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A strange recursion operator demystified

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Abstract

We show that a new integrable two-component system of KdV type studied by Karasu (Kalkanlı) et al. (arXiv: `nlin.SI/0203036`) is bihamiltonian, and its recursion operator, which has a highly unusual structure of nonlocal terms, can be written as a ratio of two compatible Hamiltonian operators. Using this, we prove that the system in question possesses an infinite hierarchy of *local* commuting generalized symmetries and conserved quantities in involution, and the evolution systems corresponding to these symmetries are bihamiltonian as well.

Using the Panilevé test, Karasu (Kalkanlı) [1] and Sakovich [2] found a new integrable evolution system of KdV type,

$$\begin{aligned} u_t &= 4u_{xxx} - v_{xxx} - 12uu_x + vu_x + 2uv_x, \\ v_t &= 9u_{xxx} - 2v_{xxx} - 12vu_x - 6uv_x + 4vv_x, \end{aligned} \tag{1}$$

and a zero curvature representation for it [2]. Notice that this system is, up to a linear transformation of u and v, equivalent to the system (16) from the Foursov's [3] list of two-component evolution systems of KdV type possessing (homogeneous) symmetries of order k, $4 \leq k \leq 9$.

Karasu (Kalkanlı), Karasu and Sakovich [4] found that (1) has a recursion operator of the form

$$
\begin{aligned}
&\mathfrak{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \\
&R_{11} = 3D_x^2 - 6u - 3u_x D_x^{-1}, \\
&R_{12} = \big[-2D_x^5 + (2u+3v)\,D_x^3 + (8v_x - 4u_x)\,D_x^2 \\
&\quad + \big(7v_{xx} - 6u_{xx} + 4u^2 - 6uv\big)\,D_x - 2u_{xxx} + 2v_{xxx} + 6uu_x - 3vu_x \\
&\quad - 4uv_x + u_x D_x^{-1} \circ v_x\big] \circ \big(3D_x^3 - 4vD_x - 2v_x\big)^{-1}, \\
&R_{21} = 6D_x^2 + 6u - 9v - 3v_x D_x^{-1}, \\
&R_{22} = \big[-3D_x^5 + (12v - 18u)\,D_x^3 + (18v_x - 27u_x)\,D_x^2 + (14v_{xx} - 21u_{xx} \\
&\quad + 12u^2 + 12uv - 9v^2\big)\,D_x - 6u_{xxx} + 4v_{xxx} + 12uu_x + 6vu_x + 6uv_x \\
&\quad - 9vv_x + v_x D_x^{-1} \circ v_x\big] \circ \big(3D_x^3 - 4vD_x - 2v_x\big)^{-1}.
\end{aligned}
$$

Here D_x is the operator of total x-derivative: $D_x = \partial/\partial x + u_x\partial/\partial u + v_x\partial/\partial v + \sum_{j=2}^{\infty}(u_{jx}\partial/\partial u_{(j-1)x} + v_{jx}\partial/\partial v_{(j-1)x})$, where $u_{kx} = \partial^k u/\partial x^k$, $v_{kx} = \partial^k v/\partial x^k$, see e.g. [5] for further details. Let also $\delta/\delta u = \partial/\partial u + \sum_{j=1}^{\infty}(-D_x)^j\partial/\partial u_{jx}$, $\delta/\delta v = \partial/\partial v + \sum_{j=1}^{\infty}(-D_x)^j\partial/\partial v_{jx}$, $\boldsymbol{u} = (u, v)^T$, and $\delta/\delta\boldsymbol{u} = (\delta/\delta u, \delta/\delta v)^T$, cf. e.g. [5]. Here and below the superscript T denotes the matrix transposition. Recall that a function that depends on x, t, u, v and a *finite* number of u_{jx} and v_{kx} is said to be *local*, see e.g. [6, 5].

Because of the nonstandard structure of nonlocal terms in $\mathfrak{R}$ the known 'direct' methods (see e.g. [7, 8, 9] and references therein) for proving the locality of symmetries generated by $\mathfrak{R}$ are not applicable, so the question of whether (1) has an infinite hierarchy of local commuting symmetries remained open for a while. It was also unknown whether (1) is a bihamiltonian system.

We have [4] $\mathfrak{R} = \mathfrak{M} \circ \mathfrak{N}^{-1}$, where $\mathfrak{M}$ and $\mathfrak{N}$ are some (non-Hamiltonian) differential operators of order five and three. Inspired by this fact, we undertook a search of Hamiltonian operators of order three and five for (1), and it turned out that such operators do exist and the system (1) *is* bihamiltonian. Namely, the following assertion holds.

Proposition 1 *The system (1) is bihamiltonian:*

$$\boldsymbol{u}_t = \mathfrak{P}_1 \delta H_0/\delta\boldsymbol{u} = \mathfrak{P}_0 \delta H_1/\delta\boldsymbol{u}, \tag{2}$$

where $H_0 = -3u + v/2$, $H_1 = 2u^2 - uv + v^2/9$, and $\mathfrak{P}_0$ and $\mathfrak{P}_1$ are compatible Hamiltonian operators of the form

$$\mathfrak{P}_0 = \begin{pmatrix} D_x^3 - 2uD_x - u_x & 0 \\ 0 & -9D_x^3 + 12vD_x + 6v_x \end{pmatrix}, \mathfrak{P}_1 = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \tag{3}$$

where $P_{11}=D_x^5-4uD_x^3-6u_xD_x^2+4(u^2-u_{xx})D_x-u_{xxx}+4uu_x-u_xD_x^{-1}\circ u_x$, $P_{12}=2D_x^5-(2u+3v)D_x^3+4(u_x-2v_x)D_x^2+(6u_{xx}-7v_{xx}-4u^2+6uv)D_x+2u_{xxx}-2v_{xxx}-6uu_x+3vu_x+4uv_x-u_xD_x^{-1}\circ v_x$, $P_{21}=2D_x^5-(2u+3v)D_x^3-(10u_x+v_x)D_x^2+(-4u^2+6uv-8u_{xx})D_x-2u_{xxx}-2uu_x+3vu_x+2uv_x-v_xD_x^{-1}\circ u_x$, $P_{22}=3D_x^5+(18u-12v)D_x^3+(27u_x-12v_x)D_x^2+(21u_{xx}-14v_{xx}-12u^2-12uv+9v^2)D_x+6u_{xxx}-4v_{xxx}-12uu_x-6vu_x-6uv_x+9vv_x-v_xD_x^{-1}\circ v_x$.

Moreover, we have $\mathfrak{R}=3\mathfrak{P}_1\circ\mathfrak{P}_0^{-1}$, *and hence* $\mathfrak{R}$ *is hereditary.*

Now we are ready to prove that (1) has infinitely many local commuting symmetries.

Proposition 2 *Define the quantities* $\boldsymbol{Q}_j$ *and* H_j *recursively by the formula* $\boldsymbol{Q}_j=\mathfrak{P}_1\delta H_j/\delta\boldsymbol{u}=\mathfrak{P}_0\delta H_{j+1}/\delta\boldsymbol{u}$, $j=0,1,2,\dots$, *where* H_0, H_1, $\mathfrak{P}_0$ *and* $\mathfrak{P}_1$ *are given in Proposition 1. Then* H_j, $j=2,3,\dots$, *are local functions that can be chosen to be independent of* x *and* t, *and* $\boldsymbol{Q}_j$ *are local commuting generalized symmetries for (1) for all* $j=1,2,\dots$.

Moreover, the evolution systems $\boldsymbol{u}_{t_j}=\boldsymbol{Q}_j$ *are bihamiltonian with respect to* $\mathfrak{P}_1$ *and* $\mathfrak{P}_0$ *by construction, and* $\mathcal{H}_j=\int H_j dx$ *are in involution with respect to the Poisson brackets associated with* $\mathfrak{P}_0$ *and* $\mathfrak{P}_1$ *for all* $j=0,1,2\dots$, *so* $\mathcal{H}_j$ *are common conserved quantities for all evolution systems* $\boldsymbol{u}_{t_k}=\boldsymbol{Q}_k$, $k=0,1,2,\dots$.

Proof. Let us use induction on j. Assume that for $\boldsymbol{Q}_j=\mathfrak{P}_1\delta H_j/\delta\boldsymbol{u}$ there exist a local function H_{j+1} such that $\boldsymbol{Q}_j=\mathfrak{P}_0\delta H_{j+1}/\delta\boldsymbol{u}$ and $\partial H_{j+1}/\partial x=\partial H_{j+1}/\partial t=0$, and let us show that then $\boldsymbol{Q}_{j+1}=\mathfrak{P}_1\delta H_{j+1}/\delta\boldsymbol{u}$ is local too and there exists a local function H_{j+2} such that $\boldsymbol{Q}_{j+1}=\mathfrak{P}_0\delta H_{j+2}/\delta\boldsymbol{u}$ and $\partial H_{j+2}/\partial x=\partial H_{j+2}/\partial t=0$.

Since H_j is independent of x, we have

$$\begin{aligned}u_x\frac{\delta H_j}{\delta u}+v_x\frac{\delta H_j}{\delta v} \quad = \quad & D_x\left(H_j-\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\left\{u_{(j-k)x}(-D_x)^k\left(\frac{\partial H}{\partial u_{jx}}\right)\right.\right.\\ &\left.\left.+\,v_{(j-k)x}(-D_x)^k\left(\frac{\partial H}{\partial v_{jx}}\right)\right\}\right).\end{aligned}$$

As H_j is local, the sum is actually finite, so $D_x^{-1}(u_x\delta H_j/\delta u+v_x\delta H_j/\delta v)$ is a local expression, and thus $\boldsymbol{Q}_{j+1}=\mathfrak{P}_1\delta H_j/\delta\boldsymbol{u}$ is local too.

Next, as $\boldsymbol{Q}_{j+1}=\mathfrak{P}_1\delta H_{j+1}/\delta\boldsymbol{u}$, $\boldsymbol{Q}_j=\mathfrak{P}_0\delta H_{j+1}/\delta\boldsymbol{u}$, and $\mathfrak{R}=3\mathfrak{P}_1\circ\mathfrak{P}_0^{-1}$, we can (formally) write $\boldsymbol{Q}_{j+1}=(1/3)\mathfrak{R}\boldsymbol{Q}_j$, cf. e.g. Section 7.3 of [5]. As $\mathfrak{R}$ is a recursion operator for (1), its Lie derivative along $\boldsymbol{Q}_0$ vanishes: $L_{\boldsymbol{Q}_0}(\mathfrak{R})=0$. By Proposition 1 $\mathfrak{R}$ is hereditary, so we have [10] $L_{\boldsymbol{Q}_{j+1}}(\mathfrak{R})=$

$(1/3)^{j+1} L_{\mathfrak{R}^{j+1}\boldsymbol{Q}_0}(\mathfrak{R}) = 0$, whence $L_{\boldsymbol{Q}_{j+1}}(\mathfrak{P}_0) = 3L_{\boldsymbol{Q}_{j+1}}(\mathfrak{R}^{-1} \circ \mathfrak{P}_1) = 3\mathfrak{R}^{-1} \circ L_{\boldsymbol{Q}_{j+1}}(\mathfrak{P}_1) = 3\mathfrak{R}^{-1} \circ L_{\mathfrak{P}_1 \delta H_{j+1}/\delta \boldsymbol{u}}(\mathfrak{P}_1) = 0$, cf. e.g. [11, 12].

In turn, $L_{\boldsymbol{Q}_{j+1}}(\mathfrak{P}_0) = 0$ implies that there exists a local function H_{j+2} such that $\boldsymbol{Q}_{j+1} = \mathfrak{P}_0 \delta H_{j+2}/\delta \boldsymbol{u}$. The proof of this fact goes along the same lines as it was done in [13] for the second Hamiltonian structure of the KdV equation. Finally, as the coefficients of $\mathfrak{P}_0$ and $\mathfrak{P}_1$ are independent of x and t, it is immediate that we always can choose H_{j+2} so that it is independent of x and t.

The induction on j starting from $j = 0$ and the use of Theorem 7.24 of Olver [5] complete the proof. □

In order to handle properly the nonlocal terms, for any local H such that $\partial H/\partial x = 0$ we shall set, in agreement with the above (see e.g. [17]–[20] for more details on dealing with nonlocalities),

$$\begin{aligned} D_x^{-1}\left(u_x \frac{\delta H_j}{\delta u} + v_x \frac{\delta H_j}{\delta v}\right) &= H_j - \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \left\{ u_{(j-k)x} (-D_x)^k \left(\frac{\partial H}{\partial u_{jx}}\right) \right. \\ &\quad \left. + v_{(j-k)x} (-D_x)^k \left(\frac{\partial H}{\partial v_{jx}}\right) \right\}. \end{aligned}$$

Then, for instance, the first commuting flow for (1) is

$$\begin{aligned} u_{t_1} &= 2u_{5x} - (5/9)v_{5x} - 20uu_{xxx} + (50/9)uv_{xxx} + (40/9)vu_{xxx} \\ &\quad - (10/9)vv_{xxx} - 50u_x u_{xx} + (125/9)u_x v_{xx} + (40/3)v_x u_{xx} - (10/3)v_x v_{xx} \\ &\quad - (40/3)vuu_x + (20/9)vuv_x + 40u^2 u_x - (80/9)u^2 v_x + (5/9)v^2 u_x, \\ v_{t_1} &= 5u_{5x} - (4/3)v_{5x} - 40uu_{xxx} + 10uv_{xxx} + (10/3)vu_{xxx} - (5/9)vv_{xxx} \\ &\quad - 120u_x u_{xx} + 30u_x v_{xx} + (80/3)v_x u_{xx} - (55/9)v_x v_{xx} + (160/3)vuu_x \\ &\quad - 20vuv_x + (40/3)u^2 v_x - (40/3)v^2 u_x + (35/9)v^2 v_x. \end{aligned}$$

By Proposition 2 this system is bihamiltonian, and indeed we can write it as

$$\boldsymbol{u}_{t_1} = \mathfrak{P}_1 \delta H_1/\delta \boldsymbol{u} = \mathfrak{P}_0 \delta H_2/\delta \boldsymbol{u},$$

where

$$H_2 = \frac{7}{162}v^3 - \frac{8}{3}u^3 - \frac{5}{9}v^2 u + \frac{20}{9}u^2 v - u_x^2 + \frac{5}{9}v_x u_x - \frac{2}{27}v_x^2.$$

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