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Reducibility of zero curvature representations with application to recursion operators

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Abstract

We present a criterion for reducibility of zero curvature representations to a solvable subalgebra, hence to a chain of conservation laws. Our results are applied to inversion of recursion operators.

1 Introduction

To establish integrability of a nonlinear partial differential equation in the sense of soliton theory [1, 22], at least in two dimensions, one usually looks for a zero curvature representation (ZCR) [26], possibly in the form of a Lax pair [13]. If depending on a non-removable (spectral) parameter, a ZCR may serve as a starting point of methods to derive infinitely many independent conservation laws and large classes of exact solutions.

However, certain ZCR's do not imply integrability because of specific degeneracy, which does not even contradict possible presence of one or more nonremovable parameters. E.g., Calogero and Nucci [2] presented a formula to assign a Lax pair to any nonlinear system possessing a single conservation law, arguing that such systems are too abundant to be all integrable. Recently Sakovich [20] observed that the Calogero–Nucci examples can be singled out by properties of their associated cyclic bases; in particular, they do not generate integrable hierarchies.

In this paper we postulate that a ZCR is degenerate if it takes values in a solvable Lie algebra or is gauge equivalent to such (Sect. 4). This idea is, of course, certainly not new (Dodd and Fordy [6, Sect. 3]). In the case of abelian Lie algebras it was shown in [1, Sect. 3.2.c] that the ZCR is equivalent to a set of local conservation laws. In the more general case of solvable algebras, it follows rather easily from the Lie theorem that every such ZCR is equivalent to a chain of nonlocal conservation laws. In particular, the result renders any attempts to generate infinitely many independent conservation laws out of a degenerate ZCR rather unrealistic.

In Sect. 5 we concentrate on the problem of detecting reducibility to a subalgebra, in particular, a solvable one. Purely algebraic criteria are insufficient since the Lie algebra a ZCR takes values in may be altered by gauge transformation. On the other hand, when trying to find the reducing gauge matrix directly, one

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encounters a rather untractable underdetermined differential system. Our idea is to restrict the manifold of gauge matrices by employing appropriate matrix decompositions, in particular, the Gram and Gauss decomposition, also known as the QR- and LU-decomposition, respectively. Remarkable connections between these decompositions and integrable systems were discovered in numerical analysis, see [4, 5, 25] and references therein.

Section 6 is devoted to recursion operators in Guthrie form, which involve a ZCR. Such a recursion operator can be expressed in the traditional terms of inverse total derivatives D^{-1} ([14, 15]) whenever the ZCR is strictly lower triangular or gauge equivalent to such. In combination with the results of previous sections, we get a recipe for inversion of recursion operators in terms of D^{-1}. In the last section of the paper, we invert the recursion operator of the Tzitzéica equation.

2 Preliminaries

Let E be a system of nonlinear partial differential equations (PDE)

$$F^l = 0 \tag{1}$$

on a number of functions u^k in two independent variables x, y. Here each F^l is a smooth function depending on a finite number of variables x, y, u^k, $u^k_x, u^k_y, \ldots, u^k_I, \ldots$, where I stands for a symmetric multiindex over the two-element set of indices $\{x, y\}$. Besides the *local* variables x, y, u^k, u^k_I, we shall also need non-local variables or pseudopotentials [24], which may be introduced as additional variables satisfying a system of equations

$$z^i_x = f^i, \qquad z^i_y = g^i, \tag{2}$$

where f^i, g^i are functions depending on a finite number of local variables as well as the pseudopotentials z^j; we require that the system (2) be compatible as a consequence of (1).

Within their geometric theory of systems of PDE's, Krasil'shchik and Vinogradov [11] introduced the notion of a covering, which separates the invariant content of nonlocality from its coordinate presentation. Pseudopotentials then correspond to a particular choice of coordinates along the fibres of the covering in question. We recall the basic facts below; details we had to leave aside may be drawn from [11] and also from [12, Ch. 6].

Let J^∞ be an infinite jet space equipped with local jet coordinates x, y, u^k, u^k_I; the functions F^l then may be interpreted as functions defined on J^∞. For simplicity we define J^∞ to be the space of jets of local sections of a fibred manifold $Y \longrightarrow M$ over a two-dimensional base manifold M equipped with local coordinates x, y, while u^k are local coordinates along the fibres. In every domain of definition of the independent variables x, y, we have two distinguished commuting vector fields on J^∞,

$$D_x = \frac{\partial}{\partial x} + \sum_{k,I} u^k_{Ix} \frac{\partial}{\partial u^k_I}, \qquad D_y = \frac{\partial}{\partial y} + \sum_{k,I} u^k_{Iy} \frac{\partial}{\partial u^k_I},$$

which are called *total derivatives*.

The *equation manifold* $\mathcal{E}$ associated with system (1) is defined to be the submanifold in J^∞ determined by the infinite system of equations $F^l = 0$ and $D_I F^l = 0$ for I running through all symmetric multiindices in x, y. The total derivatives D_x, D_y are vector fields on J^∞ tangent to $\mathcal{E}$, therefore they admit a restriction to $\mathcal{E}$. The restricted fields then generate the *Cartan distribution* $\mathcal{C}$ on $\mathcal{E}$. The pair $(\mathcal{E}, \mathcal{C})$, called a *diffiety*, is an invariant geometric object associated with system (1). In what follows, we shall not make significant difference between equations and the corresponding diffieties.

Mappings between diffieties that preserve the Cartan distributions are called *morphisms* of diffieties; they map solutions to solutions. We additionally require that morphisms commute with projections to the base manifold M; these will be called morphisms *over* M. Bijective morphisms are called *isomorphisms*; their inverses are isomorphisms, too.

A *covering* over a diffiety $\mathcal{E}$ consists of another diffiety $\mathcal{E}'$ and a surjective submersion $\mathcal{E}' \longrightarrow \mathcal{E}$ over M such that the Cartan distribution on $\mathcal{E}'$ is projected onto the Cartan distribution on $\mathcal{E}$.

The system formed by equation (1) and the $2k$ additional equations (2) generates a covering, where $\mathcal{E}'$ is the product $\mathcal{E} \times \mathbb{R}^k$ and $z^1, \dots, z^k$ provide coordinates along $\mathbb{R}^k$. In particular, the projection preserves the coordinates x, y. If f^i, g^i are functions defined on E' such that the vector fields

$$D'_x = D_x + \sum_{i=1}^{k} f^i \frac{\partial}{\partial z^i}, \qquad D'_y = D_x + \sum_{i=1}^{k} g^i \frac{\partial}{\partial z^i} \tag{3}$$

commute (which is a geometric way of saying that equations (2) are compatible), then $\mathcal{E}'$ equipped with the vector fields (3) is a diffiety and a k-dimensional covering over $\mathcal{E}$. Recall from [11] that every finite-dimensional covering is locally of this form.

Two coverings $\mathcal{E}'$ and $\mathcal{E}''$ are said to be *isomorphic over* $\mathcal{E}$ if there exists an isomorphism of the diffieties $\mathcal{E}' \cong \mathcal{E}''$ that commutes with the projections to $\mathcal{E}$. Isomorphic coverings result from invertible transformations of nonlocal variables. A k-dimensional covering is said to be *trivial* if it is isomorphic to one with $f^i = g^i = 0$; such a covering is essentially a family of identical copies of $\mathcal{E}$.

The simplest yet useful covering (2) may be associated with a single nontrivial conservation law $\alpha = f\,dx + g\,dy$, i.e., a pair of functions f, g defined on $\mathcal{E}$ and satisfying $D_y f = D_x g$ on $\mathcal{E}$:

Definition 1 A *one-dimensional abelian covering* associated with a conservation law $\alpha = f\,dx + g\,dy$ is defined to be the product projection $\mathcal{E} \times \mathbb{R} \longrightarrow \mathcal{E}$, equipped with total derivatives

$$D'_x = D_x + f\frac{\partial}{\partial z}, \qquad D'_y = D_x + g\frac{\partial}{\partial z},$$

where z denotes the coordinate along $\mathbb{R}$.

One easily checks that the vector fields D'_x, D'_y on $\mathcal{E}'$ commute if and only if $D_y f = D_x g$. In this case, the variable z is called the *potential* of the conservation law α. We have $D'_x z = f$, $D'_y z = g$ or briefly $z_x = f$, $z_y = g$.

Recall that a conservation law is said to be *trivial* if there exists a (local) function h on $\mathcal{E}$ such that $f = D_x h$, $g = D_y h$. A covering associated to a trivial conservation law is isomorphic to a trivial covering through the invertible change of variables $z = z' + h$.

A covering $\bar{\mathcal{E}} \longrightarrow \mathcal{E}$ is said to be *trivializing* for a conservation law $\alpha = f\,dx + g\,dy$, if the pullback $\bar{\alpha}$ of α along the projection $\bar{\mathcal{E}} \longrightarrow \mathcal{E}$ is a trivial conservation law on $\bar{\mathcal{E}}$. Obviously, the one-dimensional abelian covering associated with the conservation law α trivializes α.

A general n-dimensional abelian covering is obtained by repeating the construction of the one-dimensional abelian covering (cf. [24, Sect. IV]):

Definition 2 An n-dimensional covering $\tilde{\mathcal{E}}$ over $\mathcal{E}$ is said to be *abelian*, if

(1) either $n = 1$ and $\tilde{\mathcal{E}}$ is a one-dimensional abelian covering over $\mathcal{E}$;

(2) or $\tilde{\mathcal{E}}$ is a one-dimensional abelian covering over an $(n-1)$-dimensional abelian covering $\mathcal{E}'$ over $\mathcal{E}$.

Let us note that Khorkova [10] introduced the *universal abelian covering*, which need not be finite-dimensional.

3 Zero-curvature representations

Pseudopotentials can also come from zero-curvature representations. Let $\mathfrak{g}$ be a matrix Lie algebra (recall that according to the Ado theorem every finite-dimensional Lie algebra has a matrix representation). By a $\mathfrak{g}$-*valued zero-curvature representation* (ZCR) for $\mathcal{E}$ we mean a $\mathfrak{g}$-valued one-form $\alpha = A\,dx + B\,dt$ defined on $\mathcal{E}$ such that

$$D_y A - D_x B + [A, B] = 0 \tag{4}$$

holds on $\mathcal{E}$, which means that (4) holds as a consequence of system (1) (we do not insist that (4) necessarily reproduces system (1), which is normally required in integrability theory).

Let $\mathcal{G}$ be the connected and simply connected matrix Lie group associated with $\mathfrak{g}$. Then for an arbitrary $\mathcal{G}$-valued function S, the form $\alpha^S = A^S\,dx + B^S\,dt$, where

$$A^S = D_x S S^{-1} + S A S^{-1}, \qquad B^S = D_y S S^{-1} + S B S^{-1} \tag{5}$$

is another ZCR, which is said to be *gauge equivalent* to the former.

A ZCR is said to be *trivial* if it is gauge equivalent to zero, i.e., if $A = D_x S S^{-1}$, $B = D_y S S^{-1}$. A covering $\bar{\mathcal{E}} \longrightarrow \mathcal{E}$ is said to *trivialize* a ZCR $\alpha = A\,dx + B\,dy$ if the pullback $\bar{\alpha}$ of α along the projection $\bar{\mathcal{E}} \longrightarrow \mathcal{E}$ is a trivial ZCR.

A trivializing covering for the ZCR α can be obtained in the following way.

Proposition 3 *For every $\mathfrak{g}$-valued ZCR α on $\mathcal{E}$ there exists a covering $\pi_\alpha : \tilde{\mathcal{E}}_\alpha \longrightarrow \mathcal{E}$ that trivializes α.*

Proof Let $\alpha = A\,dx + B\,dy$ be a ZCR, where A and B are $n \times n$ matrices belonging the algebra $\mathfrak{g}$. Put $\tilde{\mathcal{E}}_\alpha = \mathcal{E} \times G$, where G is the matrix Lie group associated with $\mathfrak{g}$. Given an element $C \in \mathfrak{g}$, denote by ξ_C the right-invariant vector field on G corresponding to C. Given a $\mathfrak{g}$-valued function C on $\mathcal{E}$, let us denote by Ξ_C the unique vector field on $\tilde{\mathcal{E}}_\alpha$ with the $\mathcal{E}$-component zero and the G-component equal to ξ_C, at each point of $\tilde{\mathcal{E}}_\alpha$. Consider the vector fields

$$\widetilde{D}_x = D_x + \Xi_A, \qquad \widetilde{D}_y = D_y + \Xi_B$$

on $\tilde{\mathcal{E}}_\alpha$, where D_x, D_y are the total derivatives on $\mathcal{E}$. Let us show that $\widetilde{D}_x, \widetilde{D}_y$ are the total derivatives for a trivializing covering $\pi_\alpha : \tilde{\mathcal{E}}_\alpha \longrightarrow \mathcal{E}$ of α.

Let $A = (a_{ij})$, $B = (b_{ij})$. Let us first consider $G = \mathrm{GL}_n$ with its natural parametrization $\mathrm{GL}_n = \{(z_{ij}) \mid \det z_{ij} \ne 0\}$. We have

$$\Xi_A = \sum_{i,j,l} a_{ij} z_{jl} \frac{\partial}{\partial z_{il}}, \qquad \Xi_B = \sum_{i,j,l} b_{ij} z_{jl} \frac{\partial}{\partial z_{il}}.$$

Then $\widetilde{D}_x, \widetilde{D}_y$ commute since

$$\begin{aligned}
[\widetilde{D}_x, \widetilde{D}_y] &= [D_x, D_y] + [D_x, \Xi_B] - [\Xi_A, D_y] + [\Xi_A, \Xi_B] \\
&= \Xi_{D_x B - D_y A - [A,B]} \\
&= 0.
\end{aligned}$$

The same holds for arbitrary $G \subseteq \mathrm{GL}_n$, since the vector fields Ξ_A, Ξ_B are tangent to G whenever A, B belong to $\mathfrak{g}$.

Now denote by Θ the projection $\tilde{\mathcal{E}}_\alpha = \mathcal{E} \times G \longrightarrow G$ viewed as a matrix-valued function on $\tilde{\mathcal{E}}_\alpha$. Then $D_x \Theta = 0$ and therefore

$$(\widetilde{D}_x \Theta)_{\mu\nu} = (\Xi_A \Theta)_{\mu\nu} = \sum_{i,j,l} a_{ij} z_{jl} \frac{\partial}{\partial z_{il}} z_{\mu\nu} = \sum_j a_{\mu j} z_{j\nu} = (A\Theta)_{\mu\nu}.$$

Thus, $\widetilde{D}_x \Theta \cdot \Theta^{-1} = A$ and similarly $\widetilde{D}_y \Theta \cdot \Theta^{-1} = B$, whence the pullback of α on $\tilde{\mathcal{E}}_\alpha$ is trivial.

The system (2) corresponding to $\tilde{\mathcal{E}}_\alpha$ can be compactly written in terms of a single matrix Θ as

$$\Theta_x = A\Theta, \qquad \Theta_y = B\Theta. \tag{6}$$

Under the gauge transformation (5), the matrix Θ becomes $S\Theta$. The coverings $\tilde{\mathcal{E}}_\alpha$ and $\tilde{\mathcal{E}}_{\alpha^S}$ are isomorphic via $\Theta \longmapsto S\Theta$ with the inverse $\Theta \longmapsto S^{-1}\Theta$.

The trivializing covering π_α just constructed has the following factorization property:

Proposition 4 *Let $p : \mathcal{E}' \longrightarrow \mathcal{E}$ over M be a trivializing covering for a ZCR α on $\mathcal{E}$. Then there exists a morphism $p^\sharp : \mathcal{E}' \longrightarrow \tilde{\mathcal{E}}_\alpha$ such that $\pi_\alpha \circ p^\sharp = p$.*

Proof Let $\alpha = A\,dx + B\,dy$. Since p is over M, we have $p^*\alpha = p^*A\,dx + p^*B\,dy$. By assumption this is a trivial ZCR, whence $p^*A = D'_x S S^{-1}$ and $p^*B = D'_y S S^{-1}$ for a suitable G-valued function S on $\mathcal{E}'$. Recall that fibres of the covering $\tilde{\mathcal{E}}_\alpha$ are isomorphic to the Lie group G. Therefore we can define $p^\sharp : \mathcal{E}' \longrightarrow \tilde{\mathcal{E}}_\alpha$ by the formula $\Theta \circ p^\sharp = S$, where, as above, Θ denotes the projection $\tilde{\mathcal{E}}_\alpha = \mathcal{E} \times G \longrightarrow G$.

4 Lower triangular ZCR's

Let $\mathfrak{t}_n$ denote the algebra of matrices

$$\begin{pmatrix} a_{11} & 0 & \cdot & \cdot & 0 \\ a_{21} & a_{22} & 0 & \cdot & \cdot \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ a_{n1} & a_{n2} & a_{n3} & \cdot & a_{nn} \end{pmatrix}. \tag{7}$$

Denote by $\mathfrak{t}_n^{(k)}$, $k \geq 1$, the derived algebra formed by matrices satisfying $a_{ij} = 0$ whenever $i - j < k$.

ZCR's with values in $\mathfrak{t}_n$ are, in a sense, equivalent to an abelian covering.

Proposition 5 *Every $\mathfrak{t}_n$-valued ZCR can be trivialized by means of an abelian covering of dimension $\leq \frac{1}{2}n(n+1)$.*

Proof Let $\alpha = A\,dx + B\,dy$ be a ZCR such that A and B are lower triangular. We shall construct an abelian covering $\mathcal{E}^{(n-1)}$ in n steps.

It follows from equation (4) that $\gamma_1 = a_{11}\,dx + b_{11}\,dy$, $\gamma_2 = a_{22}\,dx + b_{22}\,dy, \dots,$ $\gamma_n = a_{nn}\,dx + b_{nn}\,dy$ are conservation laws. Let us denote by $\mathcal{E}^{(0)}$ the associated abelian covering with potentials $h_1, \dots, h_n$ satisfying

$$h_{i,x} = a_{ii}, \quad h_{i,y} = b_{ii} \qquad \text{for } i = 1, \dots, n.$$

Then

$$H = \begin{pmatrix} e^{-h_1} & 0 & 0 & \cdot & 0 \\ 0 & e^{-h_2} & 0 & \cdot & 0 \\ 0 & 0 & e^{-h_3} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & e^{-h_n} \end{pmatrix},$$

is a matrix defined on $\mathcal{E}^{(0)}$, with the property that all diagonal entries of $A' = A^H$ vanish:

$$A' = \begin{pmatrix} 0 & \cdot & \cdot & 0 & 0 \\ a'_{21} & 0 & \cdot & \cdot & 0 \\ a'_{31} & a'_{32} & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a'_{n1} & a'_{n2} & \cdot & a'_{n,n-1} & 0 \end{pmatrix}, \tag{8}$$

and similarly for B'. Hence, A', B' take values in $t_n^{(1)}$.

By the same equation (4), $\gamma'_2 = a'_{21}\,dx + b'_{21}\,dy$, $\gamma'_3 = a'_{32}\,dx + b'_{32}\,dy, \dots,$ $\gamma'_n = a'_{n-1,n}\,dx + b'_{n-1,n}\,dy$ are conservation laws on $\mathcal{E}^{(0)}$. Let us introduce a covering $\mathcal{E}'$ over $\mathcal{E}^{(0)}$ with potentials $h'_2, \dots, h'_n$ satisfying

$$h'_{i,x} = a'_{i,i-1}, \quad h'_{i,y} = b'_{i,i-1} \quad \text{for } i = 2, \dots, n.$$

Denoting

$$H' = \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 \\ -h'_2 & 1 & \cdot & \cdot & 0 \\ 0 & -h'_3 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & -h'_n & 1 \end{pmatrix},$$

the gauge equivalent matrices $A'' = A'^{H'}$ and $B'' = B'^{H'}$ take values in $t_n^{(2)}$ now. Compared with (8), A'' and B'' have one more subdiagonal of zeroes. The next step is similar: $\gamma''_3 = a''_{31}\,dx + b''_{31}\,dy$, $\gamma''_4 = a''_{42}\,dx + b''_{42}\,dy$, $\ldots$, $\gamma''_n = a''_{n-2,n}\,dx + b''_{n-2,n}\,dy$ are conservation laws on $\mathcal{E}'$. Let us introduce a covering $\mathcal{E}''$ over $\mathcal{E}'$ with potentials $h''_2, \ldots, h''_n$ satisfying

$$h''_{i,x} = a''_{i,i-2}, \quad h''_{i,y} = b''_{i,i-2} \quad \text{for } i = 3, \ldots, n.$$

Denoting

$$H'' = \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 1 & \cdot & \cdot & 0 & 0 \\ -h''_3 & 0 & 1 & \cdot & \cdot & 0 \\ 0 & -h''_4 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & -h''_n & 0 & 1 \end{pmatrix}$$

we observe that $A''' = A''^{H''}$, $B''' = B''^{H''}$ take values in $t_n^{(3)}$, and so on. Continuing the process until $A^{(n)}$, $B^{(n)}$ become zero, we end up with a sequence of $\frac{1}{2}n(n+1)$ conservation laws

$$\begin{array}{ccccc} \gamma_1 & \gamma_2 & \gamma_3 & \ldots & \gamma_n \\ & \gamma'_2 & \gamma'_3 & \ldots & \gamma'_n \\ & & \gamma''_3 & \ldots & \gamma''_n \\ & & & \ldots & \\ & & & \gamma_{n-1}^{(n-2)} & \gamma_n^{(n-2)} \\ & & & & \gamma_n^{(n-1)}, \end{array} \tag{9}$$

where (a) $\gamma_1, \ldots, \gamma_n$ are conservation laws on $\mathcal{E}$; (b) $\gamma_{n-\iota+1}^{(n-\iota)}, \ldots, \gamma_n^{(n-\iota)}$ are conservation laws defined on the abelian covering $\mathcal{E}^{(n-\iota-1)}$ associated with the conservation laws of all the previous levels.

Finally, $\alpha^{HH'\cdots H^{(n-1)}} = \alpha^{(n)} = 0$, where each $H^{(\iota)}$ is defined on $\mathcal{E}^{(\iota)}$. Summing up, the covering $\mathcal{E}^{(n-1)}$ trivializes α.

The sequence (9) will be called an *n-fold chain of conservation laws*; the associated abelian covering $\mathcal{E}^{(n-1)}$ of dimension $\frac{1}{2}n(n+1)$ will be called an *n-fold chain covering.*

Obviously, every finite-dimensional abelian covering is a reduction of an n-fold chain covering for a sufficiently large n (one may add as many trivial conservation laws as needed).

Proposition 6 *Let α be a $\mathfrak{t}_n$-valued ZCR, then the associated covering π_α is isomorphic to an abelian covering of dimension $\leq \frac{1}{2}n(n+1)$.*

Proof According to Proposition 5, there is an abelian covering $p : \mathcal{E}^{(n-1)} \longrightarrow \mathcal{E}$ that is trivializing for α; namely, we have $\alpha^K = 0$, where $K = HH' \cdots H^{(n-1)}$ (see proof of Proposition 5). Hence, $\alpha = 0^{K^{-1}}$ and, according to Proposition 4, there is a morphism of diffieties $p^\sharp : \mathcal{E}^{(n-1)} \longrightarrow \tilde{\mathcal{E}}_\alpha$, given by $\Theta = K^{-1}$. Here Θ represents the totality of coordinates along the fibres of the covering $\tilde{\mathcal{E}}_\alpha$, while K is parametrised by coordinates $h_s^{(\iota)}$ along the fibres of the covering $\mathcal{E}^{(n-1)}$. It follows that $p^\sharp$ is bijective on the fibres, hence isomorphism.

5 Reducibility

A $\mathfrak{g}$-valued ZCR is said to be *reducible* if it is gauge equivalent to a ZCR taking values in a proper subalgebra $\mathfrak{h} \subset \mathfrak{g}$; otherwise it is said to be *irreducible*.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. We present a simple criterion for reducibility of a $\mathfrak{g}$-valued ZCR to $\mathfrak{h}$. Let $\mathcal{H} \subset \mathcal{G}$ be the Lie subgroup corresponding to the subalgebra $\mathfrak{h}$. We call $\mathcal{H}$ a *right factor* if there exists a submanifold $\mathcal{K} \subset \mathcal{G}$ (possibly with singularities) such that the multiplication map

$$\mu : \mathcal{K} \times \mathcal{H} \longrightarrow \mathcal{G}, \qquad (K, H) \longmapsto KH \tag{10}$$

is a surjective local diffeomorphism. The manifold $\mathcal{K}$ will be called a *cofactor*. By surjectivity, every element $S \in \mathcal{G}$ can be decomposed as a product $S = KH$, where $K \in \mathcal{K}$ and $H \in \mathcal{H}$, possibly non-uniquely. The map μ being a local diffeomorphism, $\mathcal{K}$ has the minimal possible dimension $\dim \mathcal{K} = \dim \mathcal{G} - \dim \mathcal{H}$. If $\mathcal{H}$ is closed, then the assignment $K \longmapsto K\mathcal{H}$ defines a local diffeomorphism of $\mathcal{K}$ onto the homogeneous space $\mathcal{G}/\mathcal{H}$.

Proposition 7 *In the above notation, a $\mathfrak{g}$-valued ZCR α is reducible to the subalgebra $\mathfrak{h}$ if and only if there exists a local $\mathcal{K}$-valued matrix function K such that α^K lies in $\mathfrak{h}$.*

Proof The gauge equivalence with respect to $H \in \mathcal{H}$ preserves the subalgebra $\mathfrak{h}$. Therefore, the gauge-equivalent ZCR $\alpha^S = (\alpha^K)^H$ lies in $\mathfrak{h}$ if and only if α^K lies in $\mathfrak{h}$.

Otherwise said, if a ZCR is reducible to $\mathfrak{h}$, then the corresponding gauge matrix can be found in $\mathcal{K}$. Understandably, different choices of the cofactor $\mathcal{K}$ may lead to different reducibility criteria.

In this paper we are primarily interested in detecting reducibility to a solvable subalgebra. By the well-known Lie theorem ([7, Sect. 9.2]), every finite-dimensional representation of a solvable Lie algebra is equivalent to a representation by lower triangular matrices. Hence, every ZCR reducible to a solvable subalgebra is reducible to $\mathfrak{t}_n$ (and can be trivialized using an abelian covering according to Proposition 5).

Let us therefore apply Proposition 7 to $\mathfrak{h} = \mathfrak{t}_n$. There are two standard ways to make $\mathfrak{t}_n$ a right factor in $\mathfrak{gl}_n$.

The QR or Gram decomposition is an alternative formulation of the Gram–Schmidt algorithm. Namely, every $n \times n$ complex matrix A can be decomposed as a product $A = QR$, where $Q \in \mathrm{SU}_n$ and $R \in \mathfrak{t}_n$ [18, Ch. 6, Sect. 1.9]. Proposition 7 then yields

Proposition 8 *A real (complex) ZCR α is reducible to lower triangular if and only if there exists an SO_n-valued (SU_n-valued) local function K such that α^K is lower triangular.*

However, the factors Q and R are unique up to a unimodular diagonal multiplier: $QR = Q\Theta \cdot \Theta^{-1} R$, where $\Theta = \mathrm{diag}(\theta_1, \dots, \theta_n) \in \mathrm{S}(\mathrm{U}_1 \times \dots \times \mathrm{U}_1)$, i.e., $|\theta_\iota| = 1$ and $\prod_{\iota=1}^n \theta_\iota = 1$. Thus, the mapping (10) is not a local diffeomorphism unless it is restricted to a suitable immersion of the orbit space $\mathrm{SU}_n/\mathrm{S}(\mathrm{U}_1 \times \dots \times \mathrm{U}_1)$ into SU_n. In the real case we have $\theta_\iota = \pm 1$ and we get a 2^{n-1}-to-one local diffeomorphism (10) with $\mathcal{K} = \mathrm{SO}_n$.

When $n = 2$, the condition is, in the real case,

Proposition 9 *A $\mathfrak{gl}_2$-valued ZCR*

$$\alpha = A\,dx + B\,dy = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} dx + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} dy$$

is reducible to lower triangular if and only if there exists a local solution ϕ of the system

$$\begin{aligned} D_x \phi &= -a_{12} \cos^2 \phi + (a_{11} - a_{22}) \sin \phi \cos \phi + a_{21} \sin^2 \phi, \\ D_y \phi &= -b_{12} \cos^2 \phi + (b_{11} - b_{22}) \sin \phi \cos \phi + b_{21} \sin^2 \phi. \end{aligned} \tag{11}$$

Proof An arbitrary SO_2 matrix is

$$K = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.$$

By Proposition 8, the ZCR α is reducible to lower triangular if and only if α^K is lower triangular, which is exactly the meaning of conditions (11).

Example 10 The ZCR

$$\begin{aligned} \alpha = &\begin{pmatrix} \frac{1}{2}\lambda & \frac{1}{4}u + \frac{1}{2}\lambda \\ \frac{1}{4}u - \frac{1}{2}\lambda & -\frac{1}{2}\lambda \end{pmatrix} dx \\ &+ \begin{pmatrix} \frac{1}{4}\lambda u & \frac{1}{4}u_x + \frac{1}{8}u^2 + \frac{1}{4}\lambda u \\ \frac{1}{4}u_x + \frac{1}{8}u^2 - \frac{1}{4}\lambda u & -\frac{1}{4}\lambda u \end{pmatrix} dt \end{aligned}$$

of the Burgers equation $u_t = u_{xx} + u u_x$ (see [3]) is reducible to lower triangular. Indeed, in this case equations (11) have a local solution $\phi = \frac{1}{4}\pi$. The SO_2 matrix to make this ZCR lower triangular comes out as

$$K = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}.$$

Observe that we could obtain the result algebraically, since K is a constant matrix.

Let us also remark that Dodd and Fordy [6] established solvability of the Wahlquist–Estabrook prolongation algebra of the Burgers and the Kaup equation.

The LU or Gauss decomposition can be derived from the Gaussian elimination algorithm. We have the following corollary of [18, Ch. 6, Sect. 1.8]:

Proposition 11 *For every non-singular matrix A there exist matrices P, U, L such that $A = PUL$, L is lower triangular, U is upper triangular with diagonal entries equal to 1, and P is a permutation matrix. The matrix P can be omitted if all principal minors of the matrix A are nonzero.*

(Recall that Gaussian elimination may require row swapping, which is where the permutation matrix P comes from.) Let $\mathcal{K}$ denote the set of all products PU where P is a permutation matrix and U is an upper triangular matrix with diagonal entries equal to 1. Then K is a union of n intersecting submanifolds, labelled by permutation matrices P. Each of these submanifolds can lead to a different condition.

Proposition 12 *A ZCR α is reducible to lower triangular if and only if there exists a permutation matrix P and a matrix-valued local function*

$$H = \begin{pmatrix} 1 & h_{12} & h_{13} & \cdot & \cdot \\ 0 & 1 & h_{23} & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 1 \end{pmatrix} \tag{12}$$

such that α^{PH} is lower triangular.

When $n = 2$, the reducibility conditions are

Proposition 13 *A $\mathfrak{gl}_2$-valued ZCR*

$$\alpha = A\,dx + B\,dy = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} dx + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} dy$$

is reducible to lower triangular if and only if

1. *either there exists a local function p on $\mathcal{E}$ such that*

$$\begin{aligned} D_x p &= -a_{12} + (a_{11} - a_{22})p + a_{21}p^2, \\ D_y p &= -b_{12} + (b_{11} - b_{22})p + b_{21}p^2; \end{aligned} \tag{13}$$

2. *or A, B are upper triangular:*

$$a_{21} = b_{21} = 0.$$

Proof An arbitrary $\mathcal{K}$-valued function is $K = PU$, where

$$U = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

and P is one of the two 2×2 permutation matrices

$$P_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Cases 1 and 2 correspond to the choices $P = P_{12}$ and $P = P_{21}$, respectively, and express the conditions that A^{PU}, B^{PU} are lower triangular.

Recall that a *quadratic* or *Riccati pseudopotential* p associated to an $\mathfrak{sl}_2$-valued ZCR α is defined by the compatible system

$$\begin{aligned} p_x &= -a_{12} + (a_{11} - a_{22})p + a_{21}p^2, \\ p_y &= -b_{12} + (b_{11} - b_{22})p + b_{21}p^2, \end{aligned} \tag{14}$$

which is essentially identical to Equations (13). Proposition 13 then says that a non-upper-triangular ZCR is reducible to lower triangular if and only if the quadratic pseudopotential exists as a local function on $\mathcal{E}$.

Example 14 Returning to Example 10, we have Case 1, while the equations (13) have a local solution $p = 1$. It follows that the upper triangular gauge matrix to make the ZCR lower triangular is

$$K = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Example 15 Let us find an explicit reduction of the Calogero–Nucci ZCR [2]

$$\begin{aligned} &\begin{pmatrix} 0 & 1 \\ \eta \dfrac{f_x}{f} + \lambda f^2 + \eta\mu f - \eta^2 & \dfrac{f_x}{f} + \mu f - 2\eta \end{pmatrix} dx \\ &\qquad + \begin{pmatrix} \eta \dfrac{g}{f} + \nu & \dfrac{g}{f} \\ \dfrac{\eta g_x}{f} + \lambda f g + \eta\mu g - \eta^2 \dfrac{g}{f} & \dfrac{g_x}{f} + \mu g - \eta \dfrac{g}{f} + \nu \end{pmatrix} dy \end{aligned} \tag{15}$$

of an arbitrary equation possessing a conservation law

$$f_t = g_x.$$

Here η, λ, μ, ν are arbitrary constants. Again, we have Case 1 and one easily finds a local solution

$$p = \frac{1}{2} \, \frac{(\mu + \sqrt{\mu^2 + 4\lambda})f - 2\eta}{\lambda f^2 + \eta\mu f - \eta^2}$$

of equations (13). Hence, the above ZCR is reducible to lower triangular; however, here one can continue and reach reduction to an abelian subalgebra. Indeed, if p is as above and

$$q = \frac{\lambda f^2 + \eta\mu f - \eta^2}{\sqrt{\mu^2 + 4\lambda}\, f},$$

$$r = \frac{(\lambda f^2 + \eta\mu f - \eta^2)(2\lambda f + (\mu - \sqrt{\mu^2 + 4\lambda})\eta)}{2\lambda f + (\mu + \sqrt{\mu^2 + 4\lambda})\eta},$$

then the product of gauge matrices

$$\begin{pmatrix} \sqrt{r}/f & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

takes the ZCR to the diagonal form

$$\begin{pmatrix} \frac{1}{2}(\mu - \sqrt{\mu^2 + 4\lambda})f - \eta & 0 \\ 0 & \frac{1}{2}(\mu + \sqrt{\mu^2 + 4\lambda})f - \eta \end{pmatrix} dx$$
$$+ \begin{pmatrix} \frac{1}{2}(\mu - \sqrt{\mu^2 + 4\lambda})g + \nu & 0 \\ 0 & \frac{1}{2}(\mu + \sqrt{\mu^2 + 4\lambda})g + \nu \end{pmatrix} dy,$$

hence to an abelian subalgebra.

In case of $\mathfrak{sl}_3$ we have six permutation matrices, hence six subcases. We list them in the following proposition.

Proposition 16 *An $\mathfrak{sl}_3$-valued ZCR $\alpha = A\,dx + B\,dy$, where $A = (a_{ij})$ and $B = (b_{ij})$, is reducible to lower triangular if and only if*

1. *either there exists local functions p, q, r on $\mathcal{E}$ such that*

$$\begin{aligned}
D_x p &= a_{21}p^2 + a_{31}pq + (a_{11} - a_{22})p - a_{32}q - a_{12}, \\
D_x q &= a_{31}q^2 + a_{21}pq + (a_{11} - a_{33})q - a_{23}p - a_{13}, \\
D_x r &= -a_{31}pr^2 + a_{32}r^2 - a_{21}pr + a_{31}qr + a_{21}q + (a_{22} - a_{33})r - a_{23}, \\
D_y p &= b_{21}p^2 + b_{31}pq + (b_{11} - b_{22})p - b_{32}q - b_{12}, \\
D_y q &= b_{31}q^2 + b_{21}pq + (b_{11} - b_{33})q - b_{23}p - b_{13}, \\
D_y r &= -b_{31}pr^2 + b_{32}r^2 - b_{21}pr + b_{31}qr + b_{21}q + (b_{22} - b_{33})r - b_{23};
\end{aligned} \tag{16}$$

2. *or there exist local functions r, s on $\mathcal{E}$ such that*

$$\begin{aligned}
&a_{31}r + a_{21} = 0, \\
&b_{31}r + b_{21} = 0, \\
&D_x r = a_{32}r^2 + (a_{22} - a_{33})r - a_{23}, \\
&D_y r = b_{32}r^2 + (b_{22} - b_{33})r - b_{23}, \\
&D_x s = -a_{31}s^2 + a_{32}rs - a_{12}r + (a_{11} - a_{33})s + a_{13}, \\
&D_y s = -b_{31}s^2 + b_{32}rs - b_{12}r + (b_{11} - b_{33})s + b_{13};
\end{aligned}$$

3. or there exist local functions p, q on $\mathcal{E}$ such that

$$
\begin{aligned}
&-a_{31}p + a_{32} = 0,\\
&-b_{31}p + b_{32} = 0,\\
&D_x p = a_{21}p^2 + (a_{11} - a_{22})p - a_{12},\\
&D_y p = b_{21}p^2 + (b_{11} - b_{22})p - b_{12},\\
&D_x q = a_{31}q^2 + a_{21}pq - a_{23}p + (a_{11} - a_{33})q - a_{13},\\
&D_y q = b_{31}q^2 + b_{21}pq - b_{23}p + (b_{11} - b_{33})q - b_{13};
\end{aligned}
$$

4. or the ZCR is $1,2$-block upper triangular:

$$a_{21} = a_{31} = b_{21} = b_{31} = 0;$$

and there exists a local function r on $\mathcal{E}$ such that

$$
\begin{aligned}
&D_x r = a_{32}r^2 + (a_{22} - a_{33})r - a_{23},\\
&D_y r = b_{32}r^2 + (b_{22} - b_{33})r - b_{23};
\end{aligned}
$$

5. or the ZCR is $2,1$-block upper triangular:

$$a_{31} = a_{32} = b_{31} = b_{32} = 0;$$

and there exists a local function p on $\mathcal{E}$ such that

$$
\begin{aligned}
&D_x p = a_{21}p^2 + (a_{11} - a_{22})p - a_{12},\\
&D_y p = b_{21}p^2 + (b_{11} - b_{22})p - b_{12};
\end{aligned}
$$

6. or the ZCR is upper triangular:

$$a_{21} = a_{31} = a_{32} = b_{21} = b_{31} = b_{32} = 0.$$

In Cases 2 and 3, if both a_{31}, b_{31} are zero, we get a subcase in Cases 4 and 5, respectively. Otherwise we get a system that is essentially algebraic in p, r, and differential in q, s.

Proof An arbitrary $\mathcal{K}$-valued function is $K = PU$, where

$$U = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$$

and P is one of the six 3×3 permutation matrices

$$P_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P_{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P_{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad P_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Cases 1 to 6 correspond to the choices $P = P_{123}$ to $P = P_{321}$, respectively. In Case 2, we set $s = pr - q$.

As already mentioned, the case of general position (when the permutation matrix P equals the identity matrix) is characterized by the property that all principal minors of the gauge matrix K are nonzero. Then we can derive explicit formulas that generalize formulas (13) and (16) to arbitrary n.

Proposition 17 *A $\mathfrak{gl}_n$-valued ZCR $\alpha = A\,dx + B\,dy$, where $A = (a_{ij})$ and $B = (b_{ij})$, is reducible to lower triangular by means of a gauge matrix with nonzero principal minors if and only if the system*

$$\begin{aligned} D_x p_{kl} = {} & - \sum_{\substack{0 \le r \le n-1 \\ i_0 < i_1 < \cdots < i_r = l}} (-1)^r a_{k i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{r-1} i_r} \\ & - \sum_{\substack{0 \le r \le n-1 \\ k < j \\ i_0 < i_1 < \cdots < i_r = l}} (-1)^r p_{kj} a_{j i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{r-1} i_r}, \\ D_y p_{kl} = {} & - \sum_{\substack{0 \le r \le n-1 \\ i_0 < i_1 < \cdots < i_r = l}} (-1)^r b_{k i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{r-1} i_r} \\ & - \sum_{\substack{0 \le r \le n-1 \\ k < j \\ i_0 < i_1 < \cdots < i_r = l}} (-1)^r p_{kj} b_{j i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{r-1} i_r} \end{aligned} \tag{17}$$

on $\frac{1}{2}(n-1)n$ unknown functions p_{kl}, $k < l$, has a local solution.

The same system may be rewritten in terms of determinants as follows:

$$D_x p_{kl} = (-1)^k \begin{vmatrix} \tilde{a}_{k1} & \tilde{a}_{k2} & \tilde{a}_{k3} & \cdots & \tilde{a}_{kk} & \tilde{a}_{kl} \\ 1 & p_{12} & p_{13} & \cdots & p_{1k} & p_{1l} \\ 0 & 1 & p_{23} & \cdots & p_{2k} & p_{2l} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{k-1,k} & p_{k-1,l} \\ 0 & 0 & 0 & \cdots & 1 & p_{kl} \end{vmatrix},$$

$$D_y p_{kl} = (-1)^k \begin{vmatrix} \tilde{b}_{k1} & \tilde{b}_{k2} & \tilde{b}_{k3} & \cdots & \tilde{b}_{kk} & \tilde{b}_{kl} \\ 1 & p_{12} & p_{13} & \cdots & p_{1k} & p_{1l} \\ 0 & 1 & p_{23} & \cdots & p_{2k} & p_{2l} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{k-1,k} & p_{k-1,l} \\ 0 & 0 & 0 & \cdots & 1 & p_{kl} \end{vmatrix},$$

where

$$\tilde{a}_{kj} = a_{kj} + \sum_{i > k} p_{ki} a_{ij}, \qquad \tilde{b}_{kj} = b_{kj} + \sum_{i > k} p_{ki} b_{ij}.$$

Proof According to Proposition 11, a general gauge matrix S with nonzero principal minors can be expressed in the form $S = LU$, where L is lower triangular and

$$U = \begin{pmatrix} 1 & p_{12} & p_{13} & \dots & p_{1n} \\ 0 & 1 & p_{23} & \dots & p_{2n} \\ 0 & 0 & 1 & \dots & p_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Its inverse is

$$U^{-1} = \begin{pmatrix} 1 & q_{12} & q_{13} & \dots & q_{1n} \\ 0 & 1 & q_{23} & \dots & q_{2n} \\ 0 & 0 & 1 & \dots & q_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where

$$\begin{aligned} q_{ij} &= \sum_{\substack{1\le r\le n-1 \\ i=i_0<i_1<\dots<i_r=j}} (-1)^r p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{r-1} i_r} \\ &= (-1)^{i+j} \begin{vmatrix} p_{i,i+1} & p_{i,i+2} & \cdot & \cdot & p_{i,j-1} & p_{i,j} \\ 1 & p_{i+1,i+2} & p_{i+1,i+3} & \cdot & \cdot & p_{i+1,j} \\ 0 & 1 & p_{i+2,i+3} & p_{i+2,i+4} & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & p_{j-2,j-1} & p_{j-2,j} \\ 0 & 0 & \cdot & \cdot & 1 & p_{j-1,j} \end{vmatrix}, \end{aligned}$$

since $q_{kl} + \sum_{k<i<l} p_{ki} q_{il} + p_{kl} = 0$ whenever $k < l$. Let us consider the gauge equivalent matrix $A^U = U_x U^{-1} + UAU^{-1}$. All terms that contain total derivatives $D_x p_{ij}$ occur in the first summand, which is

$$U_x U^{-1} = \begin{pmatrix} 0 & z_{12} & z_{13} & \dots & z_{1n} \\ 0 & 0 & z_{23} & \dots & z_{2n} \\ 0 & 0 & 0 & \dots & z_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where

$$\begin{aligned} z_{kl} &= \sum_{\substack{1\le r\le n-1 \\ k=i_0<i_1<\dots<i_r=l}} (-1)^{r-1} D_x p_{i_0 i_1} \cdot p_{i_1 i_2} \cdots p_{i_{r-1} i_r} \\ &= (-1)^{k+l+1} \begin{vmatrix} D_x p_{k,k+1} & D_x p_{k,k+2} & \cdot & D_x p_{k,l-1} & D_x p_{k,l} \\ 1 & p_{k+1,k+2} & p_{k+1,k+3} & \cdot & p_{k+1,l} \\ 0 & 1 & p_{k+2,k+3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & p_{l-1,l} \end{vmatrix} \end{aligned}$$

for all $k < l$. Denoting $A^U =: A' = (a'_{ij})$, we have

$$a'_{kl} := z_{kl} + a_{kl} + \sum_{j<l} a_{kj} q_{jl} + \sum_{\substack{k<i \\ j<l}} p_{ki} a_{ij} q_{jl} + \sum_{k<i} p_{ki} a_{il}.$$

The condition of A' being lower triangular, $a'_{kl} = 0$ for all $k < l$, constitutes a system of equations in total derivatives $D_x p_{ij}$. The equivalent system $a'_{kl} + \sum_{k<h<l} a'_{kh} p_{hl} = 0$ is resolved with respect to the total derivatives, since derivatives occur only in the summands containing z_{ij}, which are $z_{kl} + \sum_{k<h<l} z_{kh} p_{hl} = D_x p_{kl}$. The remaining summands then simplify to the expressions given in the statement of the proposition.

6 Guthrie's formulation of recursion operators

In 1994, G.A. Guthrie [8] suggested a general definition of a recursion operator, free of some weaknesses of the standard definition in terms of pseudodifferential operators. Geometrically, Guthrie's recursion operator for an equation $\mathcal{E}$ is a Bäcklund autotransformation for the linearized equation $V\mathcal{E}$ (see [16]). At the level of diffieties, the linearization $V\mathcal{E}$ can be introduced as the vertical vector bundle $V\mathcal{E} \longrightarrow \mathcal{E}$ with respect to the projection $\mathcal{E} \longrightarrow M$ on the base manifold. The manifold $V\mathcal{E}$ carries a natural diffiety structure.

At the level of systems of PDE, the linearized system is

$$F^l = 0, \qquad \ell_{F^l}[U] = 0, \tag{18}$$

where

$$\ell_F[U] = \sum_{k,I} \frac{\partial F}{\partial u^k_I} U^k_I. \tag{19}$$

Here we assume summation over all k, I such that the functions F^l depend on u^k_I. The U^k's are coordinates along the fibres of the projection $V\mathcal{E} \longrightarrow \mathcal{E}$ and serve as additional dependent variables (velocities), one for each u^k. Morphisms $\mathcal{E} \longrightarrow V\mathcal{E}$ that are sections of the bundle $V\mathcal{E} \longrightarrow \mathcal{E}$ are in one-to-one correspondence with local symmetries of the equation E.

Recall that nonlocal symmetries (more precisely, their *shadows* [11]) correspond to morphisms $\tilde{\mathcal{E}} \longrightarrow V\mathcal{E}$ over $\mathcal{E}$, where $\tilde{\mathcal{E}}$ is a covering of the original equation. In full generality, Guthrie's definition includes such a covering. Let us denote by $\widetilde{V\mathcal{E}} \longrightarrow \tilde{\mathcal{E}}$ the pullback of the vertical bundle $V\mathcal{E} \longrightarrow \mathcal{E}$ along the covering projection $\tilde{\mathcal{E}} \longrightarrow \mathcal{E}$. Then nonlocal symmetries correspond to morphisms $\tilde{\mathcal{E}} \longrightarrow \widetilde{V\mathcal{E}}$ that are sections of the projection $\widetilde{V\mathcal{E}} \longrightarrow \tilde{\mathcal{E}}$. In coordinates, if the covering $\tilde{\mathcal{E}}$ is determined by equations $z^j_x = f^j$, $z^j_y = g^j$, then the diffiety $\widetilde{V\mathcal{E}}$ corresponds to the system

$$F^l = 0, \quad z^j_x = f^j, \quad z^j_y = g^j, \quad \ell_{F^l}[U] = 0. \tag{20}$$

Definition 18 ([8]) A *recursion operator* for the equation (1) is given by the following data:

(1) a $\mathfrak{gl}_s$-valued zero-curvature representation $\bar\alpha = \bar A\,dx + \bar B\,dy$ for $\tilde{\mathcal{E}}$;

(2) an s-dimensional covering $K : \mathcal{R} \longrightarrow \widetilde{V\mathcal{E}}$ with nonlocal variables W^j, $j = 1,\dots,s$, subject to equations

$$W^j_x = \bar A^j_i W^i + A^j_\circ, \qquad W^j_y = \bar B^j_i W^i + B^j_\circ, \tag{21}$$

where $A^j_\circ$ and $B^j_\circ$ are functions on $\widetilde{V\mathcal{E}}$ linear on the fibres (i.e., linear in the variables U^k_I);

(3) a linear mapping $L : \mathcal{R} \longrightarrow \widetilde{V\mathcal{E}}$, $L(U)^l = \bar C^l_j W^j + C^l_\circ$, where $\bar C^l_j$ are functions on $\tilde{\mathcal{E}}$ and $C^l_\circ$ are functions on $\widetilde{V\mathcal{E}}$ linear on the fibres (i.e., linear in the variables U^k_I).

The following condition is supposed to hold: If $U = (U^k)$ satisfies the linearized equation $\widetilde{V\mathcal{E}}$, then so does $U' = L(U)$.

The recursion operator defined by these data will be denoted as LK^{-1}.

Once $\bar\alpha$ is a ZCR, equations (21) determine a covering if and only if the s-vectors $A_\circ = (A^j_\circ)$, $B_\circ = (B^j_\circ)$ satisfy

$$(D_y - \bar B)A_\circ = (D_x - \bar A)B_\circ \tag{22}$$

on $\widetilde{V\mathcal{E}}$, see [8, Eq. (3.2)]. This is easily verified by cross differentiation.

Recursion operators are gauge invariant: If S is a function on E with values in $GL(s)$, then the data

$$\begin{aligned}
&\bar A' = \bar A^S = \widetilde D_x S S^{-1} + S\bar A S^{-1}, \qquad && A'_\circ = SA_\circ,\\
&\bar B' = \bar B^S = \widetilde D_y S S^{-1} + S\bar B S^{-1}, \qquad && B'_\circ = SB_\circ,\\
&\bar C' = \bar C S^{-1}, \qquad C'_\circ = C_\circ
\end{aligned} \tag{23}$$

determine the same recursion operator as a mapping $U \longmapsto U'$.

Coverings (21) with $\bar\alpha = 0$ are associated with conservation laws, since for them Eq. (22) reads $D_y A_\circ = D_x B_\circ$. Examples are provided by recursion operators that can be written in the traditional pseudodifferential form ([19])

$$U^{l\prime} = \sum_{i=0}^{r} R^{li}_k D^i_x U^k + C^l_j D^{-1}_x p^j_k U^k.$$

Upon the obvious identification $D^I_x U^k = U^k_I$ and introduction of nonlocal variables $W^j = D^{-1}_x p^j_k U^k$, the Guthrie form of this operator is

$$\begin{aligned}
W^j_x &= p^{jI}_k U^k_I,\\
W^j_y &= q^{jI}_k U^k_I,\\
U^{l\prime} &= C^l_j W^j + R^{lI}_k U^k_I,
\end{aligned}$$

where $p^{jI}_k U^k_I\,dx + q^{jI}_k U^k_I\,dy$ is a conservation law of the linearized equation $V\mathcal{E}$ (typically a linearized conservation law of the equation $\mathcal{E}$).

Example 19 The Lenard recursion operator $D_{xx} + 4u + 2u_x D_x^{-1}$ for the KdV equation $u_t = u_{xxx} + 6uu_x$ has the following Guthrie form:

$$\begin{aligned} W_x &= U, \\ W_t &= U_{xx} + 6uU, \\ U' &= U_{xx} + 4uU + 2u_x W. \end{aligned} \tag{24}$$

Indeed, if U satisfies the linearized equation $V\mathcal{E}$, i.e.,

$$U_t = U_{xxx} + 6uU_x + 6u_x U, \tag{25}$$

then so does U' (for the same u). Here $\tilde{\mathcal{E}} = \mathcal{E}$ and consequently $\widetilde{V\mathcal{E}} = V\mathcal{E}$. The conservation law $U\,dx + (U_{xx} + 6uU)\,dt$ is a linearization of the conservation law $u\,dx + (u_{xx} + 3u^2)\,dt$ of KdV.

A recursion operator is *invertible* if the morphism L of Definition 18 is a covering. The recursion operator LK^{-1} is then simply a pair of linear coverings $K, L : \mathcal{R} \longrightarrow \widetilde{V\mathcal{E}}$. The inverse of the recursion operator LK^{-1} is the recursion operator KL^{-1}.

As an obvious consequence, invertible recursion operators built upon a covering $\tilde{\mathcal{E}}$ possess inverses that are built upon the same covering $\tilde{\mathcal{E}}$. In practice usually $\tilde{\mathcal{E}} = \mathcal{E}$, but even then it may be useful to make a pullback to a nontrivial covering. We shall return to this point later.

Concerning the Guthrie form of inverse recursion operators of systems $\mathcal{E}$ integrable in the sense of soliton theory, one observes that the ZCR $\bar{\alpha}$ is usually equal to the adjoint representation of the standard ZCR, while $A_\circ = \ell_A[U]$, $B_\circ = \ell_B[U]$.

Proposition 20 *Let $\alpha = A\,dx + B\,dy$ be a $\mathfrak{g}$-valued ZCR of equation $\mathcal{E}$. Then the trivial vector bundle $\mathfrak{g} \times V\mathcal{E} \longrightarrow V\mathcal{E}$ carries a covering structure determined by the condition that an arbitrary element W of the Lie algebra $\mathfrak{g}$ be subject to equations*

$$\begin{aligned} W_x &= [A, W] + \ell_A[U], \\ W_y &= [B, W] + \ell_B[U]. \end{aligned} \tag{26}$$

Proof By comparison with Eq. (21), in this case $\bar{\alpha}$ is simply the adjoint representation of α. Formulas (22) follow from the fact that $A \longmapsto \ell_A[U]$ is a differentiation.

Remark 21 (1) Let R be a recursion operator of an integrable system, let id denote the identity map. The inverse recursion operator $(R + \lambda\,\mathrm{id})^{-1}$ has a Guthrie form that depends on λ, which may be related to the spectral parameter of the standard ZCR of the system.

(2) Let us also note that the formulas (26) can serve as a starting point of a method to find the inverse recursion operator of an integrable system without finding the standard recursion operator first. One simply computes a morphism $\mathcal{R} \longrightarrow V\mathcal{E}$, where $\mathcal{R}$ is the covering determined by (26), see [17] for an example of the stationary Nizhnik–Veselov–Novikov equation. Remarkably enough, the recursion operator obtained in loc. cit. turned out to be noninvertible for the zero value of the spectral parameter λ.

The symmetries generated by inverse recursion operators usually exhibit a non-abelian nonlocality, e.g., of Riccati type as in [14, 15]. Moreover, upon introduction of the corresponding pseudopotentials one can express the inverse recursion operators in the traditional terms of inverse total derivatives D_x^{-1}. In the rest of this section we shall demonstrate that the above-mentioned nonlocalities are closely related to reduction of ZCR's to triangular form.

Given a recursion operator

$$V\mathcal{E} \leftarrow \mathcal{R} \longrightarrow V\mathcal{E},$$

the obvious pullback along a covering $\tilde{\mathcal{E}} \longrightarrow \mathcal{E}$ yields a recursion operator

$$\widetilde{V\mathcal{E}} \leftarrow \widetilde{\mathcal{R}} \longrightarrow \widetilde{V\mathcal{E}}.$$

In this construction, $\tilde{\mathcal{E}} \longrightarrow \mathcal{E}$ can be, e.g., the trivializing covering of the ZCR $\bar{\alpha}$. This was the case in the work on the KdV equation by Guthrie and Hickman [9]; by using formal power series in the spectral parameter λ, the authors were able to describe large algebras of nonlocal symmetries resulting from iterated application of the inverse recursion operator.

Alternatively, $\tilde{\mathcal{E}} \longrightarrow \mathcal{E}$ can be a covering such that the pullback of the ZCR $\bar{\alpha}$ on $\tilde{\mathcal{E}}$ is strictly lower triangular (belongs to $\mathfrak{t}^{(1)}$). The covering (21) is then abelian by similar argument as in Proposition 5. It follows that the recursion operator $\widetilde{V\mathcal{E}} \leftarrow \widetilde{\mathcal{R}} \longrightarrow \widetilde{V\mathcal{E}}$ can be expressed in terms of inverse total derivatives $\widetilde{D}_x^{-1}$. Summing up, we have the following

Construction 22 Step 1. Construct a covering $\mathcal{E}'$ with nonlocal variables h_{ij}, $j > i$, such that $\alpha' = \alpha^{PH}$ is lower triangular, where P is a suitable permutation matrix P and H is the matrix (12). Proposition 17 yields the corresponding formulas.

Step 2. Let a'_{ii}, b'_{ii} be the diagonal entries of the lower triangular matrices A^{PH}, B^{PH}, respectively. Then $a'_{ii}\,dx + b'_{ii}\,dy$ are conservation laws; if they are nontrivial, then construct the abelian covering $\mathcal{E}''$ over $\mathcal{E}'$ with the corresponding potentials z_i.

Step 3. Compute $S = ZPH$, where Z is the diagonal matrix $\operatorname{diag}(\mathrm{e}^{-z_i})$.

Obviously, $\alpha'' = \alpha^S$ is then strictly lower triangular, and so is its adjoint representation $\overline{\alpha''}$. Finally,

$$\overline{\alpha''} = \bar{\alpha}^{\bar{S}},$$

where $\bar{S}$ is the image of S in the adjoint representation of the group G.

If omitting Step 2, the recursion operator will be expressible in terms of inverses $(D_x - a'_{ii})^{-1}$.

7 Examples

Example 23 Continuing Example 19, let us invert the Lenard operator. The result is, of course, well known. Lou [15] obtained the inverse recursion operator for the whole AKNS hierarchy.

Equations (24) and (25) imply

$$\begin{pmatrix} U \\ U_x \\ W \end{pmatrix}_x = \begin{pmatrix} 0 & 1 & 0 \\ -4u & 0 & -2u_x \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ U_x \\ W \end{pmatrix} + \begin{pmatrix} 0 \\ U' \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} U \\ U_x \\ W \end{pmatrix}_t = \begin{pmatrix} 0 & 2u & -2u_{xx} \\ -2u_{xx} - 8u^2 & 2u_x & -2u_{xxx} - 4uu_x \\ 2u & 0 & -2u_x \end{pmatrix} \begin{pmatrix} U \\ U_x \\ W \end{pmatrix} + \begin{pmatrix} U'_x \\ U'_{xx} + 2uU' \\ U' \end{pmatrix},$$

which is the Guthrie form of the inverted operator $KL^{-1} : U' \longmapsto U$. It is clear now that $L : \mathcal{R} \longrightarrow V\mathcal{E}$, formerly given by $U' = U_{xx} + 4uU + 2u_x W$, constitutes a three-dimensional covering with nonlocal variables U, U_x and W.

The associated $\mathfrak{sl}_3$-valued ZCR

$$\bar{\alpha} = \begin{pmatrix} 0 & 1 & 0 \\ -4u & 0 & -2u_x \\ 1 & 0 & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & 2u & -2u_{xx} \\ -2u_{xx} - 8u^2 & 2u_x & -2u_{xxx} - 4uu_x \\ 2u & 0 & -2u_x \end{pmatrix} dy$$

is gauge equivalent to the adjoint representation of the standard $\mathfrak{sl}_2$-valued ZCR

$$\alpha = \begin{pmatrix} 0 & u \\ -1 & 0 \end{pmatrix} dx + \begin{pmatrix} u_x & u_{xx} + 2u \\ -2u & -u_x \end{pmatrix} dy$$

of the KdV equation, the corresponding gauge matrix being

$$S = \begin{pmatrix} 0 & 1 & 2u \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Using formulas (23) we get the following alternative formula for the same operator:

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix}_x = \begin{pmatrix} 0 & -2u & 0 \\ 1 & 0 & u \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + \begin{pmatrix} U' \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix}_t = \begin{pmatrix} 2u_x & -2u_{xx} - 4u^2 & 0 \\ 2u & 0 & u_{xx} + 2u^2 \\ 0 & -4u & -2u_x \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + \begin{pmatrix} U'_{xx} + 4uU' \\ U'_x \\ -2U' \end{pmatrix},$$

$$U = Q.$$

The covering here is the covering (26) with $\mathfrak{sl}_2$ parametrized as

$$\begin{pmatrix} Q & P \\ R & -Q \end{pmatrix}.$$

To express the inverted recursion operator in terms of D_x^{-1}, we need to make the ZCR $\bar{\alpha}$ strictly lower triangular. According to Construction 22, as the first step we construct a covering $\mathcal{E}' \longrightarrow \mathcal{E}$ with the quadratic pseudopotential $h = h_{11}$ defined by Eq. (14), i.e.,

$$h_x = -h^2 - u,$$
$$h_t = -2uh^2 + 2u_x h - u_{xx} - 2u^2.$$

Then using the gauge matrix

$$H = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

we get the lower triangular ZCR

$$\alpha' = \alpha^H = \begin{pmatrix} -h & 0 \\ -1 & h \end{pmatrix} dx + \begin{pmatrix} u_x - 2uh & 0 \\ -2u & -u_x + 2uh \end{pmatrix} dy$$

with $-h, h$ on the diagonal. As the second step, we construct the abelian covering $\mathcal{E}'' \longrightarrow \mathcal{E}'$ with the potential z satisfying

$$z_x = -h, \qquad z_y = u_x - 2uh.$$

The gauge matrix

$$Z = \begin{pmatrix} \mathrm{e}^{-z} & 0 \\ 0 & \mathrm{e}^{z} \end{pmatrix}$$

then leads to the strictly lower triangular ZCR

$$\alpha'' = \alpha^{ZH} = \begin{pmatrix} 0 & 0 \\ -\mathrm{e}^{2z} & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & 0 \\ -2\mathrm{e}^{2z} u & 0 \end{pmatrix} dy.$$

The full gauge matrix and its adjoint representation are

$$S = \begin{pmatrix} \mathrm{e}^{-z} & h\mathrm{e}^{-z} \\ 0 & \mathrm{e}^{z} \end{pmatrix}, \qquad \bar{S} = \begin{pmatrix} \mathrm{e}^{-2z} & -2h\mathrm{e}^{-2z} & -h^2 e^{-2z} \\ 0 & 1 & h \\ 0 & 0 & \mathrm{e}^{2z} \end{pmatrix}.$$

Acting by $\bar{S}$ on our operator, we get

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 \\ e^{2z} & 0 & 0 \\ 0 & -2e^{2z} & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + \begin{pmatrix} \mathrm{e}^{-2z} U' \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & 0 \\ 2ue^{2z} & 0 & 0 \\ 0 & -4ue^{2z} & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$$

$$+ \begin{pmatrix} \mathrm{e}^{-2z} U'_{xx} - 2\mathrm{e}^{-2z} h U'_x + (2h^2 + 4u)\mathrm{e}^{-2z} U' \\ U'_x - 2hU' \\ -2\mathrm{e}^{2z} U' \end{pmatrix},$$

$$U = Q - h\mathrm{e}^{-2z} R.$$

In the x-part, we get $P = D_x^{-1}(\mathrm{e}^{-2z} U')$, $Q = D_x^{-1}(\mathrm{e}^{2z} P)$, $R = D_x^{-1}(\mathrm{e}^{-2z} Q)$, hence

$$U = D_x^{-1} \mathrm{e}^{2z} D_x^{-1} \mathrm{e}^{-2z} U' - h\mathrm{e}^{-2z} D_x^{-1} \mathrm{e}^{-2z} D_x^{-1} \mathrm{e}^{2z} D_x^{-1} \mathrm{e}^{-2z} U'.$$

Example 24 Let us consider the Tzitzéica equation [23]

$$u_{xy} = \mathrm{e}^{u} - \mathrm{e}^{-2u},$$

also known from the Zhiber–Shabat list [27]. Its ZCR

$$\alpha = \begin{pmatrix} -u_x & 0 & \lambda \\ \lambda & u_x & 0 \\ 0 & \lambda & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & \mathrm{e}^{-2u}/\lambda & 0 \\ 0 & 0 & \mathrm{e}^{u}/\lambda \\ \mathrm{e}^{u}/\lambda & 0 & 0 \end{pmatrix} dy \tag{27}$$

as well as the Bäcklund transformation were essentially found by Tzitzéica himself.

One could invert the known recursion operator [21], but it is easier to compute the inverse recursion operator directly by the procedure outlined in Remark 21(2). Namely, we consider the eight-dimensional covering (26), where $\bar{A}, \bar{B}, A_\circ$ and $B_\circ$ are found from the formula (27) to be

$$\bar{A} = \begin{pmatrix} 0 & -\lambda & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & -2u_x & -\lambda & 0 & 0 & 0 & 0 & \lambda \\ -2\lambda & 0 & -u_x & 0 & -\lambda & 0 & 0 & 0 \\ \lambda & 0 & 0 & 2u_x & -\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & -\lambda & 0 & u_x & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & u_x & -\lambda \\ \lambda & 0 & 0 & 0 & 2\lambda & 0 & 0 & -u_x \end{pmatrix},$$

$$\bar{B} = \begin{pmatrix} 0 & 0 & -\mathrm{e}^{u}/\lambda & \mathrm{e}^{-2u}/\lambda & 0 & 0 & 0 & 0 \\ -\mathrm{e}^{-2u}\lambda & 0 & 0 & 0 & \mathrm{e}^{-2u}/\lambda & 0 & 0 & 0 \\ 0 & -\mathrm{e}^{u}/\lambda & 0 & 0 & 0 & \mathrm{e}^{-2u}/\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{e}^{u}/\lambda & \mathrm{e}^{u}/\lambda & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{-2u}/\lambda & 0 & 0 & 0 & \mathrm{e}^{u}/\lambda \\ -\mathrm{e}^{u}/\lambda & 0 & 0 & 0 & -2\mathrm{e}^{u}/\lambda & 0 & 0 & 0 \\ 2\mathrm{e}^{u}/\lambda & 0 & 0 & 0 & \mathrm{e}^{u}/\lambda & 0 & 0 & 0 \\ 0 & \mathrm{e}^{u}/\lambda & 0 & 0 & 0 & 0 & -\mathrm{e}^{-2u}/\lambda & 0 \end{pmatrix},$$

$$A_\circ = \begin{pmatrix} -U_x \\ 0 \\ 0 \\ 0 \\ U_x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad B_\circ = \begin{pmatrix} 0 \\ -2\mathrm{e}^{-2u}U/\lambda \\ 0 \\ 0 \\ 0 \\ \mathrm{e}^{u}U/\lambda \\ \mathrm{e}^{u}U/\lambda \\ 0 \end{pmatrix}; \quad \text{and} \quad W = \begin{pmatrix} W_{11} \\ W_{12} \\ W_{13} \\ W_{21} \\ W_{22} \\ W_{23} \\ W_{31} \\ W_{32} \end{pmatrix}$$

is a column of pseudopotentials $W_{11}, W_{12}, W_{13}, W_{21}, W_{22}, W_{23}, W_{31}, W_{32}$. One easily finds that $W_{11} - W_{22}$ is a symmetry of the Tzitzéica equation if so is U. Thus, we have obtained the 'inverse' recursion operator of the Tzitzéica equation in the Guthrie form.

Let us express it in terms of D_x^{-1}. As the first step we introduce pseudopotentials p, q, r satisfying

$$
\begin{aligned}
p_x &= \lambda p^2 - 2pu_x - \lambda q, & p_y &= \frac{\mathrm{e}^u}{\lambda} pq - \frac{1}{\mathrm{e}^{2u}\lambda}, \\
q_x &= \lambda pq - qu_x - \lambda, & q_y &= \frac{\mathrm{e}^u}{\lambda}(q^2 - p), \\
r_x &= -\lambda pr + \lambda q + \lambda r^2 + u_x r, & r_y &= \frac{\mathrm{e}^u}{\lambda}(-pr^2 + qr - 1).
\end{aligned}
$$

to make the ZCR (27) lower triangular by providing a solution to equations (16). Indeed, acting on α by the gauge matrix

$$
H = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}
$$

we get

$$
\begin{aligned}
\alpha^H = {} & \begin{pmatrix} -u_x + \lambda p & 0 & 0 \\ \lambda & u_x - \lambda p + \lambda r & 0 \\ 0 & \lambda & -\lambda r \end{pmatrix} dx \\
& + \begin{pmatrix} \mathrm{e}^u q/\lambda & 0 & 0 \\ \mathrm{e}^u r/\lambda & -\mathrm{e}^u pr/\lambda & 0 \\ \mathrm{e}^u/\lambda & -\mathrm{e}^u p/\lambda & \dfrac{\mathrm{e}^u(pr - q)}{\lambda} \end{pmatrix} dy.
\end{aligned}
$$

In the second step we remove the diagonal. To this end we introduce pseudopotentials s, t by

$$
\begin{aligned}
s_x &= -u_x + \lambda p, & s_y &= \frac{\mathrm{e}^u}{\lambda} q, \\
t_x &= u_x - \lambda p + \lambda r, & t_y &= -\frac{\mathrm{e}^u}{\lambda} pr.
\end{aligned}
$$

Acting on α^H by the gauge matrix

$$
Z = \begin{pmatrix} \mathrm{e}^{-s} & 0 & 0 \\ 0 & \mathrm{e}^{-t} & 0 \\ 0 & 0 & \mathrm{e}^{s+t} \end{pmatrix}
$$

we finally get

$$
\begin{aligned}
\alpha^{ZH} = {} & \begin{pmatrix} 0 & 0 & 0 \\ \lambda \mathrm{e}^{s-t} & 0 & 0 \\ 0 & \lambda \mathrm{e}^{s+2t} & 0 \end{pmatrix} dx \\
& + \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{e}^{u+s-t} r/\lambda & 0 & 0 \\ \mathrm{e}^{u+2s+t}/\lambda & -\mathrm{e}^{u+s+2t} p/\lambda & 0 \end{pmatrix} dy.
\end{aligned}
$$

Denoting $S = ZH$, we compute the adjoint representation $\bar{S}$ to be

$$\bar{S} = \begin{pmatrix} e^{-2s-t} & -e^{-2s-t}r & e^{-2s-t}p & e^{-2s-t}(pr-2q) & -e^{-2s-t}(pr+q) & e^{-2s-t}p(pr-q) & -e^{-2s-t}qr & e^{-2s-t}q(pr-q) \\ 0 & e^{-s+t} & 0 & -e^{-s+t}p & e^{-s+t}p & -e^{-s+t}p^2 & e^{-s+t}q & -e^{-s+t}pq \\ 0 & 0 & e^{-s-2t} & -e^{-s-2t}r & -2e^{-s-2t}r & e^{-s-2t}(pr-q) & -r^2e^{-s-2t} & e^{-s-2t}(pr-q)r \\ 0 & 0 & 0 & 1 & 0 & p & 0 & q \\ 0 & 0 & 0 & 0 & 1 & -p & r & -pr \\ 0 & 0 & 0 & 0 & 0 & e^{s-t} & 0 & e^{s-t}r \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{s+2t} & -e^{s+2t}p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2s+t} \end{pmatrix}.$$

Acting by $\bar{S}$ on the above recursion operator we get

$$\bar{A}^{\bar{S}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda e^{s+2t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda e^{s-t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda e^{s-t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda e^{s-t} & -\lambda e^{s+2t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda e^{s-t} & -\lambda e^{s-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda e^{s+2t} & 2\lambda e^{s+2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda e^{s+2t} & -\lambda e^{s-t} & 0 \end{pmatrix}$$

and

$$\bar{S}A_\circ = \begin{pmatrix} e^{-2s-t}(-2pr+q)U_x \\ 2e^{-s+t}pU_x \\ -e^{-s-2t}rU_x \\ -U_x \\ U_x \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(we omit the matrices $\bar{B}^{\bar{S}}$ and $B_\circ$).

Thus, the inverse recursion operator for the Tzitzéica equation in terms of D^{-1} is

$$V = W_{21} - W_{22} - 2e^{-s+t}pW_{23} + e^{-s-2t}rW_{31} + e^{-2s-t}(2pr-q)W_{32},$$

where

$$
\begin{aligned}
W_{11} &= D^{-1}[\mathrm{e}^{-2s-t}(-2pr+q)U_x],\\
W_{12} &= D^{-1}[2\mathrm{e}^{-s+t}pU_x - \mathrm{e}^{s+2t}\lambda W_{11}],\\
W_{13} &= D^{-1}[-\mathrm{e}^{-s-2t}rU_x + \mathrm{e}^{s-t}\lambda W_{11}],\\
W_{22} &= D^{-1}[U_x + \mathrm{e}^{s-t}\lambda W_{12} - \mathrm{e}^{s+2t}\lambda W_{13}],\\
W_{21} &= D^{-1}[-U_x - \mathrm{e}^{s-t}\lambda W_{12}],\\
W_{31} &= D^{-1}[\mathrm{e}^{s+2t}\lambda(W_{21}+2W_{22})],\\
W_{23} &= D^{-1}[-\mathrm{e}^{s-t}\lambda(-W_{21}+W_{22})],\\
W_{32} &= D^{-1}[\lambda(\mathrm{e}^{s+2t}W_{23} - \mathrm{e}^{s-t}W_{31})].
\end{aligned}
$$

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References

[1] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[2] F. Calogero and M.C. Nucci, Lax pair galore, *J. Math. Phys.* **32** (1991) (1) 72–74.

[3] S.-S. Chern and K. Tenenblat, Pseudospherical surfaces and evolution equations, *Stud. Appl. Math.* **74** (1986) 55–83.

[4] M.T. Chu and L.K. Norris, Isospectral flows and abstract matrix factorizations, *SIAM J. Numer. Analysis* **25** (1988) 1383–1391.

[5] P. Deift, T. Nanda and C. Tomei, Differential equations for the symmetric eigenvalue problem, *SIAM J. Numer. Analysis* **20** (1983) 1–22.

[6] R. Dodd and A. Fordy, The prolongation structures of quasi-polynomial flows, *Proc. R. Soc. London A* **385** (1983) 389–429.

[7] W. Fulton and J. Harris, *Representation Theory. A First Course* (Springer, Berlin, 2001).

[8] G.A. Guthrie, Recursion operators and non-local symmetries, *Proc. R. Soc. London A* **446** (1994) 107–114.

[9] G.A. Guthrie and M.S. Hickman, Nonlocal symmetries of the KdV equation, *J. Math. Phys.* **34** (1993) 193–205.

[10] N.G. Khor'kova, Conservation laws and nonlocal symmetries, *Mat. Zametki* **44** (1988) (1) 134–144, 157 (in Russian); *Math. Notes* **44** (1988), (1–2) 562–568 (English translation).

[11] I.S. Krasil'shchik and A.M. Vinogradov, Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations, *Acta Appl. Math.* **15** (1989) 161–209.

[12] I.S. Krasil'shchik and A.M. Vinogradov, eds., *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics,* Translations of Mathematical Monographs 182 (American Mathematical Society, Providence, 1999).

[13] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968) 467–490.

[14] S. Lou, Recursion operator and symmetry structure of the Kawamoto-type equation, *Phys. Lett. A* **181** (1993) 13–16.

[15] S. Lou and W. Chen, Inverse recursion operator of the AKNS hierarchy, *Phys. Lett. A* **179** (1993) 271–274.

[16] M. Marvan, Another look on recursion operators, in: *Differential Geometry and Applications*, Proc. Conf. Brno, 1995 (Masaryk University, Brno, 1987) 393–402; ELibEMS http://www.emis.de/proceedings.

[17] M. Marvan and A. Sergyeyev, Recursion operator for the Nizhnik–Veselov–Novikov equation, *J. Phys. A: Math. Gen.* **36** (2003) L87–L92.

[18] M.A. Naĭmark, *Théorie des Représentations des Groupes,* (Éditions Mir, Moscow, 1979).

[19] P.J. Olver, Evolution equations possessing infinitely many symmetries, *J. Math. Phys.* **18** (1977) 1212–1215.

[20] S.Yu. Sakovich, True and fake Lax pairs: how to distinguish them, preprint nlin.SI/0112027 (2001).

[21] V.V. Sokolov and A.B. Shabat, (L, A)-pary i zamena tipa Rikatti, *Funkc. Anal. Prilozh.* **14** (1980) 79–80.

[22] L.A. Takhtadzhyan and L.D. Faddeev, *Hamiltonian Methods in the Theory of Solitons* (Springer, Berlin et al., 1987).

[23] G. Tzitzéica, Sur une nouvelle classe de surface, *C.R. Acad. Sci. Paris* **150** (1910) 955–956, 1227–1229.

[24] H.D. Wahlquist and F.B. Estabrook, Prolongation structures and nonlinear evolution equations I, II, *J. Math. Phys.* **16** (1975) 1–7; **17** (1976) 1293–1297.

[25] D.S. Watkins, Isospectral flows, *SIAM Review* **26** (1984) 379–391.

[26] V.E. Zakharov and A.B. Shabat, Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II, *Funct. Anal. Appl.* **13** (1980) 166–174; translation from *Funkc. Anal. Prilozh.* **13** (1979) (3) 13–22.

[27] A.V. Zhiber and A.B. Shabat, Klein–Gordon equations with nontrivial group, *Doklady AN SSSR* **247** (1979) 1103–1107 (in Russian).