Preprint Series in Global Analysis
Preprint GA 2/2002, Mathematical Institute
Silesian University in Opava, Czech Republic
April 2002, pp. 7

THE STRUCTURE OF VARIATIONAL FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

Alžběta Haková
Mathematical Institute of the Silesian University in Opava
Bezručovo nám. 13, 746 01 Opava
e-mail: alhak@centrum.cz

1. Introduction

The aim of this work is to analyze variationality of systems of first-order partial differential equations by methods of differential geometry. It is known that existence of a Lagrangian for a system of PDE (of any order) is closely connected with the possibility to represent the equations by means of a *closed form*, which generally is local and non-unique (see e.g. [8], [9]). We prove that for a system of first-order PDE on a fibred manifold, variationality is equivalent with the existence of a *global* and *unique* closed $(n+1)$-form (where n is the dimension of the base manifold), and provide an *explicit construction* of such a form. As a consequence we obtain an *explicit characterization* of general systems of variational first-order PDE and of their Lagrangians. Our results are a generalization of [4], where *quasilinear* first-order PDE were studied. Also the method we use is completely different. Contrary to the above mentioned paper, where variationality properties are studied by tools of the *theory of formal integrability of PDE* (as e.g. in [3]), the proof is long and rather complicated, and the main result is local and obtained for a system of C^ω equations, we use a geometric setting, representing a system of differential equations by means of a dynamical form over a fibred manifold, and obtain *more complete results*—for first-order PDE in *general* and valid for the C^∞ case—without tedious calculations and in a straightforward way based on the *Poincaré Lemma*. The setting and methods we use are similar to those applied in our previous work [5], where we have discussed quasilinear PDE. Some other closely related results can be found also in [6] and [10].

The paper is organized as follows. In Section 2 we introduce notations and recall necessary concepts and results concerning the calculus of variations on fibred manifolds. The main result, Theorem 2, is stated and proved in Section 3.

2. Basic definitions and known results

In what follows, all manifolds and mappings are smooth, and summation over repeated indices is undestood. We consider a fibred manifold $\pi : Y \to X$, $\dim X = n$, $\dim Y = m+n$. We denote J^1 the 1-jet prolongation functor, $\pi_1 : J^1Y \to X$, $\pi_{1,0} : J^1Y \to Y$. Let us recall

Práce SVOČ 2002, vedoucí práce doc. RNDr. Olga Krupková, DrSc., Matematický ústav Slezské univerzity v Opavě

some basic definitions. A mapping $\gamma : U \to Y$, where $U \subset X$ is an open subset, is called a *section* of π, if $\pi \circ \gamma = id_U$. A vector field ξ on Y is said to be π-*vertical*, if $T\pi.\xi = 0$. Similarly, a vector field ξ on J^1Y is called π_1-*vertical* (resp. $\pi_{1,0}$-*vertical*), if $T\pi_1.\xi = 0$ (resp. $T\pi_{1,0}.\xi = 0$). A q-form η on J^1Y is called π_1-*horizontal* (resp. $\pi_{1,0}$-*horizontal*), if $i_\xi\eta = 0$ for every π_1-vertical (resp. $\pi_{1,0}$-vertical) vector field ξ on J^1Y. We denote by h the *horizontalization* of differential forms. h is defined to be an R-linear wedge-product preserving mapping such that for a q-form η on Y $h\eta$ is a q-form on J^1Y, and

$$hdx^i = dx^i, \quad hdy^\sigma = y^\sigma_j dx^j, \quad hf = f \circ \pi_{(1,0)}. \tag{2.1}$$

Its easy to see, that

$$hdf = d_i f dx^i, \quad where \quad d_i f = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^\sigma} y^\sigma_i. \tag{2.2}$$

η is called *contact*, if $J^1\gamma^*\,\eta = 0$ for every section γ of π. A contact $\pi_{1,0}$-horizontal q-form η is called 1-*contact*, if for every π_1-vertical vector field ξ on J^1Y, the form $i_\xi\eta$ is π_1-horizontal; η is called k-*contact*, $2 \le k \le q$, if $i_\xi\eta$ is $(k-1)$-contact. Recall that *for every $\pi_{1,0}$-horizontal q-form on J^1Y there is a unique decomposition $\eta = \eta_0 + \eta_1 + \cdots + \eta_q$, where η_0 is a π_1-horizontal form, and η_i, $1 \le i \le q$, is a i-contact form on J^1Y*; we set $h\eta = \eta_0$, $p_i\eta = \eta_i$, and call it the *horizontal* and *i-contact part* of η, respectively. Consequently, every q-form on Y can be uniquely decomposed as follows

$$\pi^*_{1,0}\eta = h\eta + p_1\eta + \cdots + p_q\eta. \tag{2.3}$$

We denote by (x^i, y^σ) (resp. $(x^i, y^\sigma, y^\sigma_j)$) local fibred coordinates on Y (resp. the associated coordinates on J^1Y), and set

$$\begin{aligned} &\omega_0 = dx^1 \wedge dx^2 \cdots \wedge dx^n, \quad \omega^\sigma = dy^\sigma - y^\sigma_k\, dx^k, \\ &\omega_j = i_{\partial/\partial x^j}\ \omega_0, \quad \omega_{j_1 j_2} = i_{\partial/\partial x^{j_2}}\, \omega_{j_1}, \quad \text{etc.} \end{aligned} \tag{2.4}$$

A 1-contact $\pi_{1,0}$-horizontal $(n+1)$-form E on J^1Y is called a *dynamical form.* In fibred coordinates, $E = E_\sigma\, \omega^\sigma \wedge \omega_0$, where $E_\sigma = E_\sigma(x^i, y^\nu, y^\nu_k)$. A section γ of π is called a *path* of E, if $E \circ J^1\gamma = 0$, i.e., if the components γ^ν of γ satisfy the following system of m first-order PDE:

$$E_\sigma\Big(x^i, \gamma^\nu, \frac{\partial \gamma^\nu}{\partial x^j}\Big) = 0, \quad 1 \le \sigma \le m. \tag{2.5}$$

By a *first-order Lagrangian* we mean a horizontal n-form λ on J^1Y. In fibred coordinates, $\lambda = L\omega_0$, where $L = L(x^i, y^\nu, y^\nu_k)$.

Let ρ be an n-form on Y. Then $\lambda = h\rho$ is a first-order Lagrangian (with the function L *polynomial of degree $\le n$ in the first-order derivatives*), and

$$\pi^*_{1,0}\rho = L\,\omega_0 + \sum_{k=1}^{n} \Big(\frac{1}{k!}\Big)^2 \frac{\partial^k L}{\partial y^{\sigma_1}_{j_1} \cdots \partial y^{\sigma_k}_{j_k}}\, \omega^{\sigma_1} \wedge \cdots \wedge \omega^{\sigma_k} \wedge \omega_{j_1 \cdots j_k} \tag{2.6}$$

(see [7] and also [2]). We denote $\rho^{\mathcal{K}}_\lambda = \pi^*_{1,0}\rho$ and call this n-form the *Krupka form* of λ. The at most 1-contact part of $\rho^{\mathcal{K}}_\lambda$, i.e.,

$$\theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y^\sigma_j} \omega^\sigma \wedge \omega_j, \tag{2.7}$$

is called the *Poincaré–Cartan form* of λ. Note that $E_\lambda = p_1 d\rho$ is a *dynamical form* on J^1Y; it is called the *Euler–Lagrange form* of λ, and the corresponding equations for paths of E_λ are called the *Euler–Lagrange equations.* Obviously, $E_\lambda = E_\sigma(L)\,\omega^\sigma \wedge \omega_0$, where

$$E_\sigma(L) = \frac{\partial L}{\partial y^\sigma} - d_j \frac{\partial L}{\partial y^\sigma_j}, \tag{2.8}$$

and the *Euler–Lagrange expressions* E_σ, $1 \leq \sigma \leq m$, are all *polynomials of degree* $\leq n$ *in the* y^ν_j*'s.*

A dynamical form E on J^1Y is called *variational*, if for every point $x \in J^1Y$ there exists a neighbourhood U and Lagrangian λ defined on U such, that $E = E_\lambda$. Thus, for variational forms equations for paths (2.2) are the Euler–Lagrange equations. It is known (see [12]) that if $E = E_\sigma \omega^\sigma \wedge \omega_0$ is a variational dynamical form on J^1Y, then to every point in J^1Y there exists a neighbourhood U such that $\lambda = L\omega_0$, where L is a function on U defined by

$$L = y^\sigma \int_0^1 E_\sigma(x^i, uy^\nu, uy^\nu_j)\, du, \tag{2.9}$$

is a Lagrangian for E, called *Vainberg–Tonti Lagrangian.*

3. Variational properties of systems of first-order PDE

Theorem 1 ([1],[8])**.** *A dynamical form E on J^1Y is variational if and only if in every fibred chart its components E_σ satisfy the following conditions*

$$\frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} + d_j \frac{\partial E_\nu}{\partial y^\sigma_j} = 0, \; \frac{\partial E_\sigma}{\partial y^\nu_i} + \frac{\partial E_\nu}{\partial y^\sigma_i} = 0, \quad 1 \leq \sigma, \nu \leq m, \; 1 \leq i \leq n. \tag{3.10}$$

Proposition 1. *Let E be a dynamical form on J^1Y, $E = E_\sigma \omega^\sigma \wedge \omega_0$. If E is variational, then the E_σ are polynomials of degree $\leq n$ in the y^ν_j's.*

Proof. If E is a variational dynamical form on J^1Y, then by the previous theorem its components E_σ satisfy the second of the conditions (3.10). Differentiating E_σ and using this property we get

$$\begin{aligned}
&\frac{\partial^{n+1} E_\sigma}{\partial y^{\nu_1}_{i_1} \dots \partial y^{\nu_{k-1}}_{i_{k-1}} \partial y^{\nu_k}_p \partial y^{\nu_{k+1}}_{i_{k+1}} \dots \partial y^{\nu_{l-1}}_{i_{l-1}} \partial y^{\nu_l}_p \partial y^{\nu_{l+1}}_{i_{l+1}} \dots \partial y^{\nu_{n+1}}_{i_{n+1}}} \\
&= - \frac{\partial^{n+1} E_{\nu_k}}{\partial y^{\nu_1}_{i_1} \dots \partial y^{\nu_{k-1}}_{i_{k-1}} \partial y^{\sigma}_p \partial y^{\nu_{k+1}}_{i_{k+1}} \dots \partial y^{\nu_{l-1}}_{i_{l-1}} \partial y^{\nu_l}_p \partial y^{\nu_{l+1}}_{i_{l+1}} \dots \partial y^{\nu_{n+1}}_{i_{n+1}}} \\
&= \frac{\partial^{n+1} E_{\nu_l}}{\partial y^{\nu_1}_{i_1} \dots \partial y^{\nu_{k-1}}_{i_{k-1}} \partial y^{\sigma}_p \partial y^{\nu_{k+1}}_{i_{k+1}} \dots \partial y^{\nu_{l-1}}_{i_{l-1}} \partial y^{\nu_k}_p \partial y^{\nu_{l+1}}_{i_{l+1}} \dots \partial y^{\nu_{n+1}}_{i_{n+1}}} \\
&= - \frac{\partial^{n+1} E_\sigma}{\partial y^{\nu_1}_{i_1} \dots \partial y^{\nu_{k-1}}_{i_{k-1}} \partial y^{\nu_l}_p \partial y^{\nu_{k+1}}_{i_{k+1}} \dots \partial y^{\nu_{l-1}}_{i_{l-1}} \partial y^{\nu_k}_p \partial y^{\nu_{l+1}}_{i_{l+1}} \dots \partial y^{\nu_{n+1}}_{i_{n+1}}},
\end{aligned}$$

since at least two of the indices $i_1, \dots, i_{n+1}$ must take the same value, say, p.

On the other hand, it holds

$$\begin{aligned}
&\frac{\partial^{n+1} E_\sigma}{\partial y^{\nu_1}_{i_1} \dots \partial y^{\nu_{k-1}}_{i_{k-1}} \partial y^{\nu_k}_p \partial y^{\nu_{k+1}}_{i_{k+1}} \dots \partial y^{\nu_{l-1}}_{i_{l-1}} \partial y^{\nu_l}_p \partial y^{\nu_{l+1}}_{i_{l+1}} \dots \partial y^{\nu_{n+1}}_{i_{n+1}}} \\
&= \frac{\partial^{n+1} E_\sigma}{\partial y^{\nu_1}_{i_1} \dots \partial y^{\nu_{k-1}}_{i_{k-1}} \partial y^{\nu_l}_p \partial y^{\nu_{k+1}}_{i_{k+1}} \dots \partial y^{\nu_{l-1}}_{i_{l-1}} \partial y^{\nu_k}_p \partial y^{\nu_{l+1}}_{i_{l+1}} \dots \partial y^{\nu_{n+1}}_{i_{n+1}}},
\end{aligned}$$

Hence, we conclude that

$$\frac{\partial^{n+1} E_\sigma}{\partial y^{\nu_1}_{i_1} \dots \partial y^{\nu_{n+1}}_{i_{n+1}}} = 0. \tag{3.11}$$

□

In view of the above proposition and the second of the conditions (3.10) we set

$$\begin{aligned} &E_\sigma = A_\sigma + B^{j_1}_{\sigma\nu_1} y^{\nu_1}_{j_1} + \dots + B^{j_1\cdots j_n}_{\sigma\nu_1\cdots\nu_n} y^{\nu_1}_{j_1} \cdots y^{\nu_n}_{j_n}, \\ &B^{j_1\cdots j_p\cdots j_q\cdots j_k}_{\sigma\nu_1\cdots\nu_p\cdots\nu_q\cdots\nu_k} = B^{j_1\cdots j_q\cdots j_p\cdots j_k}_{\sigma\nu_1\cdots\nu_q\cdots\nu_p\cdots\nu_k}, \quad B^{j_1\cdots j_k}_{\sigma\nu_1\cdots\nu_p\cdots\nu_k} = -B^{j_1\cdots j_k}_{\nu_p\nu_1\cdots\sigma\cdots\nu_k}, \quad 1 \le k \le n. \end{aligned} \tag{3.12}$$

Taking into account (2.9), we can see immediately that the following proposition holds:

Proposition 2. *The following conditions are equivalent:*

(1) *A Lagrangian $\lambda = L\omega_0$ on J^1Y is a polynomial of degree r in y^ν_j, where $1 \le r \le n$.*
(2) *The Euler–Lagrange expresssions $E_\sigma(L)$ are polynomials of degree r in y^ν_j, where $1 \le r \le n$.*
(3) *The form $\rho^{\mathcal{K}}_\lambda$ is projectable onto Y.*

Let us turn to analyze variationality of first-order PDE.

Theorem 2. *Let E be a dynamical form on J^1Y. The following conditions are equivalent:*

(1) *E is variational.*
(2) *There is a unique closed $(n+1)$-form α on Y such that $E = p_1\alpha$.*

Proof. First, suppose that E is variational. Then we have a family of local Vainberg–Tonti Lagrangians (2.9) defined on open subsets of J^1Y. Since the components E_σ of E take the form (3.12) by Proposition 1, the corresponding Lagrangians are also polynomials in the y^ν_k's. Accordingly, we obtain a family of local n-forms $\rho^{\mathcal{K}}_\lambda$ (2.6), which are $\pi_{1,0}$-projectable. The forms $d\rho^{\mathcal{K}}_\lambda$ are closed and $p_1 d\rho^{\mathcal{K}}_\lambda = E$. We have to show that the local forms $d\rho^{\mathcal{K}}_\lambda$ give rise to global form α on Y, and that this form is unique.

Let α be a form on Y such that $E = p_1\alpha$. We have $\pi^*_{1,0}\alpha = E + F$ where F is an at least 2-contact form. Put

$$F = \sum_{k=1}^{n} F^{j_1\cdots j_k}_{\sigma\nu_1\cdots\nu_k}\, \omega^\sigma \wedge \omega^{\nu_1} \wedge \dots \wedge \omega^{\nu_k} \wedge \omega_{j_1\cdots j_k}, \tag{3.13}$$

where the components are completely antisymmetric in both the superscripts and the subscripts. The condition $d\alpha = 0$ then after a straightforward computation gives

$$F^{j_1\cdots j_k}_{\sigma\nu_1\cdots\nu_k} = \frac{1}{k!(k+1)!}\, \frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{i_1} \cdots \partial y^{\nu_k}_{i_k}}, \tag{3.14}$$

and the identities (3.10). Then the form α in fibred coordinates is of the form

$$\alpha = E_\sigma \omega^\sigma \wedge \omega_0 + \sum_{k=1}^{n} \frac{1}{k!(k+1)!} \left(\frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{i_1} \cdots \partial y^{\nu_k}_{i_k}} \right) \omega^\sigma \wedge \omega^{\nu_1} \wedge \dots \wedge \omega^{\nu_k} \wedge \omega_{j_1\cdots j_k}. \tag{3.15}$$

Now, in the base $(dx^i, dy^\sigma, dy^\sigma_j)$ we obtain after a straightforward computation

$$\begin{aligned} \alpha = {} & A_\sigma dy^\sigma \wedge \omega_0 + \frac{1}{2!}\, B^{j_1}_{\sigma\nu_1} dy^\sigma \wedge dy^{\nu_1} \wedge \omega_{j_1} + \dots \\ & + \frac{1}{(n+1)!}\, B^{j_1\cdots j_n}_{\sigma\nu_1\cdots\nu_n} dy^\sigma \wedge dy^{\nu_1} \wedge \dots \wedge dy^{\nu_n} \wedge \omega_{j_1\cdots j_n}. \end{aligned} \tag{3.16}$$

Since the form F is completely determined by the components of E, it is globally defined and unique. Consequently, α is defined on Y, and is unique.

Conversely, suppose (2). Then $\pi_{1,0}^*\alpha = E + F$, where F is at least 2-contact. Computing $d(E+F) = 0$ we get

$$d_i F^i_{\sigma\nu} + \tfrac{1}{2}\Big(\frac{\partial E_\nu}{\partial y^\sigma} - \frac{\partial E_\sigma}{\partial y^\nu}\Big) = 0, \quad 2F^k_{\sigma\nu} - \frac{\partial E_\sigma}{\partial y^\nu_k} = 0. \tag{3.17}$$

Decomposing the second of the equations (3.17) to the symmetric and antisymmetric part in σ, ν we conclude

$$\frac{\partial E_\sigma}{\partial y^\nu_j} + \frac{\partial E_\nu}{\partial y^\sigma_j} = 0, \quad F^k_{\sigma\nu} = \frac{1}{4}\left(\frac{\partial E_\sigma}{\partial y^\nu_k} - \frac{\partial E_\nu}{\partial y^\sigma_k}\right). \tag{3.18}$$

Substituting $F^k_{\sigma\nu}$ in the first of the equations (3.17) we get

$$\frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} + d_j \frac{\partial E_\nu}{\partial y^\sigma_j} = 0. \tag{3.19}$$

□

Remark. By the above theorem, first-order variational dynamical forms are in one-to-one correspondence with closed $(n+1)$-forms on Y. A similar property are known to possess variational ordinary differential equations of any order [11].

Corollary 1. *A dynamical form E on J^1Y is variational if and only if its components are of the form (3.12), and the $(n+1)$-form*

$$\begin{aligned}\alpha = {} & A_\sigma dy^\sigma \wedge \omega_0 + \frac{1}{2!}\, B^{j_1}_{\sigma\nu_1} dy^\sigma \wedge dy^{\nu_1} \wedge \omega_{j_1} + \dots \\ & + \frac{1}{(n+1)!}\, B^{j_1\cdots j_n}_{\sigma\nu_1\cdots\nu_n} dy^\sigma \wedge dy^{\nu_1} \wedge \dots \wedge dy^{\nu_n} \wedge \omega_{j_1\cdots j_n}\end{aligned} \tag{3.20}$$

on Y is closed.

Corollary 2. *A system of C^∞ first-order partial differential equations is variational if and only if for some r, $1 \le r \le n$, it is of the form*

$$B^{j_1\cdots j_r}_{\sigma\nu_1\cdots\nu_r} \frac{\partial y^{\nu_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{\nu_r}}{\partial x^{j_r}} + \ \dots\ + B^{j_1 j_2}_{\sigma\nu_1\nu_2} \frac{\partial y^{\nu_1}}{\partial x^{j_1}} \frac{\partial y^{\nu_2}}{\partial x^{j_2}} + B^{j_1}_{\sigma\nu_1} \frac{\partial y^{\nu_1}}{\partial x^{j_1}} + A_\sigma = 0, \tag{3.21}$$

where

$$B^{j_1\cdots j_p\cdots j_q\cdots j_k}_{\sigma\nu_1\cdots\nu_p\cdots\nu_q\cdots\nu_k} = B^{j_1\cdots j_q\cdots j_p\cdots j_k}_{\sigma\nu_1\cdots\nu_q\cdots\nu_p\cdots\nu_k}, \quad B^{j_1\cdots j_k}_{\sigma\nu_1\cdots\nu_p\cdots\nu_k} = -B^{j_1\cdots j_k}_{\nu_p\nu_1\cdots\sigma\cdots\nu_k}, \quad 1 \le k \le r, \tag{3.22}$$

and the $(n+1)$-form

$$\begin{aligned}\alpha = {} & A_\sigma dy^\sigma \wedge \omega_0 + \frac{1}{2!}\, B^{j_1}_{\sigma\nu_1} dy^\sigma \wedge dy^{\nu_1} \wedge \omega_{j_1} + \dots \\ & + \frac{1}{(r+1)!}\, B^{j_1\cdots j_r}_{\sigma\nu_1\cdots\nu_r} dy^\sigma \wedge dy^{\nu_1} \wedge \dots \wedge dy^{\nu_r} \wedge \omega_{j_1\cdots j_r}\end{aligned} \tag{3.23}$$

on Y is closed. In this case, α is the exterior derivative of the Krupka form (2.3) associated with the corresponding Vainberg–Tonti Lagrangian L (which is a polynomial of degree r in the variables y^ν_j).

Acknowledgements

It is my pleasure to express deep gratitude to Prof. Olga Krupková for stimulating discussions on the subject of this work and her kind help. I acknowledge the financial support from the Grant GAČR 201/00/0724.

References

1. I. Anderson and T. Duchamp, *On the existence of global variational principles*, Am. J. Math. **102** (1980) 781–867.
2. D. E. Betounes, *Extension of the classical Cartan form*, Phys. Rev. D 29 (1984) 599–606.
3. R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt and P.A. Griffiths, *Exterior Differential Systems*, Springer, New York, 1991.
4. J. Grifone, J. Muñoz Masqué and L.M. Pozo Coronado, *Variational first-order quasilinear equations*, in: *Steps in Differential Geometry*, Proc. Colloq., Debrecen, July 2000, L. Kozma and L. Tamássy. eds. (University of Debrecen, Debrecen, 2001) 131–138.
5. A.Haková, *Vztah mezi variačností a uzavřeností pro* $(n+1)$*-formy 1.řádu*, Práce SVOČ 2001 (3.cena v celostátním kole); Preprint GA 2/2001 (Mathematical Institute of the Silesian University in Opava, Opava, 2001) 7pp.
6. A.Haková and O. Krupková, *Variational first-order partial differential equations*, submitted.
7. D. Krupka, *A map associated to the Lepagean forms of the calculus of variations in fibered manifolds*, Czechoslovak Math. J. 27 (1977) 114–118.
8. D. Krupka, *On the local structure of the Euler-Lagrange mapping of the calculus of variations*, in: Proc. Conf. on Diff. Geom. and Its Appl. 1980, O. Kowalski, ed. (Universita Karlova, Prague, 1981) 181–188.
9. D. Krupka, *Lepagean forms in higher order variational theory*, in: *Modern Developments in Analytical Mechanics I: Geometrical Dynamics*, Proc. IUTAM-ISIMM Symposium, Torino, Italy 1982, S. Benenti, M. Francaviglia and A. Lichnerowicz, eds. (Accad. delle Scienze di Torino, Torino, 1983) 197–238.
10. D. Krupka, *Variational principles for energy-momentum tensors*, Reports on Math. Phys. 49 (2002).
11. O.Krupková, *The Geometry of Ordinary Variational Equations*, Lecture Notes in Mathematics 1678, Springer, Berlin, 1997.
12. E. Tonti, *Variational formulation of nonlinear differential equations I, II*, Bull. Acad. Roy. Belg. Cl. Sci. 55 (1969) 137–165, 262–278.

Alžběta Haková

THE STRUCTURE OF VARIATIONAL FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

(Práce SVOČ 2002)

Abstrakt

Cílem práce je analyzovat variačnost systémů parciálních diferenciálních rovnic 1. řádu na hladkých varietách metodami diferenciální geometrie. Je známo, že variačnost systému parciálních diferenciálních rovnic (libovolného řádu) úzce souvisí s existencí jistých uzavřených diferenciálních forem, které obecně nejsou určeny jednoznačně. Zároveň netriviální topologická stuktura podkladové variety vede i k nutnosti zkoumat definiční obory těchto uzavřených forem, tj. k otázce *globální existence*.

Pro případ obyčejných diferenciálních rovnic libovolného řádu na fibrovaných prostorech bylo dokázá-no, že variačnost je ekvivalentní s existencí jisté uzavřené formy, a že tato forma je globální a jediná. Hlavním výsledkem předložené práce je důkaz, že analogické tvrzení platí i pro parciální diferenciální rovnice 1. řádu. Výsledkem je rovněž explicitní konstrukce této diferenciální formy. Jako důsledek pak získáváme explicitní charakteristiku všech systémů PDR 1. řádu, které jsou variační a jejich Lagrangiánů.

Vztah práce k dřívějším pracím SVOČ a k práci diplomové

Práce navazuje na práci "Alžběta Haková: Vztah mezi variačností a uzavřeností pro $(n+1)$-formy 1. řádu", prezentovanou v rámci SVOČ v r. 2001 (práce získala 3. cenu v celostátním kole), v níž byl studován stejný problém pro speciální případ *kvazilineárních* rovnic. Výsledek předložené práce představuje zobecnění na libovolný systém PDR 1. řádu.

Tento výsledek byl prezentován formou posteru na mezinárodní zimní škole "Geometry and Physics" v Srní v lednu 2002 a představuje část práce A. Haková, O. Krupková: Variational first-order partial differential equations", která byla nabídnuta do recenzního řízení v mezinárodním matematickém časopise.

Předložená práce SVOČ není prací diplomovou.