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Normal forms of sl_3-valued zero curvature representations

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Abstract

We find normal forms for an sl_3-valued zero curvature representation.

1 Introduction

Zero curvature representations (ZCR) rank among the most important attributes of integrable partial differential equations [6]. A ZCR is usually treated as a special case of the Wahlquist–Estabrook prolongation structure [8], but the famous Wahlquist–Estabrook procedure is not sufficient for obtaining a complete classification of integrable systems. The main obstacle consists in the presence of a large group of gauge transformations. Thus we are naturally led to the problem of introduction of normal forms of ZCR's such that every orbit of the gauge action contains at least one normal form.

In nineties, independently M. Marvan [2] and S. Yu. Sakovich [5] introduced a characteristic element of a ZCR, which is a matrix that transforms by conjugation during gauge transformations of the ZCR. It follows that one can reduce the gauge freedom by putting the characteristic element in the Jordan normal form. There is a remaining gauge freedom, which can be used for further reduction of one of the matrices constituting the ZCR. This is rather similar to classification of pairs of matrices under simultaneous conjugation, developed by Belitskiĭ [1].

In case of the Lie algebra sl_2 a solution of the problem can be found in [3]. This made possible the subsequent complete classification of second-order evolution equations possessing an sl_2-valued ZCR [4].

In this work we try to obtain such a classification in case of sl_3. The number of possible normal forms is much higher than in case of sl_2. As examples, we consider the Tzitzéica equation [7], whose ZCR is known since 1910, Sawada-Kotera equation and the Kupershmidt equation.

2 Preliminaries

Let us consider a system of nonlinear differential equations

$$F^l(t,x,u^k,\ldots,u^k_I,\ldots)=0, \tag{1}$$

in two independent variables t and x, a finite number of dependent variables u^k and their derivatives u^k_I, where I denotes a finite symmetric multiindex over t and x.

Let J^∞ be an infinite-dimensional jet space such that t, x, u^k, u^k_I are local jet coordinates on J^∞. We have two distinguished vector fields on J^∞

$$D_t=\frac{\partial}{\partial t}+\sum_{k,I}u^k_{It}\frac{\partial}{\partial u^k_I},\qquad D_x=\frac{\partial}{\partial x}+\sum_{k,I}u^k_{Ix}\frac{\partial}{\partial u^k_I},$$

which are called *total derivatives.* Let g be a matrix Lie algebra. By a g-valued *zero curvature representation* (ZCR) for (1) we mean two g-valued functions A,B which satisfy

$$D_tA-D_xB+[A,B]=0$$

as a consequence of (1). Let G be the connected and simply connected matrix Lie group associated with g. Then for every G-valued function W we define the *gauge transformation* of ZCR (A,B) by the formulas

$$A^W:=D_xW\cdot W^{-1}+W\cdot A\cdot W^{-1},$$
$$B^W:=D_tW\cdot W^{-1}+W\cdot B\cdot W^{-1}.$$

As is well known, (A^W,B^W) is a ZCR too, and we say that it is *gauge equivalent* to (A,B).

We define a new differential operator $\widehat{D}_I$:

$$\widehat{D}_xM=D_xM-[A,M],\qquad \widehat{D}_tM=D_tM-[B,M]$$

and $\widehat{D}_I=D_{i_1}\cdots D_{i_\kappa}$ where $I=(i_1\cdots i_\kappa)$ as usual. A *characteristic element* R is a g-valued function defined in [2]. The following assertion holds:

Proposition 2.1 ([2])
1) Gauge equivalent ZCR's have conjugate characteristic elements.
2) The characteristic element R satisfies

$$\sum_{k,I}(-1)^{|I|}\widehat{D}_I\left(\frac{\partial F^l}{\partial u^k_I}R_l\right)=0.$$

If a ZCR (A, B) is gauge equivalent to another ZCR with coefficients in a proper subalgebra of g, then we say that ZCR is *reducible.* Otherwise it is said to be *irreducible.* A ZCR gauge equivalent to zero is called *trivial.* A very important case is a ZCR with coefficients in a non-solvable Lie algebra. The simplest case of a non-solvable Lie algebra is the algebra sl_2. In [3] the following proposition is obtained:

Proposition 2.2 *Let (A, B) be an irreducible sl_2-valued ZCR, let $R \neq 0$ be its characteristic element. Then we have one of the two following normal forms for R and A :*
– Nilpotent case

$$R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}.$$

– Diagonal case

$$R = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}, \qquad A = \begin{pmatrix} a_1 & 1 \\ a_3 & -a_1 \end{pmatrix}.$$

3 Normal forms

In this section we define the normal form of g-valued ZCR and explain the method to find them. The main idea is taken from the first part of proposition 2.1. Gauge equivalent ZCR's have conjugate characteristic elements, therefore we can restrict ourselves to the characteristic elements in the Jordan normal form. Since the gauge transformation is a group action, it is possible to consider the stabilizer group of the characteristic element, which is a proper subgroup of G. The stabilizer is usually rather small (see Table 1), therefore we can compute its action on the matrix A and find the corresponding normal forms rather easily. We aim at finding the minimal set of normal forms. In the case of the diagonal characteristic element R we can achieve substantial reduction by taking into account permutations of the Jordan blocks.

In this work we distinguish between *normal forms* and *seminormal forms*. We say, that we have the normal form if we have just finite number of possibilities of a choice of the corresponding gauge matrix (see section 5). If our choice of the corresponding gauge matrix depend on at least one arbitrary function, we say, that we have the seminormal form. In this case we may use the residual gauge freedom to transform the matrix B.

The following table lists all possible Jordan forms J_i of sl_3-matrices and the corresponding stabilizers W_i, where w_j denote arbitrary complex numbers such that all algebraic operations make sense. J_2 and J_4 are degenerate cases of J_1

and J_3, respectively, when the two eigenvalues coincide and the dimension of the stabilizer raises from two to four. Cases J_2 and J_4 are treated at the end of this work.

$$
\begin{aligned}
J_1 &= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1-\lambda_2 \end{pmatrix}; \quad \lambda_1 \neq \lambda_2, & W_1 &= \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_1 w_2^{-1} \end{pmatrix}, \\
J_2 &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}; \quad \lambda \neq 0, & W_2 &= \begin{pmatrix} w_{11} & w_{12} & 0 \\ w_{21} & w_{22} & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix}, \\
J_3 &= \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}; \quad \lambda \neq 0, & W_3 &= \begin{pmatrix} w_1 & 0 & 0 \\ w_2 & w_1 & 0 \\ 0 & 0 & w_1^{-2} \end{pmatrix}, \\
J_4 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & W_4 &= \begin{pmatrix} w_1 & 0 & 0 \\ w_2 & w_1 & w_3 \\ w_4 & 0 & w_1^{-2} \end{pmatrix}, \\
J_5 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & W_5 &= \begin{pmatrix} 1 & 0 & 0 \\ w_2 & 1 & 0 \\ w_3 & w_2 & 1 \end{pmatrix},
\end{aligned}
$$

where $\quad Z = w_{11}w_{22} - w_{12}w_{21}.$

Table 1: Jordan forms and the corresponding stabilizers

4 Subalgebras of algebra sl_3

For further reference, we list here several subalgebras of sl_3. Two subalgebras a, b are said to be conjugate, if there exist $S \in SL_3$ such that $a = SbS^{-1}$. Note that for constant matrices $S \in SL_3$ conjugation and gauge equivalence coincide. One obvious automorphismus of sl_3 is also $A \longmapsto -A^\top$, which we call *transposition*. We introduce six permutation matrices

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$P_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The following four types of subalgebras appear in this work:

Type 1. Six 6-dimensional subalgebras consisting of traceless matrices A of either the form:

$$A = \begin{pmatrix} . & . & . \\ . & . & . \\ 0 & 0 & . \end{pmatrix}, \begin{pmatrix} . & . & . \\ 0 & . & . \\ 0 & . & . \end{pmatrix}, \begin{pmatrix} . & 0 & 0 \\ . & . & . \\ . & . & . \end{pmatrix}, \begin{pmatrix} . & . & 0 \\ . & . & 0 \\ . & . & . \end{pmatrix}, \begin{pmatrix} . & 0 & . \\ . & . & . \\ . & 0 & . \end{pmatrix}, \begin{pmatrix} . & . & . \\ 0 & . & 0 \\ . & . & . \end{pmatrix}.$$

This six subalgebras are mutually isomorphic via transposition or conjugation.

Type 2. Two subalgebras consisting of traceless matrices A of either the form:

$$A = \begin{pmatrix} . & . & 0 \\ . & . & 0 \\ 0 & 0 & . \end{pmatrix}, \begin{pmatrix} . & 0 & 0 \\ 0 & . & . \\ 0 & . & . \end{pmatrix}.$$

This two subalgebras are isomorphic to the algebra gl_2.

Type 3. Two subalgebras consisting of all lower(upper)-triangular 3×3 traceless matrices A and four subalgebras mutually isomorphic via conjugation of either the form:

$$A = \begin{pmatrix} . & 0 & 0 \\ . & . & 0 \\ . & . & . \end{pmatrix}, \begin{pmatrix} . & 0 & 0 \\ . & . & . \\ . & 0 & . \end{pmatrix}, \begin{pmatrix} . & 0 & . \\ . & . & . \\ 0 & 0 & . \end{pmatrix}, \begin{pmatrix} . & . & 0 \\ 0 & . & 0 \\ . & . & . \end{pmatrix}, \begin{pmatrix} . & . & . \\ 0 & . & 0 \\ 0 & . & . \end{pmatrix}, \begin{pmatrix} . & . & . \\ 0 & . & . \\ 0 & 0 & . \end{pmatrix}.$$

Type 4. The abelian subalgebra consisting of all diagonal 3×3 traceless matrices.

5 Case J_1

In this section we solve the classification problem in case of the characteristic element R whose Jordan normal form is diagonal (case J_1). The diagonal Jordan normal form is unique up to the order of the elements on the diagonal, i.e., up to conjugation with respect to one of the permutation matrix $P_0, \ldots, P_5$. Given a matrix A, the corresponding gauge equivalent matrices will be $A_i = D_x P_i . P_i^{-1} + P_i A P_i^{-1} = P_i A P_i^{-1}$, $i = 0, 1, \ldots, 5$, namely

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{11} & a_{13} & a_{12} \\ a_{31} & a_{33} & a_{32} \\ a_{21} & a_{23} & a_{22} \end{pmatrix},$$
$$A_2 = \begin{pmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ a_{12} & a_{13} & a_{11} \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{pmatrix},$$
$$A_4 = \begin{pmatrix} a_{33} & a_{31} & a_{32} \\ a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \end{pmatrix}, \quad A_5 = \begin{pmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{pmatrix}.$$

Remark 5.1 Note that A_i is gauge equivalent to A for every $i = 0, 1, \ldots, 5$.

The following algorithm assigns a normal form to the matrix A. The input is the matrix A. Dots denote arbitrary elements.

Case 1. If there exists $i = 0, 1, \ldots, 5$ such that $a_{21} \neq 0$ and $a_{32} \neq 0$ in $A = A_i$, then the *normal form* is

$$N_1^1 = \begin{pmatrix} . & . & . \\ 1 & . & . \\ . & 1 & . \end{pmatrix}.$$

The gauge matrix which sends A to N_1^1 is

$$W_1^1 = \begin{pmatrix} a_{32}^{1/3} a_{21}^{2/3} & 0 & 0 \\ 0 & a_{32}^{1/3} a_{21}^{-1/3} & 0 \\ 0 & 0 & a_{32}^{2/3} a_{21}^{-1/3} \end{pmatrix}.$$

One easily sees that the matrix W_1^1 is unique up to the choice of cubic roots, hence N_1^1 is the *normal form* (see section 3).

Case 2. Otherwise, if there exists $i = 0, 1, \ldots, 5$ such that $a_{21} \neq 0$, $a_{32} = 0$ and $a_{31} \neq 0$ in $A = A_i$, then we may assume that $a_{23} = 0$ as well. Indeed, if

$a_{23} \neq 0$ in A, then a_{32} and a_{21} are nonzero in $A_j = P_1 A P_1^{-1}$ and we would have the first case. The normal form is

$$N_1^2 = \begin{pmatrix} . & . & . \\ 1 & . & 0 \\ 1 & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_1^2 = \begin{pmatrix} a_{31}^{1/3} a_{21}^{1/3} & 0 & 0 \\ 0 & a_{31}^{1/3} a_{21}^{-2/3} & 0 \\ 0 & 0 & a_{21}^{1/3} a_{31}^{-2/3} \end{pmatrix}.$$

Case 3. Otherwise, if there exists $i = 0, 1, \ldots, 5$ such that $a_{21} \neq 0$, $a_{32} = 0$, $a_{31} = 0$ and $a_{23} \neq 0$ in $A = A_i$, then we may assume that $a_{13} = 0$. Indeed, when $a_{13} \neq 0$ in A, then $a_{21} \neq 0$ in $A_j = P_4 A P_4^{-1}$ and nonzero a_{21} in A imply nonzero a_{32} in A_j, and we would have the first case again. The normal form is

$$N_1^3 = \begin{pmatrix} . & . & 0 \\ 1 & . & 1 \\ 0 & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_1^3 = \begin{pmatrix} a_{21}^{2/3} a_{23}^{-1/3} & 0 & 0 \\ 0 & a_{23}^{-1/3} a_{21}^{-1/3} & 0 \\ 0 & 0 & a_{23}^{2/3} a_{21}^{-1/3} \end{pmatrix}.$$

The matrix N_1^3 belongs to the subalgebra of Type 1.

Case 4. Otherwise, if there exists $i = 0, 1, \ldots, 5$ such that $a_{21} \neq 0$, $a_{32} = 0$, $a_{31} = 0$ and $a_{23} = 0$ in $A = A_i$, then we obtain a *seminormal form*

$$N_1^4 = \begin{pmatrix} . & . & 0 \\ 1 & . & 0 \\ 0 & 0 & . \end{pmatrix}.$$

Indeed, using the same argument as in the Case 3 we may assume that $a_{13} = 0$, the corresponding gauge matrix being, for example,

$$W_1^4 = \begin{pmatrix} a_{21} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{21}^{-1} \end{pmatrix}.$$

However, the most general gauge matrix is

$$\begin{pmatrix} a_{21} & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & a_{21}^{-1} \end{pmatrix}$$

and depends on the choice of one arbitrary function w_2. If we set $w_2 = 1$, then we obtain W_1^4. Hence N_1^4 is the seminormal form. The matrix N_1^4 belongs to the subalgebra of Type 2.

Case 5. If $a_{21} = 0$ for all A_i, then all the off-diagonal elements must be zero, therefore the seminormal form is

$$A = N_1^5 = \begin{pmatrix} . & 0 & 0 \\ 0 & . & 0 \\ 0 & 0 & . \end{pmatrix}.$$

The matrix N_1^5 belongs to the subalgebra of Type 4.

As a matter of fact, we have proved:

Theorem 5.2 *In a ZCR such that its characteristic element has the diagonal Jordan normal form J_1 the matrix A has one of the above normal forms N_1^1, N_1^2, N_1^3, or seminormal forms N_1^4, N_1^5. If A does not belong to a proper subalgebra of sl_3, then A has one of the above normal forms N_1^1, N_1^2.*

Example 5.3 The Tzitzéica equation [7]:

$$u_{tx} = e^u - e^{-2u}.$$

The corresponding ZCR, which depends on a parameter $m \neq 0$, is

$$A = \begin{pmatrix} -u_x & 0 & m \\ m & u_x & 0 \\ 0 & m & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \dfrac{\mathrm{e}^{-2u}}{m} & 0 \\ 0 & 0 & \dfrac{\mathrm{e}^{u}}{m} \\ \dfrac{\mathrm{e}^{u}}{m} & 0 & 0 \end{pmatrix}.$$

The matrix A belongs to Case 1 with the normal form N_1^1. Namely, the normal forms of the characteristic element R and the matrix A are

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -u_x & 0 & m^3 \\ 1 & u_x & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

6 Case J_3

In this section we solve the classification problem for characteristic element in the form J_3 with the corresponding stabilizer W_3 (see Table 1). We first find all relevant normal forms and then we select a minimal set of normal forms which can occur in orbits of the gauge action.

As a result we obtain the following algorithm which assigns a normal form to the general sl_3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{11} - a_{22} \end{pmatrix}.$$

Case 1. If $a_{13} \neq 0$, then the normal form is

$$N_3^1 = \begin{pmatrix} . & . & 1 \\ . & . & 0 \\ . & . & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_3^1 = \begin{pmatrix} a_{13}^{-1/3} & 0 & 0 \\ -\dfrac{a_{23}}{a_{13}^{4/3}} & a_{13}^{-1/3} & 0 \\ 0 & 0 & a_{13}^{2/3} \end{pmatrix}.$$

Case 2. Otherwise, if $a_{13} = 0, a_{32} \neq 0$, then the normal form is

$$N_3^2 = \begin{pmatrix} . & . & 0 \\ . & . & . \\ 0 & 1 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_3^2 = \begin{pmatrix} a_{32}^{1/3} & 0 & 0 \\ \dfrac{a_{31}}{a_{32}^{2/3}} & a_{32}^{1/3} & 0 \\ 0 & 0 & a_{32}^{-2/3} \end{pmatrix}.$$

Case 3. If $a_{13} = 0, a_{32} = 0, a_{23} \neq 0, a_{12} \neq 0$, then the normal form is

$$N_3^3 = \begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_3^3 = \begin{pmatrix} a_{23}^{-1/3} & 0 & 0 \\ \dfrac{-D_x a_{23} + 3a_{23}a_{11}}{3a_{12}a_{23}^{4/3}} & a_{23}^{-1/3} & 0 \\ 0 & 0 & a_{23}^{2/3} \end{pmatrix}.$$

Case 4. If $a_{13} = 0, a_{32} = 0, a_{23} \neq 0, a_{12} = 0$, then the seminormal form is

$$N_3^4 = \begin{pmatrix} . & 0 & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_3^4 = \begin{pmatrix} a_{23}^{-1/3} & 0 & 0 \\ 0 & a_{23}^{-1/3} & 0 \\ 0 & 0 & a_{23}^{2/3} \end{pmatrix}.$$

The matrix N_3^4 belongs to the subalgebra of Type 3. Indeed, applying the permutation matrix P_1 to matrix N_3^4 by conjugation we obtain a lower triangular matrix.

Case 5. If $a_{13} = 0, a_{32} = 0, a_{23} = 0, a_{31} \neq 0, a_{12} \neq 0$, then the normal form is

$$N_3^5 = \begin{pmatrix} 0 & . & 0 \\ . & . & 0 \\ 1 & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_3^5 = \begin{pmatrix} a_{31}^{1/3} & 0 & 0 \\ \dfrac{D_x a_{31} + 3a_{31}a_{11}}{3a_{12}a_{31}^{2/3}} & a_{31}^{1/3} & 0 \\ 0 & 0 & a_{31}^{-2/3} \end{pmatrix}.$$

The matrix N_3^5 belongs to the subalgebra of Type 1.

Case 6. If $a_{13} = 0, a_{32} = 0, a_{23} = 0, a_{31} \neq 0, a_{12} = 0$, then the seminormal form is

$$N_3^6 = \begin{pmatrix} . & 0 & 0 \\ . & . & 0 \\ 1 & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_3^6 = \begin{pmatrix} a_{31}^{1/3} & 0 & 0 \\ 0 & a_{31}^{1/3} & 0 \\ 0 & 0 & a_{31}^{-2/3} \end{pmatrix}.$$

The matrix N_3^6 belongs to the subalgebra of Type 3.

Case 7. Otherwise, if $a_{13} = 0, a_{32} = 0, a_{23} = 0, a_{31} = 0, a_{12} \neq 0$, then the seminormal form is

$$N_3^7 = \begin{pmatrix} 0 & . & 0 \\ . & . & 0 \\ 0 & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_3^7 = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{a_{11}}{a_{12}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Obviously, N_3^7 belongs to the subalgebra of Type 2.

Case 8. Otherwise, if $a_{13} = 0, a_{32} = 0, a_{23} = 0, a_{31} = 0, a_{12} = 0$, then the seminormal form is

$$N_3^8 = \begin{pmatrix} . & 0 & 0 \\ . & . & 0 \\ 0 & 0 & . \end{pmatrix}.$$

Matrices of this form constitute a 3-dimensional solvable subalgebra of sl_3.

Whence we have proved the next theorem:

Theorem 6.1 *In a ZCR such that its characteristic element has the Jordan normal form in the form J_3 the matrix A has one of the above normal forms $N_3^1, N_3^2, N_3^3, N_3^5$ or seminormal forms $N_3^4, N_3^6, N_3^7, N_3^8$. If A does not belong to a proper subalgebra of sl_3, then A has one of the above normal forms N_3^1, N_3^2, N_3^3.*

7 Case J_5

In this section we solve the classification problem for characteristic element in the form J_5 (see Table 1). Similarly as in the previous case, we first find all relevant normal forms an then we select a minimal set of them. The matrix A is considered in the same form as in the case of J_3. As a result we obtain the following:

Case 1. If $a_{13} \neq 0$, then the normal form is

$$N_5^1 = \begin{pmatrix} 0 & 0 & . \\ . & . & . \\ . & . & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_5^1 = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{a_{12}}{a_{13}} & 1 & 0 \\ \dfrac{a_{11}}{a_{13}} & \dfrac{a_{12}}{a_{13}} & 1 \end{pmatrix}$$

Case 2. Otherwise, if $a_{13} = 0, a_{12} \neq 0$, then normal form is

$$N_5^2 = \begin{pmatrix} 0 & . & 0 \\ . & . & . \\ . & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_5^2 = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{a_{11}}{a_{12}} & 1 & 0 \\ w_3 & \dfrac{a_{11}}{a_{12}} & 1 \end{pmatrix},$$

where $w_3 = (a_{11}D_x a_{12} - a_{12}D_x a_{11} + a_{23}a_{11}^2 - a_{32}a_{12}^2 - 2a_{22}a_{12}a_{11} - a_{12}a_{11}^2)/a_{12}^3$.

Case 3. If $a_{13} = 0, a_{12} = 0, a_{23} \neq 0$, then normal form is

$$N_5^3 = \begin{pmatrix} . & 0 & 0 \\ 0 & . & . \\ . & . & 0 \end{pmatrix},$$

the corresponding gauge matrix being

$$W_5^3 = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{a_{22} + a_{11}}{a_{23}} & 1 & 0 \\ w_3 & \dfrac{a_{22} + a_{11}}{a_{23}} & 1 \end{pmatrix},$$

where $w_3 = (a_{23}D_x a_{22} - a_{11}D_x a_{23} - a_{22}D_x a_{23} + a_{23}D_x a_{11} + 2a_{22}a_{23}a_{11} + a_{23}^2 a_{21} + 2a_{23}a_{11}^2)/a_{23}^3$. The matrix N_5^3 belongs to the subalgebra of Type 1.

Case 4. Otherwise, if $a_{13} = 0, a_{12} = 0, a_{23} = 0$, then the seminormal form is

$$N_5^4 = \begin{pmatrix} . & 0 & 0 \\ . & . & 0 \\ . & . & . \end{pmatrix}.$$

Matrices of this form fall to the subalgebra of Type 3.

Again we have, in fact, proved the next theorem:

Theorem 7.1 *In a ZCR such that its characteristic element has the Jordan normal form J_5 the matrix A has one of the above normal forms N_5^1, N_5^2, N_5^3 or seminormal form N_5^4. If A does not belong to a proper subalgebra of sl_3, then it has one of the above normal forms N_5^1, N_5^2.*

Example 7.2 The Kupershmidt equation:

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x$$

The corresponding ZCR, which depends on a parameter $m \neq 0$, is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -u & 0 & 1 \\ m & -u & 0 \end{pmatrix},$$

the matrix B is very large, hence omitted. The matrix A belongs to Case 2 with the normal form N_5^2, namely,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -2u & 0 & 1 \\ u_x + m & 0 & 0 \end{pmatrix}.$$

8 Case J_2

The following two cases J_2 and J_4 of characteristic elements are singular and the number of parameters increase in the corresponding stabilizer subgroups from two to four (see Table 1). We begin with the classification problem for characteristic element in the form J_2.

Let $K = a_{13}D_x a_{23} - a_{23}D_x a_{13} + a_{11}a_{13}a_{23} - a_{21}a_{13}^2 + a_{12}a_{23}^2 - a_{22}a_{13}a_{23}$, $L = a_{32}D_x a_{31} - a_{31}D_x a_{32} + a_{11}a_{32}a_{31} + a_{21}a_{32}^2 - a_{12}a_{31}^2 - a_{22}a_{32}a_{31}$, and $R = a_{13}a_{31} + a_{23}a_{32}$.

Case 1. If $K \neq 0$, then the normal form is

$$N_2^1 = \begin{pmatrix} 0 & 1 & 0 \\ . & . & 1 \\ . & . & . \end{pmatrix}.$$

The corresponding gauge matrix W_2^1 is found to be

$$w_{11} = -\frac{a_{23}}{K^{2/3}}, \quad w_{12} = \frac{a_{13}}{K^{2/3}},$$

$$w_{21} = \frac{\frac{2}{3}\, a_{23} K^{-1} D_x K - D_x a_{23} - a_{11} a_{23} + a_{13} a_{21}}{K^{2/3}},$$

$$w_{22} = \frac{-\frac{2}{3}\, a_{13} K^{-1} D_x K + D_x a_{13} - a_{12} a_{23} + a_{22} a_{13}}{K^{2/3}}.$$

Case 2. If $K = 0, L \neq 0, R \neq 0$, then the normal form is

$$N_2^2 = \begin{pmatrix} . & 0 & 0 \\ 1 & . & . \\ 0 & 1 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_2^2 = \begin{pmatrix} \dfrac{L^{2/3} a_{23}}{R} & -\dfrac{L^{2/3} a_{13}}{R} & 0 \\ \dfrac{a_{31}}{L^{1/3}} & \dfrac{a_{32}}{L^{1/3}} & 0 \\ 0 & 0 & L^{-1/3} \end{pmatrix}.$$

Indeed, applying W_2^2 to general sl_3 matrix A (see section 6) we obtain

$$A^{W_2^2} = \begin{pmatrix} . & \dfrac{KL}{R^2} & 0 \\ 1 & . & . \\ 0 & 1 & . \end{pmatrix},$$

and we see that for $K = 0$ we have $A^{W_2^2} = N_2^2$.

The normal form N_2^2 falls to the subalgebra of Type 1.

Case 3. If $K = 0, L \neq 0, R = 0$, then the normal form is

$$N_2^3 = \begin{pmatrix} 0 & . & 0 \\ 1 & . & 0 \\ 0 & 1 & . \end{pmatrix},$$

the corresponding gauge matrix W_2^3 is found to be

$$\begin{aligned}
w_{11} &= \frac{L^{2/3} + a_{31}w_{12}}{a_{32}}, \\
w_{12} &= \frac{-\frac{2}{3}\,a_{32}L^{-1}D_xL + D_xa_{32} - a_{12}a_{31} + a_{11}a_{32}}{L^{1/3}}, \\
w_{21} &= \frac{a_{31}}{L^{1/3}}, \quad w_{22} = \frac{a_{32}}{L^{1/3}}.
\end{aligned}$$

Note that a_{32} in the denominator of w_{11} cancels out after evaluating D_xL in w_{12}. E.g., for $a_{32} = 0$ we obtain

$$w_{11} = \frac{1}{3}\,\frac{3a_{31}^3a_{12}a_{22} + a_{31}^2a_{12}D_xa_{31} + 2a_{31}^3D_xa_{12}}{(a_{31}^2a_{12})^{4/3}},$$

while $a_{31}a_{12} \neq 0$ is just a consequence of $L \neq 0$. Indeed, applying W_2^3 to general sl_3 matrix A we obtain

$$A^{W_2^3} = \begin{pmatrix} 0 & . & K + RC \\ 1 & . & R \\ 0 & 1 & . \end{pmatrix},$$

for appropriate C and we see that for $K = 0, R = 0$ we have $A^{W_2^3} = N_2^3$.

The normal form N_2^3 falls to the subalgebra of Type 1.

Case 4. If $K = 0, L = 0, R \neq 0$, then the seminormal form is

$$N_2^4 = \begin{pmatrix} . & 0 & 0 \\ 0 & . & . \\ 0 & 1 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_2^4 = \begin{pmatrix} a_{23} & -a_{13} & 0 \\ \dfrac{a_{31}}{\sqrt{R}} & \dfrac{a_{32}}{\sqrt{R}} & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{R}} \end{pmatrix}.$$

Indeed, applying W_2^4 to general sl_3 matrix A we obtain

$$A^{W_2^4} = \begin{pmatrix} . & -\dfrac{K}{R^{1/2}} & 0 \\ \dfrac{L}{R^{3/2}} & . & . \\ 0 & 1 & . \end{pmatrix},$$

and we see that for $K=0, L=0$ we have $A^{W_2^4}=N_2^4$.

The seminormal form N_2^4 falls to the subalgebra of Type 2.

For $K=0, L=0, R=0$ we have three subcases:

Case 5a. If $a_{13}\neq 0$ or $a_{23}\neq 0$, then the seminormal form is

$$N_2^{5a}=\begin{pmatrix} . & 0 & 0\\ . & . & .\\ . & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_2^{5a}=\begin{pmatrix} a_{23} & -a_{13} & 0\\ w_{21} & w_{22} & 0\\ 0 & 0 & \dfrac{1}{w_{21}a_{13}+w_{22}a_{23}} \end{pmatrix}$$

for arbitrary nonzero parameters w_{21} and w_{22}. Indeed, applying W_2^{5a} to general sl_3 matrix A we obtain

$$A^{W_2^{5a}}=\begin{pmatrix} . & \dfrac{K}{w_{21}a_{13}+w_{22}a_{23}} & 0\\ . & . & .\\ . & \dfrac{R}{(w_{21}a_{13}+w_{22}a_{23})^2} & . \end{pmatrix},$$

and we see that for $K=0, R=0$ we have $A^{W_2^{5a}}=N_2^{5a}$. Note, that in this case $L=K(a_{32}/a_{13})^2$ or $L=K(a_{31}/a_{23})^2$.

The seminormal form N_2^{5a} falls to the subalgebra of Type 3.

Case 5b. If $a_{31}\neq 0$ or $a_{32}\neq 0$, then the seminormal form is

$$N_2^{5b}=\begin{pmatrix} . & . & .\\ 0 & . & 0\\ 0 & . & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_2^{5b}=\begin{pmatrix} w_{11} & w_{12} & 0\\ a_{31} & a_{32} & 0\\ 0 & 0 & \dfrac{1}{w_{11}a_{32}-w_{12}a_{31}} \end{pmatrix}$$

for arbitrary nonzero parameters w_{11} and w_{12}. Indeed, applying W_2^{5b} to general sl_3 matrix A we obtain

$$A^{W_2^{5b}} = \begin{pmatrix} . & . & . \\ \dfrac{L}{w_{11}a_{32} - w_{12}a_{31}} & . & R(w_{11}a_{32} - w_{12}a_{31}) \\ 0 & . & . \end{pmatrix},$$

and we see that for $L = 0, R = 0$ we have $A^{W_2^{5b}} = N_2^{5b}$. Note, that in this case $K = L(a_{23}/a_{31})^2$ or $K = L(a_{13}/a_{32})^2$.

The seminormal form N_2^{5b} falls to the subalgebra of Type 3.

Case 5c. If $a_{13} = 0, a_{23} = 0, a_{31} = 0, a_{32} = 0$, then the seminormal form is

$$N_2^{5c} = \begin{pmatrix} . & . & 0 \\ . & . & 0 \\ 0 & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being just the unit matrix. The seminormal form N_2^{5c} falls to the subalgebra of Type 2.

Again we have, in fact, proved the next theorem:

Theorem 8.1 *In a ZCR such that its characteristic element has the diagonal Jordan normal form J_2 the matrix A has one of the above normal forms N_2^1, N_2^2, N_2^3 or seminormal forms $N_2^4, N_2^{5a}, N_2^{5b}, N_2^{5c}$. If A does not belong to a proper subalgebra of sl_3, then A has the above normal form N_2^1.*

9 Case J_4

In this section we solve the classification problem for characteristic element in the form J_4 (see Table 1).

Let $M = a_{12}D_x a_{13} - a_{13}D_x a_{12} - 2a_{12}a_{13}a_{22} + a_{23}a_{12}^2 - a_{32}a_{13}^2 - a_{11}a_{12}a_{13}$ and $N = a_{12}D_x a_{32} - a_{32}D_x a_{12} + 2a_{11}a_{12}a_{32} - a_{31}a_{12}^2 + a_{13}a_{32}^2 + a_{12}a_{22}a_{32}$.

Case 1. If $a_{12} \neq 0, M \neq 0$, then the normal form is

$$N_4^1 = \begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}.$$

The corresponding gauge matrix W_4^1 is obtained in the following way:

$$w_1 = \frac{a_{12}^{2/3}}{\sqrt[3]{M}}, \qquad w_2 = \frac{D_x w_1 + a_{11} w_1}{a_{12}},$$
$$w_3 = \frac{w_1 a_{13}}{a_{12}}, \qquad w_4 = -\frac{a_{32}}{w_1^2 a_{12}}.$$

Case 2. If $a_{12} \neq 0, M = 0, N \neq 0$, then the normal form is

$$N_4^2 = \begin{pmatrix} 0 & . & 0 \\ . & . & 0 \\ 1 & 0 & . \end{pmatrix}.$$

The corresponding gauge matrix W_4^2 is obtained in the following way:

$$w_1 = -\frac{\sqrt[3]{N}}{a_{12}^{2/3}}, \qquad w_2 = \frac{D_x w_1 + a_{11} w_1}{a_{12}},$$
$$w_3 = \frac{w_1 a_{13}}{a_{12}}, \qquad w_4 = -\frac{a_{32}}{w_1^2 a_{12}}.$$

The normal form N_4^2 falls to the subalgebra of Type 1.

Case 3. If $a_{12} \neq 0, M = 0, N = 0$, then the seminormal form is

$$N_4^3 = \begin{pmatrix} 0 & . & 0 \\ . & . & 0 \\ 0 & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_4^3 = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{a_{11}}{a_{12}} & 1 & \dfrac{a_{13}}{a_{12}} \\ -\dfrac{a_{32}}{a_{12}} & 0 & 1 \end{pmatrix}.$$

The seminormal form N_4^3 falls to th subalgebra of Type 2.

Case 4. If $a_{12} = 0, a_{13} \neq 0, a_{32} \neq 0$, then the normal form is

$$N_4^4 = \begin{pmatrix} 0 & 0 & 1 \\ . & 0 & 0 \\ . & . & 0 \end{pmatrix}.$$

The corresponding gauge matrix W_4^4 is obtained in the following way:

$$w_1 = \frac{1}{a_{13}^{1/3}}, \qquad w_3 = \frac{D_x a_{13} - 3a_{13}a_{22}}{3a_{32}a_{13}^{4/3}}, \qquad w_4 = -\frac{D_x a_{13} - 3a_{11}a_{13}}{3a_{13}^{4/3}},$$

$$w_2 = -\frac{w_3 D_x a_{13}}{3a_{13}^2} - \frac{D_x w_3 - a_{32}a_{13}^{1/3}w_3^2 - a_{11}w_3 - 2a_{22}w_3 + a_{23}a_{13}^{-1/3}}{a_{13}}.$$

Case 5. If $a_{12} = 0, a_{13} = 0, a_{32} \neq 0$, then the seminormal form is

$$N_4^5 = \begin{pmatrix} . & 0 & 0 \\ . & . & . \\ 0 & 1 & 0 \end{pmatrix},$$

the corresponding gauge matrix being

$$W_4^5 = \begin{pmatrix} a_{32}^{1/3} & 0 & 0 \\ \frac{a_{31}}{a_{32}^{2/3}} & a_{32}^{1/3} & -\frac{2D_x a_{32} + 3a_{32}a_{11} + 3a_{32}a_{22}}{3a_{32}^{5/3}} \\ 0 & 0 & a_{32}^{-2/3} \end{pmatrix}.$$

The seminormal form N_4^5 falls to the subalgebra of Type 1.

Case 6. If $a_{12} = 0, a_{13} \neq 0, a_{32} = 0$, then the seminormal form is

$$N_4^6 = \begin{pmatrix} 0 & 0 & 1 \\ . & . & 0 \\ . & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_4^6 = \begin{pmatrix} a_{13}^{-1/3} & 0 & 0 \\ -\frac{a_{23}}{a_{13}^{4/3}} & a_{13}^{-1/3} & 0 \\ \frac{-D_x a_{13} + 3a_{13}a_{11}}{3a_{13}^{4/3}} & 0 & a_{13}^{2/3} \end{pmatrix}.$$

The seminormal form N_4^6 falls to the subalgebra of Type 1.

Case 7. If $a_{12} = 0, a_{13} = 0, a_{32} = 0, a_{23} \neq 0$, then the seminormal form is

$$N_4^7 = \begin{pmatrix} . & 0 & 0 \\ 0 & . & . \\ . & 0 & . \end{pmatrix},$$

the corresponding gauge matrix being

$$W_4^7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{a_{21}}{a_{23}} & 0 & 1 \end{pmatrix}.$$

The seminormal form N_4^7 falls to the subalgebra of Type 3. Indeed, applying the permutation matrix P_1 to N_4^7 by conjugation we obtain a lower triangular matrix.

Case 8. If $a_{12} = 0, a_{13} = 0, a_{32} = 0, a_{23} = 0$, then the seminormal form is

$$N_4^8 = \begin{pmatrix} . & 0 & 0 \\ . & . & 0 \\ . & 0 & . \end{pmatrix}.$$

The seminormal form N_4^8 falls to the subalgebra of Type 3.

Again we have, in fact, proved the next theorem:

Theorem 9.1 *In a ZCR such that its characteristic element has the Jordan normal form J_4 the matrix A has one of the above normal forms N_4^1, N_4^2, N_4^4 or seminormal forms $N_4^3, N_4^5, \dots, N_4^8$. If A does not belong to a proper subalgebra of sl_3, then A has one of the normal forms N_4^1, N_4^4.*

Example 9.2 Sawada-Kotera equation:

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x$$

The corresponding ZCR, which depends on a parameter $m \neq 0$, is

$$A = \begin{pmatrix} 0 & -1 & 0 \\ u & 0 & -m \\ 1 & 0 & 0 \end{pmatrix},$$

the matrix B is very large. The matrix A belongs to Case 1 with the normal form N_4^1, namely,

$$A = \begin{pmatrix} 0 & -1 & 0 \\ u & 0 & 1 \\ -m & 0 & 0 \end{pmatrix}.$$

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