorem A is its particular case.

Proof. We follow the idea of the proof of Theorem 2.4 in [ScSm].

(A) This result was already proved in [M1] as Theorem 2.2.

(B) Let $\{\omega_i\}_{i=1}^{\infty}$ be system of the minimal periodic portions of all basic sets. (This system is nonempty since topological entropy is positive [M1] and countable by (iv) of Theorem 2.1). Denote by $\tilde{\omega}_u$ the maximal ω-limit set containing $\omega_f(u)$.

(B1) Let $D = \{F_{uv};\ u \text{ and } v \text{ are synchronous}\}$ and $E = \{F_{uv};\ u, v \in \omega_i,\ \text{for some } i\}$. It is easy to see that $E \subset D$. To prove $D \subset E$ take $F_{uv} \in D$. If $\tilde{\omega}_u$ $(= \tilde{\omega}_v)$ is a solenoid then trajectories of u and v enter into periodical decomposition of arbitrarily high order and consequently $F_{uv} = \chi_{(0,\infty)} \in E$. For details see [M1]. If $\tilde{\omega}_u$ is a basic set then by Lemma 2.5 there are $u^*, v^* \in$

$\tilde{\omega}_u$ such that $F_{uv} = F_{u^*v^*}$. Thus $D = E$ and Lemma 4.4 gives the result on the spectrum $\Sigma(f)$.

Now let $D_w = \{F_{uv};\ u, v$ satisfy $\liminf_{i\to\infty} \delta_{uv}(i) = 0\}$ and $E_w = \{F_{uv};\ u \in \omega_i, v \in \omega_j$ and $\omega_i \cap \omega_j \neq \emptyset,$ for some i and $j\}$. Let $F_{uv} \in D_w$. Similarly as before either $F_{uv} = \chi_{(0,\infty)}$ or there are are points $u^* \in \tilde{\omega}_i$, $v^* \in \tilde{\omega}_j$ such that $\lim_{i\to\infty} \delta_{u^*u}(i) = 0$, $\lim_{i\to\infty} \delta_{v^*v}(i) = 0$ and this with

$$\liminf_{i\to\infty} \delta_{uv}(i) = 0 \tag{16}$$

gives $\tilde{\omega}_u \cap \tilde{\omega}_v \neq \emptyset$. Thus $F_{u,v} = F_{u^*v^*} \in E_w$. Consequently $D_w \subset E_w$.

To prove $E_w \subset D_w$, take $F_{uv} \in E_w$, $u \in \tilde{\omega}_i$, $v \in \tilde{\omega}_j$, $w \in \tilde{\omega}_i \cap \tilde{\omega}_j$ and $Q = \{(u, v), (w, w)\}$. Now apply Lemma 4.1 to get x, y such that $F_{xy} \le F_{uv}$ and $F^*_{xy} = \chi_{(0,\infty)}$. Since $\liminf_{i\to\infty} \delta_{xy}(i) = 0$ we have $\omega_f(x) \cap \omega_f(y) \neq \emptyset$. Thus $F_{xy} \in D_w$. Since $D_w \subset E_w$ and E_w has the lower bounds in D_w, both D_w and E_w have the same system $\Sigma_w(f)$ of minimal elements and application of Lemma 4.4 on E_w completes the proof.

(B2) For any $k \le m$ there are $x, y \in \omega_i$ for some i such that $F_{xy} = F_k$ (see proof (B1)). Existence of P_k now follows by Lemma 4.1 with $Q = \{(x, y), (x, x)\}$ and Lemmas 3.3 and 4.3.

(B4) Let any $u, v \in S$ satisfy (16). If for some $u \in S$ $\tilde{\omega}_u$ is solenoid then similarly as in the proof of (B1) we get $F_{uv} = \chi_{(0,\infty)}$ for any $u, v \in S$ and $S = S_0 \cup \emptyset \cup \emptyset$ is the corresponding decomposition.

Assume that for every $u \in S$ the set $\tilde{\omega}_u$ is a basic set. For every $u \in S$ denote by $u^* \in \tilde{\omega}_u$ a point such that $\lim_{n\to\infty} \delta_{u^*u}(n) = 0$ (Lemma 2.5). Let $T_i = \{u \in S;\ u^* \in \omega_i\}$, Lemma 2.5 shows that $S = \bigcup_{i=1}^{\infty} T_i$. Suppose that four distinct sets $T_{j(1)}$, $T_{j(2)}$, $T_{j(3)}$ and $T_{(j(4))}$ are nonempty. Any two sets $\omega_{j(r)}$, $\omega_{j(s)}$, $1 \le r, s \le 3$, have a point in common (since $\liminf_{i\to\infty} \delta_{u(r)u(s)}(i) = 0$, for $u(i) \in T_i$). Now by (v) of Theorem 2.1 two of sets $\omega_{j(1)}, \omega_{j(2)}, \omega_{j(3)}$ and $\omega_{j(4)}$ must coincide (say $\omega_{j(3)}$ and $\omega_{j(4)}$).

By Lemma 4.4, for any $1 \le k \le 3$ and any $u, v \in T_{j(k)}$ there is $j \le m$ such that $F_{uv} \ge G_{j(k)j(k)} \ge F_j$. Consequently $S = T_{j(1)} \cup T_{j(2)} \cup T_{j(3)}$.

(B3) If $j > m$ and $F_{uv} = F_j$ for any distinct $u, v \in S$ then u^* and v^* belong to different $T_{j(k)}$. Elsewhere F_{uv} cannot be minimal. This shows that every $T_{j(k)}$ contains one point. □

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