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VARIATIONAL INTEGRATING FACTORS FOR FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

Given a dynamical form E', we can ask if there is a locally variational form, equivalent with E'. The integrating factor G such that $E = GE'$ is a locally variational form is then called a variational integrating factor.

A complete solution of the problem of searching for variational integrating factors in general is yet not known. There have been achieved some particular results concerning mainly second-order ODE (see e.g. [2], [4], [12], [14]). Concerning PDE, there is only one paper containing a short remark on a solution of the multiplier problem for a single second order partial differential equation (see [2]).

The aim of this work is to study the problem of variational integrating factors for a dynamical form, which represents a system of first order PDE. We prove that if an everywhere regular matrix G is a variational integrating factor for a regular variational form E', then $E = GE'$ is regular and the associated dynamical differential ideals coincide. With help of the variationality conditions for PDE (see [1], [9]) we find a system of equations for variational integrating factors by the assumption that E' is a polynomial in the first derivatives. Finally we compute concrete conditions for variational integrating factors in two special cases, namely when E' represents quasilinear equations with constant coefficients and 2 independent and 1 dependent variable (resp. 2 independent and 2 dependent variables).

In this work we use our recently obtained results concerning variationality of a system of PDE (see [5],[6],[7]).

The paper is organized as follows. In Section 2 we introduce notations and necessary concepts and results concerning the calculus of variations on fibred manifolds. In Section 3 we recall some results concerning variational propertis of systems of first-order PDE. Main results concerning integrating factors are stated and proved in Section 4.

2. Basic definitions and known results

In what follows, all manifolds and mappings are smooth, and summation over repeated indices is understood. We consider a fibred manifold $\pi : Y \to X$, $\dim X = n$, $\dim Y = m+n$. We denote J^1 the 1-jet prolongation functor, $\pi_1 : J^1Y \to X$, $\pi_{1,0} : J^1Y \to Y$. Let us recall

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some basic definitions. A mapping $\gamma : U \to Y$, where $U \subset X$ is an open subset, is called a *section* of π, if $\pi \circ \gamma = id_U$. A vector field ξ on Y is said to be π-*vertical*, if $T\pi.\xi = 0$. Similarly, a vector field ξ on J^1Y is called π_1-*vertical* (resp. $\pi_{1,0}$-*vertical*), if $T\pi_1.\xi = 0$ (resp. $T\pi_{1,0}.\xi = 0$). A q-form η on J^1Y is called π_1-*horizontal* (resp. $\pi_{1,0}$-*horizontal*), if $i_\xi \eta = 0$ for every π_1-vertical (resp. $\pi_{1,0}$-vertical) vector field ξ on J^1Y. We denote by h the *horizontalization* of differential forms. h is defined to be an R-linear wedge-product preserving mapping such that for a q-form η on Y $h\eta$ is a q-form on J^1Y, and

$$hdx^i = dx^i, \quad hdy^\sigma = y^\sigma_j dx^j, \quad hf = f \circ \pi_{1,0}. \tag{2.1}$$

It's easy to see, that

$$hdf = d_i f dx^i, \quad where \quad d_i f = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^\sigma} y^\sigma_i. \tag{2.2}$$

η is called *contact*, if $J^1\gamma^* \eta = 0$ for every section γ of π. A contact $\pi_{1,0}$-horizontal q-form η is called 1-*contact*, if for every π_1-vertical vector field ξ on J^1Y the form $i_\xi \eta$ is π_1-horizontal; η is called k-*contact*, $2 \le k \le q$, if $i_\xi \eta$ is $(k-1)$-contact. Recall that *for every $\pi_{1,0}$-horizontal q-form on J^1Y there is a unique decomposition $\eta = \eta_0 + \eta_1 + \cdots + \eta_q$, where η_0 is a π_1-horizontal form, and η_i, $1 \le i \le q$, is a i-contact form on J^1Y*; we set $h\eta = \eta_0$, $p_i\eta = \eta_i$, and call it the *horizontal* and *i-contact part* of η, respectively. Consequently, every q-form on Y can be uniquely decomposed as follows

$$\pi^*_{1,0}\eta = h\eta + p_1\eta + \cdots + p_q\eta. \tag{2.3}$$

We denote by (x^i, y^σ) (resp. $(x^i, y^\sigma, y^\sigma_j)$) local fibred coordinates on Y (resp. the associated coordinates on J^1Y), and set

$$\begin{aligned} &\omega_0 = dx^1 \wedge dx^2 \cdots \wedge dx^n, \quad \omega^\sigma = dy^\sigma - y^\sigma_k \, dx^k, \\ &\omega_j = i_{\partial/\partial x^j} \, \omega_0, \quad \omega_{j_1 j_2} = i_{\partial/\partial x^{j_2}} \, \omega_{j_1}, \quad \text{etc.} \end{aligned} \tag{2.4}$$

A 1-contact $\pi_{1,0}$-horizontal $(n+1)$-form E on J^1Y is called a *dynamical form*. In fibred coordinates, $E = E_\sigma \, \omega^\sigma \wedge \omega_0$, where $E_\sigma = E_\sigma(x^i, y^\nu, y^\nu_k)$. A section γ of π is called a *path* of E, if $E \circ J^1\gamma = 0$, i.e., if the components γ^ν of γ satisfy the following system of m first-order PDE:

$$E_\sigma\Big(x^i, \gamma^\nu, \frac{\partial \gamma^\nu}{\partial x^j}\Big) = 0, \quad 1 \le \sigma \le m. \tag{2.5}$$

By a *first-order Lagrangian* we mean a horizontal n-form λ on J^1Y. In fibred coordinates, $\lambda = L\omega_0$, where $L = L(x^i, y^\nu, y^\nu_k)$.

Let ρ be an n-form on Y. Then $\lambda = h\rho$ is a first-order Lagrangian (with the function L *polynomial of degree $\le n$ in the first-order derivatives*), and

$$\pi^*_{1,0}\rho = L\,\omega_0 + \sum_{k=1}^{n} \Big(\frac{1}{k!}\Big)^2 \frac{\partial^k L}{\partial y^{\sigma_1}_{j_1} \cdots \partial y^{\sigma_k}_{j_k}} \omega^{\sigma_1} \wedge \cdots \wedge \omega^{\sigma_k} \wedge \omega_{j_1 \cdots j_k} \tag{2.6}$$

(see [8] and also [3]). We denote $\rho^{\mathcal{K}}_\lambda = \pi^*_{1,0}\rho$ and call this n-form the *Krupka form* of λ. The at most 1-contact part of $\rho^{\mathcal{K}}_\lambda$, i.e.,

$$\theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y^\sigma_j} \omega^\sigma \wedge \omega_j, \tag{2.7}$$

is called the *Poincaré–Cartan form* of λ. Note that $E_\lambda = p_1 d\rho$ is a *dynamical form* on J^1Y; it is called the *Euler–Lagrange form* of λ, and the corresponding equations for paths of E_λ are called the *Euler–Lagrange equations.* Obviously, $E_\lambda = E_\sigma(L)\,\omega^\sigma \wedge \omega_0$, where

$$E_\sigma(L) = \frac{\partial L}{\partial y^\sigma} - d_j \frac{\partial L}{\partial y^\sigma_j}, \tag{2.8}$$

and the *Euler–Lagrange expressions* E_σ, $1 \le \sigma \le m$, are all *polynomials of degree* $\le n$ *in the* y^ν_j*'s.*

A dynamical form E on J^1Y is called *variational*, if for every point $x \in J^1Y$ there exists a neighbourhood U and Lagrangian λ defined on U such, that $E = E_\lambda$. Thus, for variational forms equations for paths (2.5) are the Euler–Lagrange equations. It is known (see [15]) that if $E = E_\sigma \omega^\sigma \wedge \omega_0$ is a variational dynamical form on J^1Y, then to every point in J^1Y there exists a neighbourhood U such that $\lambda = L\omega_0$, where L is a function on U defined by

$$L = y^\sigma \int_0^1 E_\sigma(x^i, uy^\nu, uy^\nu_j)\, du, \tag{2.9}$$

is a Lagrangian for E, called *Vainberg–Tonti Lagrangian.*

For more details see [10], [11], [13].

3. Variational properties of systems of first-order PDE

In the sequel, we recall some properties of systems of first-order PDE on manifolds as obtained in [6],[7].

First of all, for *any* system of first-order PDE to be variational, polynomiality in the first-order derivatives is a *necessary* property:

Proposition 3.1. *Let E be a dynamical form on J^1Y, $E = E_\sigma\, \omega^\sigma \wedge \omega_0$. If E is locally variational, then the E_σ are polynomials of degree $\le n$ in the y^ν_j's.*

In view of the above proposition, the components E_σ of a locally variational form E on J^1Y are polynomials of degree at most n in the y^ν_k's with the coefficients completely antisymmetric in both the upper and lower indices. We set

$$\begin{gathered} E_\sigma = B_\sigma + B^{j_1}_{\sigma\nu_1} y^{\nu_1}_{j_1} + \cdots + B^{j_1\cdots j_n}_{\sigma\nu_1\cdots\nu_n} y^{\nu_1}_{j_1}\cdots y^{\nu_n}_{j_n}, \\ B^{j_1\cdots j_p\cdots j_q\cdots j_k}_{\sigma\nu_1\cdots\nu_p\cdots\nu_q\cdots\nu_k} = B^{j_1\cdots j_q\cdots j_p\cdots j_k}_{\sigma\nu_1\cdots\nu_q\cdots\nu_p\cdots\nu_k}, \quad B^{j_1\cdots j_k}_{\sigma\nu_1\cdots\nu_p\cdots\nu_k} = -B^{j_1\cdots j_k}_{\nu_p\nu_1\cdots\sigma\cdots\nu_k}, \quad 1 \le k \le n. \end{gathered} \tag{3.1}$$

Next, first-order locally variational forms are *equivalent* to *closed* $(n+1)$*-forms on* Y.

Theorem 3.1. *Let E be a dynamical form on J^1Y. The following conditions are equivalent:*

(1) *In every fibered chart the components E_σ of E satisfy the following conditions:*

$$\frac{\partial E_\sigma}{\partial y^\nu_j} + \frac{\partial E_\nu}{\partial y^\sigma_j} = 0, \quad \frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} + d_i \frac{\partial E_\nu}{\partial y^\sigma_i} = 0, \quad 1 \le \sigma, \nu \le m,\ 1 \le j \le n. \tag{3.2}$$

(2) *There exists a unique closed $(n+1)$-form α on Y such that $E = p_1\alpha$.*
(3) *E is locally variational.*

Taking into account the relation between dynamical forms and partial differential equations, we obtain an explicit characterization of variational first order PDE and their Lagrangians:

Theorem 3.2. *A system of C^∞ first-order partial differential equations is variational if and only if for some r, $1 \le r \le n$, it is of the form*

$$B^{j_1\cdots j_r}_{\sigma\nu_1\cdots\nu_r} \frac{\partial y^{\nu_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{\nu_r}}{\partial x^{j_r}} + \dots + B^{j_1 j_2}_{\sigma\nu_1\nu_2} \frac{\partial y^{\nu_1}}{\partial x^{j_1}} \frac{\partial y^{\nu_2}}{\partial x^{j_2}} + B^{j_1}_{\sigma\nu_1} \frac{\partial y^{\nu_1}}{\partial x^{j_1}} + B_\sigma = 0, \tag{3.3}$$

where the coefficients are functions of (x^i, y^ν), completely antisymmetric in the upper and lower indices, and the $(n+1)$-form

$$\begin{aligned} \alpha = B_\sigma dy^\sigma \wedge \omega_0 + \frac{1}{2!} B^{j_1}_{\sigma\nu_1} dy^\sigma \wedge dy^{\nu_1} \wedge \omega_{j_1} + \dots \\ + \frac{1}{(r+1)!} B^{j_1\cdots j_r}_{\sigma\nu_1\cdots\nu_r} dy^\sigma \wedge dy^{\nu_1} \wedge \cdots \wedge dy^{\nu_r} \wedge \omega_{j_1\cdots j_r} \end{aligned} \tag{3.4}$$

on Y is closed. In this case, α is the exterior derivative of the Krupka form ρ_λ (2.6) associated with the corresponding Vainberg–Tonti Lagrangian L (which is a polynomial of degree r in the variables y^ν_j).

Let E be a dynamical form on J^1Y. By a *Lepage class of E* we mean the equivalence class $[\alpha]$ of (possibly local) $(n+1)$-forms on J^1Y such that

$$\alpha \in [\alpha] \quad \Longleftrightarrow \quad p_1\alpha = E. \tag{3.5}$$

This means that every element of the class $[\alpha]$ is of the form $\alpha = E + F$ where F is an at least 2-contact form.

By definition, $(n+1)$-forms belonging to the Lepage class of a first-order dynamical form E are defined on open subsets of J^1Y. We say that E is Y*-pertinent* if to every point in Y there exists a neighborhood U and a form α_U belonging to the Lepage class of E, projectable onto U. In other words, E is Y-pertinent if it can be represented by a Lepage class defined on Y.

In [7] the following proposition is proved

Proposition 3.2. *Let E be a dynamical form on J^1Y.*

The following four conditions are equivalent:

(1) *E is Y-pertinent.*
(2) *In every fiber chart, E is of the form $E = E_\sigma dy^\sigma \wedge \omega_0$, where*

$$\begin{aligned} &E_\sigma = B_\sigma + B^{j_1}_{\sigma\nu_1} y^{\nu_1}_{j_1} + \cdots + B^{j_1\cdots j_n}_{\sigma\nu_1\cdots\nu_n} y^{\nu_1}_{j_1} \cdots y^{\nu_n}_{j_n}, \\ &B^{j_1\cdots j_p\cdots j_q\cdots j_k}_{\sigma\nu_1\cdots\nu_p\cdots\nu_q\cdots\nu_k} = B^{j_1\cdots j_q\cdots j_p\cdots j_k}_{\sigma\nu_1\cdots\nu_q\cdots\nu_p\cdots\nu_k}, \quad B^{j_1\cdots j_k}_{\sigma\nu_1\cdots\nu_p\cdots\nu_k} = -B^{j_1\cdots j_k}_{\nu_p\nu_1\cdots\sigma\cdots\nu_k}, \quad 1 \le k \le n. \end{aligned} \tag{3.6}$$

(3) *There exists a unique $(n+1)$-form α on Y such that $E = p_1\alpha$.*
(4) *The $(n+1)$-form*

$$\mathfrak{Lep}_2(E) = E_\sigma \omega^\sigma \wedge \omega_0 + \sum_{k=1}^{n} \frac{1}{k!(k+1)!} \frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{j_1} \cdots y^{\nu_k}_{j_k}} \omega^\sigma \wedge \omega^{\nu_1} \wedge \cdots \wedge \omega^{\nu_k} \wedge \omega_{j_1\cdots j_k}, \tag{3.7}$$

is projectable onto Y.

The mapping $\mathfrak{Lep}_2$, defined by (3.7) is a bijection between Y-pertinent dynamical forms on J^1Y and $(n+1)$-forms on Y. The inverse to $\mathfrak{Lep}_2$ is the mapping p_1.

In view of Proposition 3.2, equations for paths of an Y-pertinent dynamical form E on J^1Y read

$$\gamma^* i_\xi \alpha_E = 0 \quad \text{for every vertical vector field } \xi \text{ on } Y, \tag{3.8}$$

where α_E is the unique Lepage form on Y, associated to E. In other words, paths of E are integral sections of the ideal of differential forms on Y, generated by the following system of n-forms:

$$\mathcal{D}_{\alpha_E} = \{ i_\xi \alpha_E \mid \xi \text{ runs over all vertical vector fields on } Y \}. \tag{3.9}$$

Computing local generators explicitly, we obtain $\mathcal{D}_{\alpha_E} = \operatorname{span}\{\eta_\sigma,\ 1 \le \sigma \le m\}$, where

$$\eta_\sigma = B_\sigma \omega_0 + \sum_{k=1}^{n} \frac{1}{k!}\, B^{j_1 \cdots j_k}_{\sigma \nu_1 \cdots \nu_k} dy^{\nu_1} \wedge \cdots \wedge dy^{\nu_k} \wedge \omega_{j_1 \cdots j_k}. \tag{3.10}$$

Definition 3.1. An Y-pertinent dynamical form E on J^1Y (respectively, equations (3.8), respectively, an $(n+1)$-form α on Y) is called *regular* if

$$\operatorname{rank} \mathcal{D}_{\alpha_E} = m. \tag{3.11}$$

Condition (3.11) obviously means that generators (3.10) of $\mathcal{D}_{\alpha_E}$ are linearly independent at each point of Y, or equivalently, that rank of the matrix

$$\mathrm{B} = \begin{pmatrix} B_\sigma & B^{j_1}_{\sigma\nu_1} & B^{j_1 j_2}_{\sigma \nu_1 \nu_2} & \cdots & B^{j_1 \cdots j_n}_{\sigma \nu_1 \cdots \nu_n} \end{pmatrix}, \tag{3.12}$$

where σ labels rows and the other sets of indices label columns, is maximal and equal to $m = \dim Y - \dim X$ at each point of Y.

The matrix (3.12) is equivalent with the matrix

$$\left(E_\sigma \quad \frac{\partial E_\sigma}{\partial y^{\nu_1}_{j_1}} \quad \frac{\partial^2 E_\sigma}{\partial y^{\nu_1}_{j_1} y^{\nu_2}_{j_2}} \quad \cdots \quad \frac{\partial^n E_\sigma}{\partial y^{\nu_1}_{j_1} \cdots y^{\nu_n}_{j_n}} \right). \tag{3.13}$$

From this fact immediately follows

Proposition 3.3. *Let E be an Y-pertinent dynamical form on J^1Y. For E be regular any of the following n conditions is sufficient:*

$$\operatorname{rank}\left(\frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{j_1} \cdots y^{\nu_k}_{j_k}} \right) = m, \quad 1 \le k \le n, \tag{3.14}$$

where σ labels rows and the other incides label columns.

4. Variational integrating factors for first-order PDE

In this section we will study the question on the existence of variational integrating factors for first-order PDE. The setting of the problem is as follows: given a *dynamical* form E', we can ask if in a neighbourhood U of every point $x \in J^1Y$ there is a *locally variational form* E, such that $E = GE'$ for a regular matrix G on $\pi_{1,0}(U) \subset Y$. If this is the case, we call E' *equivalent* with E and G a *variational integrating factor*, or *variational multiplier* for E'.

We shall discuss properties of the ideals $\mathcal{D}_{\alpha_E}$ and $\mathcal{D}_{\alpha_{E'}}$, regularity conditions, and conditions for an integrating factor G to be variational.

In what follows, we denote by E_σ the components of E, and by E'_ν the components of E'. In fibered coordinates $E_\sigma = G^\nu_\sigma E'_\nu$, where G^ν_σ, $1 \le \sigma, \nu \le m$, are functions of the variables (x^i, y^κ).

Taking into account Definiton 3.1 it is easy to show that the assumption of regularity of the matrix G means that the differential systems D_{α_E} and $D_{\alpha_{E'}}$ are of the same rank.

Proposition 4.1. *Let E,E' be two Y-pertinent dynamical forms on $U \subset J^1Y$, $E = GE'$ for an (mxm)-matrix G. If G is regular then* $\operatorname{rank} D_{\alpha_E} = \operatorname{rank} D_{\alpha_{E'}}$.

Proof. Using the fact that G^ν_σ are functions of the variables (x^i, y^κ), and the relation $E_\sigma = G^\nu_\sigma E'_\nu$, we get

$$\frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{j_1} \dots \partial y^{\nu_k}_{j_k}} = \frac{\partial^k G^\nu_\sigma E'_\nu}{\partial y^{\nu_1}_{j_1} \dots \partial y^{\nu_k}_{j_k}} = G^\nu_\sigma \frac{\partial^k E'_\nu}{\partial y^{\nu_1}_{j_1} \dots \partial y^{\nu_k}_{j_k}}, \quad 1 \le k \le n. \tag{4.1}$$

Hence, if G is regular, we obtain by (3.12), $\operatorname{rank} \mathcal{D}_{\alpha_E} = \operatorname{rank} \mathcal{D}_{\alpha_{E'}}$. □

Remark 4.1. Similar result is valid for systems of ODE of any order (see [12], [13]).

Denote by $\mathcal{I}(\mathcal{D}_{\alpha_E})$ the ideal generated by the system of n-forms D_{α_E}.

Proposition 4.2. *Let E,E' be two Y-pertinent dynamical forms on J^1Y, $E = GE'$ on $U \subset J^1Y$. If G is regular then $\mathcal{I}(\mathcal{D}_{\alpha_E}) = \mathcal{I}(\mathcal{D}_{\alpha_{E'}})$.*

Proof. This assertion follows from the fact that $\mathcal{I}(\mathcal{D}_{\alpha_E})$ and $\mathcal{I}(\mathcal{D}_{\alpha_{E'}})$ are generated by the same system of n-forms.

Indeed,

$$i_\xi \alpha_E = i_\xi \alpha_{GE'} = G i_\xi \alpha_{E'}. \tag{4.2}$$

□

Let us prove the main result of this section.

Theorem 4.1. *Consider an Y-pertinent dynamical form E' on J^1Y. Set*

$$E'_\nu = D_\nu + \sum_{k=1}^{n} D^{j_1 \cdots j_k}_{\nu\kappa_1\cdots\kappa_k} y^{\kappa_1}_{j_1} \cdots y^{\kappa_k}_{j_k} \tag{4.3}$$

Let $x \in J^1Y$ be a point, G a regular matrix defined in a neighbourhood of $\pi_{1,0}(x)$. G is a variational integrating factor for E' if and only if it satisfies the following system of equations:

$$\begin{gathered} G^\nu_\sigma D^l_{\nu\rho} + G^\nu_\rho D^l_{\nu\sigma} = 0 \\ G^\nu_\sigma D^{lj_2\cdots j_k}_{\nu\rho\kappa_2\cdots\kappa_k} + G^\nu_\rho D^{lj_2\cdots j_k}_{\nu\sigma\kappa_2\cdots\kappa_k} = 0, \quad 2 \le k \le n, \end{gathered} \tag{4.4}$$

$$D_\nu\Big(\frac{\partial G^\nu_\sigma}{\partial y^\rho}-\frac{\partial G^\nu_\rho}{\partial y^\sigma}\Big)+G^\nu_\sigma\frac{\partial D_\nu}{\partial y^\rho}-G^\nu_\rho\frac{\partial D_\nu}{\partial y^\sigma}+D^i_{\nu\sigma}\frac{\partial G^\nu_\rho}{\partial x^i}+G^\nu_\rho\frac{\partial D^i_{\nu\sigma}}{\partial x^i}=0,$$

$$\begin{aligned}(4.5)\qquad &D^{lj_2\cdots j_k}_{\nu\kappa\kappa_2\cdots\kappa_k}\Big(\frac{\partial G^\nu_\sigma}{\partial y^\rho}-\frac{\partial G^\nu_\rho}{\partial y^\sigma}\Big)+D^{lj_2\cdots j_k}_{\nu\sigma\kappa_2\cdots\kappa_k}\frac{\partial G^\nu_\rho}{\partial y^\kappa}+G^\nu_\sigma\frac{\partial D^{lj_2\cdots j_k}_{\nu\kappa\kappa_2\cdots\kappa_k}}{\partial y^\rho}\\ &-G^\nu_\rho\frac{\partial D^{lj_2\cdots j_k}_{\nu\kappa\kappa_2\cdots\kappa_k}}{\partial y^\sigma}+G^\nu_\rho\frac{\partial D^{lj_2\cdots j_k}_{\nu\sigma\kappa_2\cdots\kappa_k}}{\partial y^\kappa}+\frac{\partial G^\nu_\rho}{\partial x^i}D^{ilj_2\cdots j_k}_{\nu\sigma\kappa\kappa_2\cdots\kappa_k}+G^\nu_\rho\frac{\partial D^{ilj_2\cdots j_k}_{\nu\sigma\kappa\kappa_2\cdots\kappa_k}}{\partial x^i}=0,\\ &2\le k\le n-1,\end{aligned}$$

$$\begin{aligned}&D^{lj_2\cdots j_n}_{\nu\kappa\kappa_2\cdots\kappa_n}\Big(\frac{\partial G^\nu_\sigma}{\partial y^\rho}-\frac{\partial G^\nu_\rho}{\partial y^\sigma}\Big)+D^{lj_2\cdots j_n}_{\nu\sigma\kappa_2\cdots\kappa_n}\frac{\partial G^\nu_\rho}{\partial y^\kappa}\\ &+G^\nu_\sigma\frac{\partial D^{lj_2\cdots j_n}_{\nu\kappa\kappa_2\cdots\kappa_n}}{\partial y^\rho}-G^\nu_\rho\frac{\partial D^{lj_2\cdots j_n}_{\nu\kappa\kappa_2\cdots\kappa_n}}{\partial y^\sigma}+G^\nu_\rho\frac{\partial D^{lj_2\cdots j_n}_{\nu\sigma\kappa_2\cdots\kappa_n}}{\partial y^\kappa}=0.\end{aligned}$$

Proof. As we have shown above, the components E_σ of a locally variational form satisfy equations (3.2). Taking into account the first of them and using the relation $E_\sigma=G^\nu_\sigma E'_\nu$ we get

$$\begin{aligned}0=&\frac{\partial E_\sigma}{\partial y^\rho_l}+\frac{\partial E_\rho}{\partial y^\sigma_l}\\ (4.6)\qquad =&G^\nu_\sigma\frac{\partial E'_\nu}{\partial y^\rho_l}+G^\nu_\rho\frac{\partial E'_\nu}{\partial y^\sigma_l}\\ =&G^\nu_\sigma\Big(D^l_{\nu\rho}+\sum_{k=2}^n D^{lj_2\cdots j_k}_{\nu\rho\kappa_2\cdots\kappa_k}y^{\kappa_2}_{j_2}\cdots y^{\kappa_k}_{j_k}\Big)+G^\nu_\rho\Big(D^l_{\nu\sigma}+\sum_{k=2}^n D^{lj_2\cdots j_k}_{\nu\sigma\kappa_2\cdots\kappa_k}y^{j_2}_{\kappa_2}\cdots y^{j_k}_{\kappa_k}\Big)\end{aligned}$$

This polynomial is equal to zero if and only if all its coefficients are equal to zero, giving us (4.4).

Next, the E_σ's satisfy also the second of the equations (3.2). Similarly as above we obtain

$$\begin{aligned}0=&\frac{\partial E_\sigma}{\partial y^\rho}-\frac{\partial E_\rho}{\partial y^\sigma}+d_i\frac{\partial E_\rho}{\partial y^\sigma_i}\\ =&E'_\nu\Big(\frac{\partial G^\nu_\sigma}{\partial y^\rho}-\frac{\partial G^\nu_\rho}{\partial y^\sigma}\Big)+G^\nu_\sigma\frac{\partial E'_\nu}{\partial y^\rho}-G^\nu_\rho\frac{\partial E'_\nu}{\partial y^\sigma}\\ (4.7)\qquad &+\frac{\partial G^\nu_\rho}{\partial x^i}\frac{\partial E'_\nu}{\partial y^\sigma_i}+G^\nu_\rho\frac{\partial^2 E'_\nu}{\partial x^i\partial y^\sigma_i}\\ &+\frac{\partial G^\nu_\rho}{\partial y^\kappa}\frac{\partial E'_\nu}{\partial y^\sigma_i}y^\kappa_i+G^\nu_\rho\frac{\partial^2 E'_\nu}{\partial y^\kappa\partial y^\sigma_i}y^\kappa_i\\ &+\frac{\partial G^\nu_\rho}{\partial y^\kappa_m}\frac{\partial E'_\nu}{\partial y^\sigma_i}y^\kappa_{mi}+G^\nu_\rho\frac{\partial^2 E'_\nu}{\partial y^\kappa_m\partial y^\sigma_i}y^\kappa_{mi}.\end{aligned}$$

Last two terms are obviously equal to zero. Differentiating E'_ν and using (4.3) we get

$$\begin{aligned}
&\Big(D_\nu + \sum_{k=1}^{n} D^{j_1\cdots j_k}_{\nu\kappa_1\cdots\kappa_k} y^{j_1}_{\kappa_1}\cdots y^{j_k}_{\kappa_k}\Big)\Big(\frac{\partial G^\nu_\sigma}{\partial y^\rho} - \frac{\partial G^\nu_\rho}{\partial y^\sigma}\Big)\\
&+ G^\nu_\sigma\Big(\frac{\partial D_\nu}{\partial y^\rho} + \sum_{k=1}^{n}\frac{\partial D^{j_1\cdots j_k}_{\nu\kappa_1\cdots\kappa_k}}{\partial y^\rho} y^{j_1}_{\kappa_1}\cdots y^{j_k}_{\kappa_k}\Big)\\
&- G^\nu_\rho\Big(\frac{\partial D_\nu}{\partial y^\sigma} + \sum_{k=1}^{n}\frac{\partial D^{j_1\cdots j_k}_{\nu\kappa_1\cdots\kappa_k}}{\partial y^\sigma} y^{j_1}_{\kappa_1}\cdots y^{j_k}_{\kappa_k}\Big)\\
&+ \frac{\partial G^\nu_\rho}{\partial x^i}\Big(D^i_{\nu\sigma} + \sum_{k=2}^{n} D^{ij_2\cdots j_k}_{\nu\sigma\kappa_2\cdots\kappa_k} y^{j_2}_{\kappa_2}\cdots y^{j_k}_{\kappa_k}\Big)\\
&+ G^\nu_\rho\Big(\frac{\partial D^i_{\nu\sigma}}{\partial x^i} + \sum_{k=2}^{n}\frac{\partial D^{ij_2\cdots j_k}_{\nu\sigma\kappa_2\cdots\kappa_k}}{\partial x^i} y^{j_2}_{\kappa_2}\cdots y^{j_k}_{\kappa_k}\Big)\\
&+ \frac{\partial G^\nu_\rho}{\partial y^\kappa}\Big(D^i_{\nu\sigma}y^\kappa_i + \sum_{k=2}^{n} D^{ij_2\cdots j_k}_{\nu\sigma\kappa_2\cdots\kappa_k} y^{j_2}_{\kappa_2}\cdots y^{j_k}_{\kappa_k} y^\kappa_i\Big)\\
&+ G^\nu_\rho\Big(\frac{\partial D^i_{\nu\sigma}}{\partial y^\kappa}y^\kappa_i + \sum_{k=2}^{n}\frac{\partial D^{ij_2\cdots j_k}_{\nu\sigma\kappa_2\cdots\kappa_k}}{\partial y^\kappa} y^{j_2}_{\kappa_2}\cdots y^{j_k}_{\kappa_k} y^\kappa_i\Big) = 0
\end{aligned} \tag{4.8}$$

It remains to express the last equation as a polynomial and compare the coefficients at the terms of the same degree. Finally we obtain (4.5). □

Corollary 4.1. *Let* $\dim X = 2$, $\dim Y = 3$. *Suppose that (4.3) is quasilinear, i.e.*

$$E'_1 = D_1 + D^1_{11}y^1_1 + D^2_{11}y^1_2, \tag{4.9}$$

where at least one of the D^k_{11}, $k \in \{1,2\}$, *is nonzero. Then* E' *has no variational integrating factor.*

Proof. The first of equations (4.4) immediately gives

$$\begin{aligned} G^1_1 D^1_{11} + G^1_1 D^1_{11} &= 0,\\ G^1_1 D^2_{11} + G^1_1 D^2_{11} &= 0. \end{aligned} \tag{4.10}$$

The only solution of this system of equations is $G = (G^1_1) = 0$ Hence, there is no non-zero integrator for E'. □

Corollary 4.2. *Let* $\dim X = 2$, $\dim Y = 4$. *Suppose that*

$$\begin{aligned} E'_1 &= 1 + y^1_1 + y^2_2,\\ E'_2 &= 1 + y^1_1 + y^2_2. \end{aligned} \tag{4.11}$$

Then every regular matrix G, *satisfying the following equations*

$$\begin{aligned} G^1_1 + G^2_1 &= 0,\\ G^1_2 + G^2_2 &= 0, \end{aligned} \tag{4.12}$$

is a variational integrating factor for E'.

Proof. In view of (4.11), the coefficients D^l_{jk} are the following

$$
\begin{aligned}
&D_1 = D_2 = 1,\\
&D^1_{11} = D^2_{12} = D^1_{21} = D^2_{22} = 1,\\
&D^1_{12} = D^2_{11} = D^1_{21} = D^2_{21} = 0,
\end{aligned}
\tag{4.13}
$$

and only the first equation from the system of equations (4.4) (resp.(4.5)) is nontrivial. For different choice of the coefficients l, σ, ρ, where $1 \leq l, \sigma, \rho \leq 2$, the first of equations (4.4) gives six equations as follows

$$
\begin{aligned}
&2G^1_1 + 2G^2_1 = 0,\\
&2G^1_2 + 2G^2_2 = 0,\\
&G^1_2 + G^2_2 = 0,\\
&G^1_1 + G^2_1 = 0.
\end{aligned}
\tag{4.14}
$$

Similarly, the second of equations (4.5) gives four equations, which vanish identically. Finally, we conclude (4.12). □

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