On normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations

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Abstract

We find normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations (ZCR) with the characteristic element possessing a single Jordan cell.

Keywords: Zero curvature representation, gauge transformation, normal form, nonlinear partial differential equation, jet space.

1 Introduction

In nineties, Marvan [3] and independently Sakovich [7] introduced characteristic elements of zero-curvature representations [12] of integrable PDE and gave various applications [1, 2, 5, 8, 9]. The most important property is that gauge equivalent zero-curvature representations have conjugated characteristic elements. This opens way to classification of zero curvature representations based on the Jordan normal form of the characteristic element. In the simplest case of $\mathfrak{sl}_2$-valued zero curvature representations Marvan [4] showed that only two classes exist which do not belong to a solvable subalgebra. The case of $\mathfrak{sl}_3$ exhibits a similar pattern (Sebestyén [11]): for every $\mathfrak{sl}_3$-matrix in Jordan normal form there are finitely many normal forms of zero-curvature representation irreducible to a solvable subalgebra. In this paper we extend these results to $\mathfrak{sl}_n$-valued zero curvature representations with the characteristic element possessing a single Jordan cell.

2 Preliminaries

Let us consider a system of nonlinear differential equations

$$F^l(t, x, u^k, \ldots, u^k_I, \ldots) = 0, \tag{1}$$

in two independent variables t and x, a finite number of dependent variables u^k and their derivatives u^k_I, where I denotes a finite symmetric multiindex over t and x. Moreover, we consider an infinite-dimensional jet space J^∞ such that t, x, u^k, u^k_I are local jet coordinates on it. We have two distinguished vector fields on J^∞: D_t and D_x, which are usual *total derivatives.*

Let $D_I = D_{i_1} \cdots D_{i_\kappa}$, where D_{i_ι} denotes D_x or D_t. Then $\sum_{l,I} D_I F^l = 0$ represent an equation manifold $\mathcal{E}$ as a submanifold of J^∞.

Let $\mathfrak{g}$ be a matrix Lie algebra. A $\mathfrak{g}$-valued *zero curvature representation* (ZCR) [12] for equations (1) is a pair (A, B) of $\mathfrak{g}$-valued functions on J^∞, which satisfy

$$D_t A - D_x B + [A, B] = 0, \tag{2}$$

when restricted to $\mathcal{E}$.

By (2) we mean that there exists $\mathfrak{g}$-valued functions K^I_l on J^∞ such that

$$D_t A - D_x B + [A, B] = \sum_{l,I} D_I F^l \cdot K^I_l. \tag{3}$$

Let G be the connected and simply connected matrix Lie group associated with $\mathfrak{g}$. Then for every G-valued function W on $\mathcal{E}$ we define the *gauge transformation* of ZCR (A, B) by the formulas

$$A^W := D_x W \cdot W^{-1} + W \cdot A \cdot W^{-1}, \tag{4}$$

$$B^W := D_t W \cdot W^{-1} + W \cdot B \cdot W^{-1}. \tag{5}$$

Then (A^W, B^W) is a ZCR again, and we say that it is *gauge equivalent* to ZCR (A, B).

We consider the differential operator $\widehat{D}_I$ defined on J^∞ by

$$\widehat{D}_x M = D_x M - [A, M], \qquad \widehat{D}_t M = D_t M - [B, M],$$

where M is arbitrary G-matrix, $\widehat{D}_I = \widehat{D}_{i_1} \cdots \widehat{D}_{i_\kappa}$ and $\widehat{D}_{i_\iota}$ is $\widehat{D}_x$ or $\widehat{D}_t$.

Definition 1 ([4],[7]) Let $\mathfrak{g}$-matrices K^I_l satisfy (3). Put

$$C_l = \sum_I (-\widehat{D})_I K^I_l.$$

Then C_l, restricted to $\mathcal{E}$, is the *characteristic element* for ZCR (A, B).

Proposition 1 ([3],[7]) *Gauge equivalent ZCR's have conjugate characteristic elements.*

Definition 2 ([6]) If a ZCR (A, B) is gauge equivalent to another ZCR with values in a proper subalgebra of $\mathfrak{g}$, then we say that the ZCR is *reducible.* Otherwise it is said to be *irreducible.* A ZCR gauge equivalent to zero is called *trivial.*

3 Normal forms with respect to gauge equivalence

Consider an $\mathfrak{sl}_n$-valued ZCR (A, B). Using Definition 1 we compute its characteristic element C. Without loss of generality we can suppose that C is in Jordan normal form. If not, then we transform C by conjugation to its Jordan normal form J and simultaneously we transform the ZCR (A, B) by gauge transformation. Following Proposition 1, we consider the stabilizer group H_J of the Jordan normal form J with respect to conjugation. The stabilizer H_J is a proper subgroup of SL_n. We compute its action on the matrix A and find the corresponding normal forms.

In this work we consider ZCR's for which the Jordan normal form J of the corresponding characteristic element C consists of a single cell. Then J and an arbitrary matrix W of the corresponding stabilizer subgroup H_J are

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ w_1 & 1 & \dots & 0 & 0 \\ w_2 & w_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{n-1} & w_{n-2} & \dots & w_1 & 1 \end{pmatrix}.$$

Here w_j are parameters of $(n-1)$-dimensional subgroup H_J. Denoting by w^i_j elements of the matrix W we have $w_k = w^l_{l-k}$ for all l. For example $w^2_1 = w^3_2 = \ldots = w^n_{n-1} = w_1$.

We denote

$$\begin{aligned} \widehat{A} := A^W &= D_x W \cdot W^{-1} + W \cdot A \cdot W^{-1} \\ \widehat{a}^i_j &= D_x w^i_s \cdot \bar{w}^s_j + w^i_s \cdot a^s_r \cdot \bar{w}^r_j, \end{aligned}$$

where $\widehat{a}^i_j$ (resp. w^i_j) are elements of $\widehat{A}$ (resp. W) and $\bar{w}^i_j$ (resp. $\bar{w}_j$) are elements of W^{-1} (resp. parameters of W^{-1} in H_J),

$$\bar{w}^i_j = \sum_{\substack{1 \le r \le n-1 \\ j = j_0 < j_1 < \dots < j_r = i}} (-1)^r w^{j_1}_{j_0} w^{j_2}_{j_1} \dots w^{j_r}_{j_{r-1}},$$

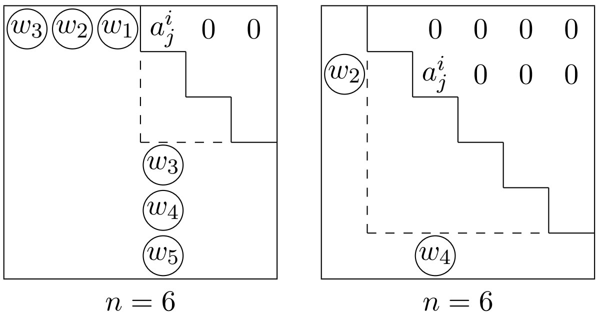
$$\bar{w}_j = \sum_{\substack{1 \le r \le j \\ 1i_1+2i_2+\cdots+ri_r=j \\ i_r \neq 0}} \Gamma_{i_1,\ldots,i_r} (w_1)^{i_1} (w_2)^{i_2} \ldots (w_r)^{i_r}, \tag{6}$$

where $\Gamma_{i_1,\ldots,i_r}$ is a nonzero integer. For $r = j$ is $\Gamma_{0,\ldots,0,1} = -1$.

The principle is to annihilate as many elements $\widehat{a}^i_j$ as possible by solving equations $\widehat{a}^i_j = 0$ for appropriate elements w_k. The next lemma shows when we can solve these equations purely algebraically.

Lemma 1 *Let $a^i_j \neq 0, i < j$ be an element of A such that all elements $a^k_l, 1 \le k \le i, n \ge l \ge j$ except a^i_j are zero. Then $\widehat{a}^i_{j-t} = a^i_j w_t + f^i_{j-t}$ for all $t = i, \ldots, j-1$ and $\widehat{a}^{i+t}_j = a^i_j w_t + g^{i+t}_j$ for all $t = n+1-j, \ldots, n-i$. The expressions f^i_{j-t}, g^{i+t}_j do not depend on w_s, $s \ge t$.*

The next picture shows two possible situations in Lemma 1.



w_t denotes a linear polynomial in w_t, independent of w_s, $s > t$.

Proof. Let $a^i_j \neq 0$ satisfy the assumption of Lemma 1. For shortness we denote $r = j - t$ for all $t = i, \ldots, j-1$.

$$\begin{aligned}
\widehat{a}^i_r &= \sum_{\alpha=1}^{n} D_x w^i_\alpha \cdot \bar{w}^\alpha_r + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} w^i_\alpha \cdot a^\alpha_\beta \cdot \bar{w}^\beta_r \\
&= \sum_{\alpha=r}^{i-1} D_x w^i_\alpha \cdot \bar{w}^\alpha_r + \sum_{\alpha=1}^{i} \sum_{\beta=r}^{j} w^i_\alpha \cdot a^\alpha_\beta \cdot \bar{w}^\beta_r .
\end{aligned}$$

The only summand to contain w_t is $w^i_i \cdot a^i_j \cdot \bar{w}^j_r = a^i_j \cdot \bar{w}^j_{j-t} = a^i_j \cdot \bar{w}_t$ and using (6) we see that w_t is the parameter with the highest index.

Let $s = i + t$ for all $t = n+1-j, \dots, n-i$. Then

$$\begin{aligned}\widehat{a}_j^s &= \sum_{\alpha=1}^{n} D_x w_\alpha^s \cdot \bar{w}_j^\alpha + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} w_\alpha^s \cdot a_\beta^\alpha \cdot \bar{w}_j^\beta \\ &= \sum_{\alpha=j}^{s} D_x w_\alpha^s \cdot \bar{w}_j^\alpha + \sum_{\alpha=i}^{s}\sum_{\beta=j}^{n} w_\alpha^s \cdot a_\beta^\alpha \cdot \bar{w}_j^\beta.\end{aligned}$$

The only summand to contain w_t is $w_i^s \cdot a_j^i \cdot \bar{w}_j^j = w_i^{i+t} \cdot a_j^i = w_t \cdot a_j^i$ and w_t is the parameter with the highest index. □

We use one obvious automorphism of $\mathfrak{sl}_n$:

$$A \mapsto -P \cdot A^\top \cdot P^{-1} =: A^*,$$

i.e., transposition $A \mapsto -A^\top$ followed by conjugation by permutation matrix P,

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

which can be described by the rule: $a_k^l \mapsto -a_r^s, r = (n+1)-l, s = (n+1)-k$. In this case the Jordan normal form of the characteristic element of the ZCR (A^*, B^*) is again J.

Definition 3 We say that the ZCR (A, B) is equivalent with a ZCR (C, D) and write $(A, B) \sim (C, D)$, if (A, B) is gauge equivalent with the ZCR (C, D) or with the ZCR (C^*, D^*).

Proposition 2 *The relation $\sim$ is reflexive, symmetric and transitive.*

Proof. Let we have two ZCRs (A, B) and (C, D). It is enough to show the proof for one matrix of a ZCR. The reflexivity is trivial. By simple computation we obtain the nontrivial part of symmetry: if $A^W = C^*$ then $C^W = A^*$, i.e., $(A^*)^{W^{-1}} = C$.

Let we have a third ZCR (E, F). For the nontrivial parts of transitivity we have: if $A^{W_1} = C^*$ and $C^{W_2} = E^*$ then $A^{W_1 \cdot W_2^{-1}} = E$. If $A^{W_1} = C^*$ and $C^{W_2} = E$ then $A^{W_1 \cdot W_2^{-1}} = E^*$. □

Definition 4 A *type* is a subset N of the set $\{(i,j) \mid i = 1,\ldots,n, j = 1,\ldots,n\}$. A matrix A is said to be of type N if for every couple $(i,j) \in N$ we have $a^i_j = 0$.

Construction 1 Let $((i_1,j_1),\ldots,(i_m,j_m))$ be an m-tuple of couples of natural numbers , $1 \le m \le \lfloor n/2 \rfloor$, satisfying the following inequalities:

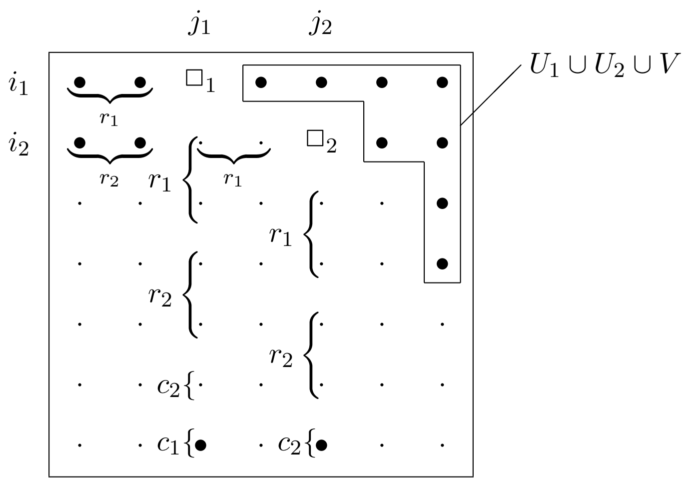
$$\begin{gathered} 1 = i_1 < i_2 < \cdots < i_m \le \lfloor n/2 \rfloor, \\ 1 < j_1 < j_2 < \cdots < j_m \le n, \\ i_\alpha < j_\alpha \text{ for } \alpha = 1,\ldots,m, \\ j_m > \lfloor n/2 \rfloor, \\ j_m \le n-1 \text{ and } j_\alpha \le \lfloor n/2 \rfloor \text{ for } \alpha = 1,\ldots,m-1 \text{ and } m > 1. \end{gathered}$$

We construct the type $N^{i_1,\ldots,i_m}_{j_1,\ldots,j_m}$ as the union of the sets:

$$\begin{aligned} U_\alpha &:= \{(k,l) \mid k \le i_\alpha, l \ge j_\alpha\} \setminus \{(i_\alpha, j_\alpha)\}, \\ V &:= \{(k,n) \mid k < n+1-i_1\}, \\ P_\alpha &:= \{(i_\alpha, q) \mid q = 1,\ldots,j_\alpha - j_{\alpha-1}\}, \\ Q_\alpha &:= \{(p, j_\alpha) \mid p = n+1-(\widehat{i}_{\alpha+1} - i_\alpha),\ldots,n\}, \end{aligned}$$

where $\alpha = 1,\ldots,m$, $j_0 := 1$, $i_{m+1} := n+1-j_m$ and $\widehat{i}_\alpha := min(i_\alpha, n+1-j_m)$. If $\widehat{i}_{\alpha+1} - i_\alpha < 1$ then $Q_\alpha = \emptyset$.

The next picture shows one of the possible cases in $\mathfrak{sl}_7$:



Here $\bullet \in N^{i_1,i_2}_{j_1,j_2}$, $\square_\alpha = (i_\alpha, j_\alpha)$, further $r_\alpha = \#P_\alpha$ and $c_\alpha = \#Q_\alpha$.

Definition 5 We say, that a ZCR (A, B) with single cell of Jordan normal form of its characteristic element is in a *normal form*, if the matrix A is of one of the types $N^{i_1,\dots,i_m}_{j_1,\dots,j_m}$ from Construction 1. For shortness, we denote the normal form by the same symbol $N^{i_1,\dots,i_m}_{j_1,\dots,j_m}$.

Theorem 1 *Let (A, B) be an irreducible $\mathfrak{sl}_n$-valued ZCR with Jordan normal form of the corresponding characteristic element equal to J. Then A either belongs to one of the subalgebras*

$$L_k = \{(a^i_j) \in \mathfrak{sl}_n \mid a^i_j = 0 \text{ for all } i = 1, \dots, k, j = k + 1, \dots, n\}, \qquad (7)$$

where $k = 1, \dots, n - 1$, or is equivalent to a matrix in the normal form.

4 Proof of Theorem 1

This section is dedicated to the proof of Theorem 1. We introduce a procedure of computation of the gauge matrix W wich sends the matrix A to the corresponding normal form by gauge transformation. The idea is based on multiple usage of Lemma 1. The procedure is a sequence of three simple algorithms. In Algorithm 1 we choose between A and A^* to lower the number of normal forms. In Algorithm 2 we establish the m-tuple from Construction 1. Algorithm 3 computes the parameters of the gauge matrix W and returns the normal form $N^{i_1,\dots,i_m}_{j_1,\dots,j_m} := A^W$. If Algorithm 2 stop with A in a subalgebra then we escape the rest of the procedure and we do not compute the normal form.

Algorithm 1 Choose between A and A^*

Input: $A \in \mathfrak{sl}_n$. *Output*: A.

Find the highest column index k in the first row of A such that $a^1_k \neq 0$; if all elements in the first row are zero then put $k = 0$
Find the highest column index l in the first row of A^* such that ${a^*}^1_l \neq 0$; if all elements in the first row are zero then put $l = 0$
if $k < l$ **then**
    $A := A^*$
end if
return A

Algorithm 2 Find the m-tuple $((i_1,j_1),\dots,(i_m,j_m))$

Input: $A\in\mathfrak{sl}_n$. *Output*: m-tuple $((i_1,j_1),\dots,(i_m,j_m))$.

```
m := 0, i_0 := 0 and j_0 := 1
repeat
  r := i_m
  repeat
    r := r + 1
    Find the highest column index k in the r-th row of A such that
    a^r_k ≠ 0; if all elements in the r-th row are zero then put k = 0
  until k > j_m or r > ⌊n/2⌋
  m := m + 1
  (i_m, j_m) := (r, k)
  if i_m ≥ j_m or i_m > j_{m−1} then
    STOP: A in subalgebra L_{i_m} or L_{i_m − 1}
  end if
until j_m > ⌊n/2⌋
return ((i_1, j_1), ..., (i_m, j_m))
```

Algorithm 3 Compute the gauge matrix W and the normal form $N^{i_1,\dots,i_m}_{j_1,\dots,j_m}$

Input: m-tuple $((i_1,j_1),\dots,(i_m,j_m))$. *Output*: $N^{i_1,\dots,i_m}_{j_1,\dots,j_m}$.

```
j_0 := 1, i_{m+1} := n + 1 − j_m
for α = 1, ..., m do
  for t = j_{α−1}, ..., j_α − 1 do
    solve â^{i_α}_{j_α − t} = 0 for w_t and insert w_t back into W
  end for
end for
for α = m, ..., 1 do
  if i_α ≥ n + 1 − j_m then
    i_α := n + 1 − j_m
  else
    for t = n + 1 − i_{α+1}, ..., n − i_α do
      solve â^{i_α + t}_{j_α} = 0 for w_t and insert w_t back into W
    end for
  end if
end for
N^{i_1,...,i_m}_{j_1,...,j_m} := A^W
return N^{i_1,...,i_m}_{j_1,...,j_m}
```

Algorithm 2 always ends. Indeed, if in all tuples (i_α, j_α), $\alpha = 1, \ldots, m$ we have $i_\alpha < j_\alpha$, then the algorithm stops at the latest in the $\lfloor n/2 \rfloor$-th row of A since $j_m > i_m$, i.e., $j_m > \lfloor n/2 \rfloor$. Moreover, all elements $a^{i_\alpha}_{j_\alpha}$ satisfy assumption of Lemma 1. If $i_m \geq j_m$ for some m then $a^k_l = 0$ for all $k \leq i_m$, $l > j_m$ and by (7) A belongs to the subalgebra L_{i_m}. If $i_m > j_{m-1}$ then $a^k_l = 0$ for all $k < i_m$, $l > j_{m-1}$ and by (7) $A \in L_{i_m - 1}$.

To prove correctness of Algorithm 3 we must, firstly, show that t runs through the sequence $1, \ldots, n-1$ without repetitions. But $t = j_0, \ldots, j_1 - 1, j_1, \ldots, j_2 - 1, \ldots, j_{m-1}, \ldots, j_m - 1$ where $j_0 = 1$. Further $t = n + 1 - i_{k+1}, \ldots, n - i_k, n + 1 - i_k, \ldots, n - i_{k-1}, \ldots, n + 1 - i_1, \ldots, n - i_1$ for some k, $m \geq k \geq 1$, where $n + 1 - i_{k+1} = j_m$ and $i_1 = 1$.

Secondly, we must show that the corresponding $\widehat{a}^{i_\alpha}_{j_\alpha - t}$ (resp. $\widehat{a}^{i_\alpha + t}_{j_\alpha}$) in step 4 (resp. 12) depends on w_t for exactly one t, moreover linearly. But this follows from Lemma 1, since in step 4 we have $j_{\alpha-1} \geq i_\alpha$ and in step 12 we have $i_{\alpha+1} \leq j_\alpha$. In all above equations w_t is the only parameter since parameters with lower indices have been already solved and inserted back in previous steps.

As a conclusion we can say that our algorithms find for every $\mathfrak{sl}_n$ matrix A the corresponding normal form, or stop when A belongs to some subalgebra L_i.

Remark 1 If the matrix A belongs to one of the subalgebras L_i, and B does not, then we can apply the procedure of computation of a normal form to the matrix B using (5) instead of (4).

Remark 2 The normal forms constructed in this paper are not the only possible. Lemma 1 allows us to construct different normal forms. But every normal form of the other type will be equivalent (in sense of Definition 3) with some normal form of our type.

Remark 3 Normal forms of a ZCR with the characteristic element possessing a single Jordan cell provide a base for the classification of all ZCRs. This, however, requires further research.

5 Examples

Example 1 The set of normal forms for $\mathfrak{sl}_7$.

$$N_7^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a_7^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad N_6^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a_6^1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \end{pmatrix},$$

$$N_5^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & a_5^1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \end{pmatrix}, \quad N_4^1 = \begin{pmatrix} 0 & 0 & 0 & a_4^1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$N_{3,6}^{1,2} = \begin{pmatrix} 0 & 0 & a_3^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & a_6^2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, N_{3,5}^{1,2} = \begin{pmatrix} 0 & 0 & a_3^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & a_5^2 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & 0 & \cdot & \cdot \end{pmatrix},$$

$$N_{3,4}^{1,2} = \begin{pmatrix} 0 & 0 & a_3^1 & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & a_4^2 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot \end{pmatrix}, N_{3,6}^{1,3} = \begin{pmatrix} 0 & 0 & a_3^1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & a_6^3 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

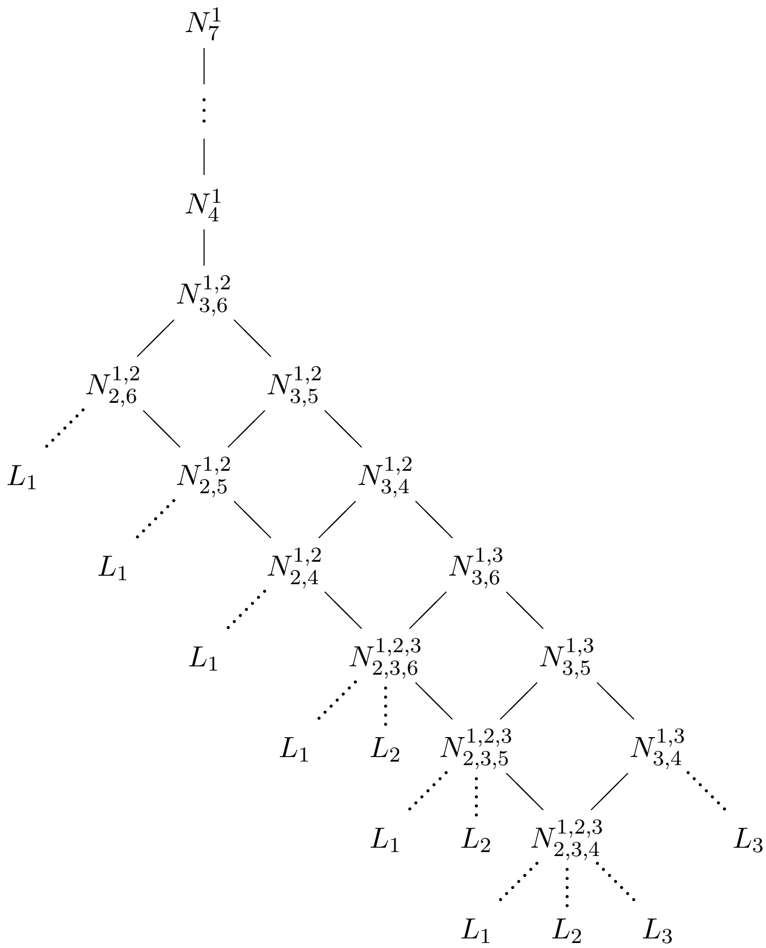
$$N^{1,3}_{3,5} = \begin{pmatrix} 0 & 0 & a_3^1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & a_5^3 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, N^{1,3}_{3,4} = \begin{pmatrix} 0 & 0 & a_3^1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & a_4^3 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot \end{pmatrix},$$

$$N^{1,2}_{2,6} = \begin{pmatrix} 0 & a_2^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & a_6^2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, N^{1,2}_{2,5} = \begin{pmatrix} 0 & a_2^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & a_5^2 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot \end{pmatrix},$$

$$N^{1,2}_{2,4} = \begin{pmatrix} 0 & a_2^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & a_4^2 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & 0 & \cdot & \cdot & \cdot \end{pmatrix}, N^{1,2,3}_{2,3,6} = \begin{pmatrix} 0 & a_2^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & a_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & a_6^3 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$N^{1,2,3}_{2,3,5} = \begin{pmatrix} 0 & a_2^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & a_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & a_5^3 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, N^{1,2,3}_{2,3,4} = \begin{pmatrix} 0 & a_2^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & a_3^2 & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & a_4^3 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & 0 & \cdot & \cdot & \cdot \end{pmatrix}.$$

Example 2 The decision graph of normal forms for $\mathfrak{sl}_7$.



Example 3 The Kupershmidt equation, $n = 3$. This example was already published in [11]. The corresponding normal form is N_2^1.

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