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On the invariant variational sequences in mechanics [1]

Jana Šeděnková

Abstract

The r-th order variational sequence is the quotient sequence of the De Rham sequence on the r-th jet prolongation of a fibered manifold, factored through its contact subsequence. In this paper, the first order variational sequence on a fibered manifold with one-dimensional base is considered. A new representation of all quotient spaces as some spaces of (global) forms is given. The factorization procedure is based on a modification of the interior Euler operator, used in the theory of (infinite) variational bicomplexes.

1 Introduction

The aim of this paper is to extend some recent results on the structure of variational sequences in mechanics (Krupka [7, 8]).

In [7] an invariant description of the classes in the "variational" terms of the sequence was given (lagrangians, Euler-Lagrange forms, Helmholtz-Sonin forms). Analogous results were obtained for the higher order field theory by Krbek, Musilová, and Kašparová [4, 5]. Musilová and Krbek [10] found a solution of the problem of the reconstruction of forms from their (invariant) classes for the "variational" terms in higher order mechanics.

In this paper we show that similar results can be obtained and substantially extended by means of the techniques known in the theory of (infinite) variational bicomplexes. We use a slight (finite order) modification of an operator I, called by Anderson the interior Euler operator (see Anderson

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[1], and Kuperschmidt [9], Dedecker and Tulczyjew [3], Bauderon [2], where this operator was denoted by τ^+, τ and D^* respectively). Our $\mathcal{I}$ is defined on forms on the underlying fibered manifold; we show that the factorization induced by $\mathcal{I}$ yealds exactly the first order variational sequence in mechanics. In this way we obtain an invariant description of all, not only the "variational", terms in the variational sequence.

Throughout this paper the standard notation of the theory of variational sequences in mechanics is used (see Krupka [7]). For generalities on the calculus of variations on fibered spaces related to the concept of the variational sequence, we refer to [1, 6, 9].

2 The variational sequence and its representations

Let $\pi : Y \to X$ be a fibered manifold over a one-dimensional base X, $\dim Y = m+1$, let J^rY be the r-jet prolongation of Y, and let $\pi^{r,s} : J^rY \to J^sY$, where $0 \leq s \leq r$, be the canonical jet projections. Let Ω^r_k be the direct image of the sheaf of smooth k-forms over J^rY by the jet projection $\pi^{r,0}$, where $k \geq 0$. Denote

$$\Omega^r_{0,c} = \{0\}, \quad \Omega^r_{k,c} = \ker p_{k-1}, \quad \Theta^r_k = \Omega^r_{k,c} + d\Omega^r_{k-1,c},$$

where $k \geq 1$, and $d\Omega^r_{k-1,c}$ is the image sheaf of $\Omega^r_{k-1,c}$ by d. Then for every open set $W \subset Y$, $\Omega^r_k W$ (resp. $\Omega^r_{k,c} W$) is the Abelian group of k-forms (resp. k-contact k-forms) on $W^r = (\pi^{r,0})^{-1}(W)$, $d\Omega^r_{k-1,c}W$ is the Abelian group of forms which can be locally expressed as differentials of $(k-1)$-contact $(k-1)$-forms on W^r, and $\Theta^r_k W$ is a subgroup of $\Omega^r_k W$. (Recall that a form ρ on J^rY is said to be *contact* if it vanishes along the r-jet prolongation $J^r\gamma$ of every section γ of Y.)

We get a sequence

$$0 \to \Theta^r_1 \to \Theta^r_2 \to \Theta^r_3 \to \ldots \to \Theta^r_M \to 0 \tag{1}$$

in which all arrows denote the exterior differentiation d, and $M = mr + 1$.

The sequence (1) is an exact subsequence of the De Rham sequence

$$0 \to \mathbb{R}_Y \to \Omega^r_0 \to \Omega^r_1 \to \Omega^r_2 \to \ldots \to \Omega^r_{N-1} \to \Omega^r_N \to 0 \tag{2}$$

where $N = \dim J^rY = 1 + m(r+1)$. The quotient sequence

$$\begin{aligned} 0 \to \mathbb{R}_Y \to \Omega^r_0 \to \Omega^r_1/\Theta^r_1 \to \Omega^r_2/\Theta^r_2 \to \ldots \\ \ldots \to \Omega^r_M/\Theta^r_M \to \Omega^r_{M+1} \to \ldots \to \Omega^r_{N-1} \to \Omega^r_N \to 0 \end{aligned} \tag{3}$$

is also exact. (3) is called the *r-th order variational sequence.* The class of a differential form $\rho \in \Omega^r_k W$ in the variational sequence (3) is denoted by $[\rho]$.

For every $\rho \in \Omega^1_{k+1} W$ there exists a unique decomposition

$$(\pi^{2,1})^* \rho = p_k \rho + p_{k+1} \rho, \tag{4}$$

where $p_k \rho$ denotes the k-contact component of ρ ($p_0 \rho = h\rho$ denotes the horizontal component of ρ).

Let (V, ψ), $\psi = (t, q^\sigma)$ be a fibered chart on Y and let (V^3, ψ^3), $\psi^3 = (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma, \dddot{q}^\sigma)$ be the associated fibered chart on $J^3 Y$. We set

$$\Xi = \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} + \ddot{q}^\sigma \frac{\partial}{\partial \dot{q}^\sigma} + \dddot{q}^\sigma \frac{\partial}{\partial \ddot{q}^\sigma}. \tag{5}$$

Ξ is a vector field on V^3. If $\rho \in \Omega^1_{k+1} V$, $k \geq 1$, we define

$$\begin{aligned} \mathcal{I}_{(V,\psi)}(\rho) &= \tfrac{1}{k} \omega^\alpha \wedge [i_{\frac{\partial}{\partial q^\alpha}} p_k \rho - \partial_\Xi \, i_{\frac{\partial}{\partial \dot{q}^\alpha}} p_k \rho], \\ &= p_k \rho - \tfrac{1}{k} \partial_\Xi (\omega^\alpha \wedge i_{\frac{\partial}{\partial \dot{q}^\alpha}} p_k \rho). \end{aligned} \tag{6}$$

For $k = 0$ and $\rho \in \Omega^1_1$, we define

$$\mathcal{I}_{(V,\psi)}(\rho) = h\rho.$$

Note that the form $\mathcal{I}_{(V,\psi)}(\rho)$ depends only on the k-contact $(k+1)$-form $p_k \rho$.

Lemma. *Let (V, ψ), $\psi = (t, q^\sigma)$, $(\bar{V}, \bar{\psi})$, $\psi = (\bar{t}, \bar{q}^\sigma)$ be two fibered charts on Y such that $V \cap \bar{V} \neq 0$. Then for every $\rho \in \Omega^1_{k+1}(V \cap \bar{V})$, $k \geq 0$,*

$$\mathcal{I}_{(V,\psi)}(\rho) = \mathcal{I}_{(\bar{V},\bar{\psi})}(\rho). \tag{7}$$

Proof. We prove the Lemma by a direct calculation. We use the following transformation formulas

$$\bar{\omega}^\sigma = \frac{\partial \bar{q}^\sigma}{\partial q^\alpha} \omega^\alpha, \quad \bar{\Xi} = \frac{dt}{d\bar{t}} \Xi + F^\alpha \frac{\partial}{\partial \dddot{q}^\alpha},$$

$$\frac{\partial}{\partial \dot{\bar{q}}^\sigma} = \frac{\partial \dot{q}^\alpha}{\partial \dot{\bar{q}}^\sigma} \frac{\partial}{\partial \dot{q}^\alpha} + \frac{\partial \ddot{q}^\alpha}{\partial \dot{\bar{q}}^\sigma} \frac{\partial}{\partial \ddot{q}^\alpha} + \frac{\partial \dddot{q}^\alpha}{\partial \dot{\bar{q}}^\sigma} \frac{\partial}{\partial \dddot{q}^\alpha},$$

where F^α are functions of $t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma, \dddot{q}^\sigma$. We also use the identity

$$\partial_{g\xi} f\eta = fg\partial_\xi \eta + d(fg) i_\xi \eta + g i_\xi (df \wedge \eta),$$

where f, g are functions, ξ is a vector field and η is a differential form.

Then by definition,

$$\begin{aligned}
\mathcal{I}_{(\bar{V},\bar{\psi})}(\rho) &= p_k\rho - \tfrac{1}{k}\partial_{\bar{\Xi}}(\bar{\omega}^\sigma \wedge i_{\frac{\partial}{\partial \dot{\bar{q}}^\sigma}} p_k\rho) \\
&= p_k\rho - \tfrac{1}{k}\frac{d\bar{t}}{dt}\frac{dt}{d\bar{t}}\partial_\Xi(\omega^\alpha \wedge i_{\frac{\partial}{\partial \dot{q}^\alpha}} p_k\rho) \\
&\quad -\tfrac{1}{k} d\left(\frac{d\bar{t}}{dt}\frac{dt}{d\bar{t}}\right) i_\Xi(\omega^\alpha \wedge i_{\frac{\partial}{\partial \dot{q}^\alpha}} p_k\rho) \\
&\quad -\tfrac{1}{k}\frac{dt}{d\bar{t}} i_\Xi \left(d\left(\frac{d\bar{t}}{dt}\right) \wedge \omega^\alpha \wedge i_{\frac{\partial}{\partial \dot{q}^\alpha}} p_k\rho \right) \\
&= p_k\rho - \tfrac{1}{k}\partial_\Xi(\omega^\alpha \wedge i_{\frac{\partial}{\partial \dot{q}^\alpha}} p_k\rho) = \mathcal{I}_{(V,\psi)}(\rho).
\end{aligned}$$

Some of the terms in this expression vanish identically because the form $p_k\rho$ contains only linear forms $dt, \omega^\sigma, \dot{\omega}^\sigma$, and $p_0\rho = h\rho$ contains only the linear form dt. This completes the proof.

Using the invariance of (6), we define $\mathcal{I}(\rho)$ to be the $(k+1)$-form defined locally by (6). In accordance with Anderson [1], we call $\mathcal{I}$ the *interior Euler-Lagrange operator.*

The following theorem shows basic properties of the operator $\mathcal{I}$.

Theorem 1. *Let $k \geq 0$ and $\rho \in \Omega^1_{k+1}V$. Then*
a) *$\mathcal{I}(\rho)$ lies in the same class as $(\pi^{3,1})^*\rho$.*
b) *$\mathcal{I}^2 = \mathcal{I}$.*

Proof. a) We can see that

$$\begin{aligned}
\mathcal{I}(\rho) &= p_k\rho - \tfrac{1}{k}\partial_\Xi(\omega^\alpha \wedge i_{\frac{\partial}{\partial \dot{q}^\alpha}} p_k\rho) \\
&= \rho - p_{k+1}\rho - \tfrac{1}{k} i_\Xi d(\omega^\alpha \wedge i_{\frac{\partial}{\partial \dot{q}^\alpha}} p_k\rho) - \tfrac{1}{k} d i_\Xi(\omega^\alpha \wedge i_{\frac{\partial}{\partial \dot{q}^\alpha}} p_k\rho).
\end{aligned}$$

The second and third terms are $(k+1)-$contact, the last term is the exterior derivative of a k-contact form, i.e. the last three terms lie in the kernel $\Theta^3_{k+1}V$, and the class $[(\pi^{3,1})^*\rho]$ is the same as the class $[\mathcal{I}(\rho)]$.

b) Condition b) means, that $\mathcal{I}$ is a projector. For $k = 0$ we have trivially $\mathcal{I}^2(\rho) = h^2\rho = h\rho = \mathcal{I}(\rho)$. For $k \geq 1$, if ρ is on higher jet prolongation then the definition of $\mathcal{I}(\rho)$ is modified. The form $\mu = \mathcal{I}(\rho)$ is on third jet prolongation and $p_k\mu = \mu$. Then we put

$$\Xi = \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \ddot{q}^\alpha \frac{\partial}{\partial \dot{q}^\alpha} + \dddot{q}^\alpha \frac{\partial}{\partial \ddot{q}^\alpha} + q_4^\alpha \frac{\partial}{\partial \dddot{q}^\alpha}$$

and the operator $\mathcal{I}$ is defined by

$$\mathcal{I}(\mu) = \tfrac{1}{k}\omega^\alpha \wedge [i_{\frac{\partial}{\partial q^\alpha}}\mu - \partial_\Xi\, i_{\frac{\partial}{\partial \dot{q}^\alpha}}\mu + \partial_\Xi\partial_\Xi i_{\frac{\partial}{\partial \ddot{q}^\alpha}}\mu],$$

where only last terms in Ξ and $\mathcal{I}(\mu)$ are new. The condition $\mathcal{I}(\mathcal{I}(\rho)) = \mathcal{I}(\rho)$ follows from a direct computation. For simplicity, denote $\eta = p_k\rho$. In the following we use the identities

$$\begin{gathered} i_{[\Xi,\theta]}\omega = \partial_\Xi i_\theta\omega - i_\theta\partial_\Xi\omega \\ [\Xi, \tfrac{\partial}{\partial q^\alpha}] = 0, \quad [\Xi, \tfrac{\partial}{\partial \dot{q}^\alpha}] = -\tfrac{\partial}{\partial q^\alpha} \quad [\Xi, \tfrac{\partial}{\partial \ddot{q}^\alpha}] = -\tfrac{\partial}{\partial \dot{q}^\alpha}. \end{gathered}$$

We have

$$\begin{aligned} \mathcal{I}(\mathcal{I}(\rho)) &= \tfrac{1}{k}\omega^\alpha \wedge [i_{\frac{\partial}{\partial q^\alpha}}(\eta - \tfrac{1}{k}\partial_\Xi(\omega^\sigma \wedge i_{\frac{\partial}{\partial \dot{q}^\sigma}}\eta)) \\ &\quad -\partial_\Xi\, i_{\frac{\partial}{\partial \dot{q}^\alpha}}(\eta - \tfrac{1}{k}\partial_\Xi(\omega^\sigma \wedge i_{\frac{\partial}{\partial \dot{q}^\sigma}}\eta)) \\ &\quad +\partial_\Xi\partial_\Xi i_{\frac{\partial}{\partial \ddot{q}^\alpha}}(\eta - \tfrac{1}{k}\partial_\Xi(\omega^\sigma \wedge i_{\frac{\partial}{\partial \dot{q}^\sigma}}\eta))] \\ &= \tfrac{1}{k}\omega^\alpha \wedge [i_{\frac{\partial}{\partial q^\alpha}}\eta - \tfrac{1}{k}\partial_\Xi i_{\frac{\partial}{\partial q^\alpha}}(\omega^\sigma \wedge i_{\frac{\partial}{\partial \dot{q}^\sigma}}\eta) \\ &\quad -\partial_\Xi\, i_{\frac{\partial}{\partial \dot{q}^\alpha}}\eta + \tfrac{1}{k}\partial_\Xi\partial_\Xi i_{\frac{\partial}{\partial \dot{q}^\alpha}}(\omega^\sigma \wedge i_{\frac{\partial}{\partial \dot{q}^\sigma}}\eta) \\ &\quad +\tfrac{1}{k}\partial_\Xi i_{\frac{\partial}{\partial q^\alpha}}(\omega^\sigma \wedge i_{\frac{\partial}{\partial \dot{q}^\sigma}}\eta) - \tfrac{1}{k}\partial_\Xi\partial_\Xi i_{\frac{\partial}{\partial \dot{q}^\alpha}}(\omega^\sigma \wedge i_{\frac{\partial}{\partial \dot{q}^\sigma}}\eta) \\ &\quad -\partial_\Xi\partial_\Xi\partial_\Xi i_{\frac{\partial}{\partial \ddot{q}^\alpha}}(\omega^\sigma \wedge i_{\frac{\partial}{\partial \dot{q}^\sigma}}\eta)] \\ &= \tfrac{1}{k}\omega^\alpha \wedge [i_{\frac{\partial}{\partial q^\alpha}}\eta - \partial_\Xi\, i_{\frac{\partial}{\partial \dot{q}^\alpha}}\eta] = \mathcal{I}(\rho). \end{aligned}$$

In particular, it follows from Theorem 1 that $\mathcal{I}(\rho)$ can be used as a representative of the class $[\rho]$ in $\Omega^3_{k+1}V$. In the following Corollary these representatives are given explicitly for the second and the third columns of the variational sequence.

Corollary. a) *Let $\rho \in \Omega^1_2 V$, where*

$$p_1\rho = A_\sigma\omega^\sigma \wedge dt + B_\sigma\dot{\omega}^\sigma \wedge dt.$$

Then

$$\mathcal{I}(\rho) = (A_\sigma - \tfrac{d}{dt}B_\sigma)\omega^\sigma \wedge dt.$$

b) *Let* $\rho \in \Omega^1_3 V$, *where*

$$p_2\rho = A_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge dt + B_{\sigma\nu}\dot{\omega}^\sigma \wedge \omega^\nu \wedge dt + C_{\sigma\nu}\dot{\omega}^\sigma \wedge \dot{\omega}^\nu \wedge dt.$$

Then

$$\begin{aligned}\mathcal{I}(\rho) &= \tfrac{1}{2}(A_{\sigma\nu} - A_{\nu\sigma} - \tfrac{d}{dt}B_{\sigma\nu})\omega^\sigma \wedge \omega^\nu \wedge dt \\ &+ \tfrac{1}{2}(B_{\sigma\nu} + B_{\nu\sigma} - \tfrac{d}{dt}(C_{\sigma\nu} - C_{\nu\sigma}))\dot{\omega}^\sigma \wedge \omega^\nu \wedge dt \\ &+ \tfrac{1}{2}(C_{\nu\sigma} - C_{\sigma\nu})\ddot{\omega}^\sigma \wedge \omega^\nu \wedge dt.\end{aligned}$$

Now consider the quotient mappings in the variational sequence,

$$E : \Omega^1_{k+1}/\Theta^1_{k+1} \ni [\rho] \to E([\rho]) = [d\rho] \in \Omega^1_{k+2}/\Theta^1_{k+2},$$

which satisfy the condition $E^2 = 0$. Every class $[\rho] \in \Omega^1_{k+1}/\Theta^1_{k+1}$, $k \geq 0$, can be represented by the form $\mathcal{I}(\rho) \in \Omega^3_{k+1}$ (or $\mathcal{I}(\rho) \in \Omega^2_1$ for $k = 0$). This induces the associated mappings

$$\bar{E} : \Omega^3_{k+1} \supset \mathcal{I}\Omega^1_{k+1} \ni \beta \to \bar{E}(\beta) = \mathcal{I}(d\beta) \in \mathcal{I}\Omega^1_{k+2} \subset \Omega^3_{k+2}.$$

We will give an independent proof of the identity $\bar{E}^2 = 0$ in the resulting sequence

$$\begin{aligned}&0 \to \mathbb{R}_Y \to \Omega^1_0 \to \mathcal{I}\Omega^1_1 \to \mathcal{I}\Omega^1_2 \to \dots \\ &\quad \dots \to \mathcal{I}\Omega^1_M \to \Omega^1_{M+1} \to \dots \to \Omega^1_{N-1} \to \Omega^1_N \to 0,\end{aligned} \tag{8}$$

where $M = m + 1$, $N = 2m + 1$.

Theorem 2. *The associated mappings*

$$\bar{E} : \mathcal{I}\Omega^1_{k+1} \ni \beta \to \bar{E}(\beta) = \mathcal{I}(d\beta) \in \mathcal{I}\Omega^1_{k+2}, \qquad k \geq 0.$$

satisfy condition $\bar{E}^2 = 0$.

Proof. Using the identity

$$p_{k+1}dp_{k+1}\rho = (-1)^{k+1}\partial_\Xi(p_{k+1}\rho \wedge dt),$$

we get by a direct calculation

$$\mathcal{I}d\mathcal{I}(\rho) = \mathcal{I}(d\rho).$$

Then

$$\bar{E}^2(\mathcal{I}(\rho)) = \bar{E}(\mathcal{I}d\mathcal{I}(\rho)) = \bar{E}(\mathcal{I}(d\rho)) = \mathcal{I}d\mathcal{I}(d\rho) = \mathcal{I}(dd\rho) = 0.$$

This completes the proof.

Remark. There are many possibilities to represent the quotient spaces Ω^1_k/Θ^1_k. One possible representation is given by identifying Ω^1_k/Θ^1_k with $\mathcal{I}\Omega^1_k \subset \Omega^3_k$. Note that the order of $\mathcal{I}(\rho)$, $\rho \in \Omega^1_k$, is in general higher than the order of ρ. The Euler-Lagrange operator $\mathcal{I}$ solves the problem of finding the *invariant* representatives for all k-forms, $k > 0$. Thus, our main result consists of finding a complete description of the first order variational sequence by (invariant) forms.

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