Testing Possible Central Projection Images

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Abstract:*This remark improves on some old contributions of Chetverukhin and Beskin to the classical criterion of Kruppa for a central axonometry to be a central projection. Several additional observations are attached.*

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§1 Preliminaries

We denote by $\mathbb{P}^3$ the real projective space of dimension 3 and by $\overline{\mathbb{E}}^3$ the projective closure of the real Euclidean space $\mathbb{E}^3$ of dimension 3.

If *A,B* are distinct points, then *AB* will designate their join line; similarly, if *A,B,C* are non-collinear points, then *ABC* will designate their join plane. For distinct coplanar lines *a,b* the symbol $a \sqcap b$ will denote their intersection point.

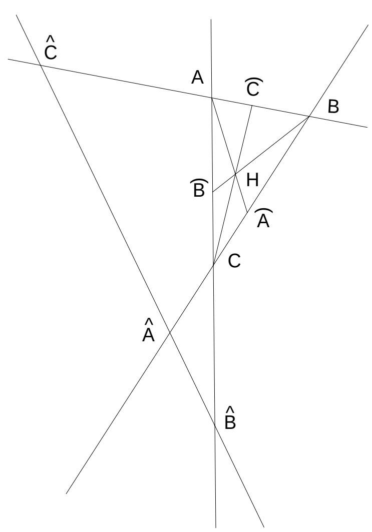

Figure 1.

Now, let *A, B, C* be non-collinear points, and *d* a line in the plane *ABC* in a general position, i.e., with *A, B, C* not lying on *d*. Then the *harmonic pole H* of *d* is defined as the common intersection point of lines $A\widehat{A}$, $B\widehat{B}$, $C\widehat{C}$, where $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ have to satisfy that $(A, B, \hat{C}, \widehat{C})$, $(B, C, \hat{A}, \widehat{A})$, $(A, C, \hat{B}, \widehat{B})$ with $\hat{C} = AB \sqcap d$, $\hat{B} = AC \sqcap d$, $\hat{A} = BC \sqcap d$ are harmonic quadruplets. The conic k in the plane ABC, for which the triplet $\{A, B, C\}$ is autopolar and *H* is the pole of *d* will be called the *fundamental conic* with respect to $\{A, B, C, d\}$.

§2 Desarguesian configurations

A *Desarguesian configuration* $\mathcal{D}$ in $\mathbb{P}^3$ is defined as a sextuple of points (A,B,C, A', B', C') satisfying the following conditions:

(i) the points A,B,C are non-collinear,

(ii) $A \neq A'$, $B \neq B'$, $C \neq C'$,

(iii) the points $BC \sqcap B'C'$, $AC \sqcap A'C'$, $AB \sqcap A'B'$ are mutually distinct and they lie on the same line d called *axis*,

(iv) the lines AA', BB', CC' are mutually distinct and they pass through the same point D called *centre*.

The configuration $\mathcal{D}$ is accompanied by the harmonic pole H of d with respect to $\{A,B,C\}$ and by the harmonic pole H' of d with respect to $\{A',B',C'\}$. The fundamental

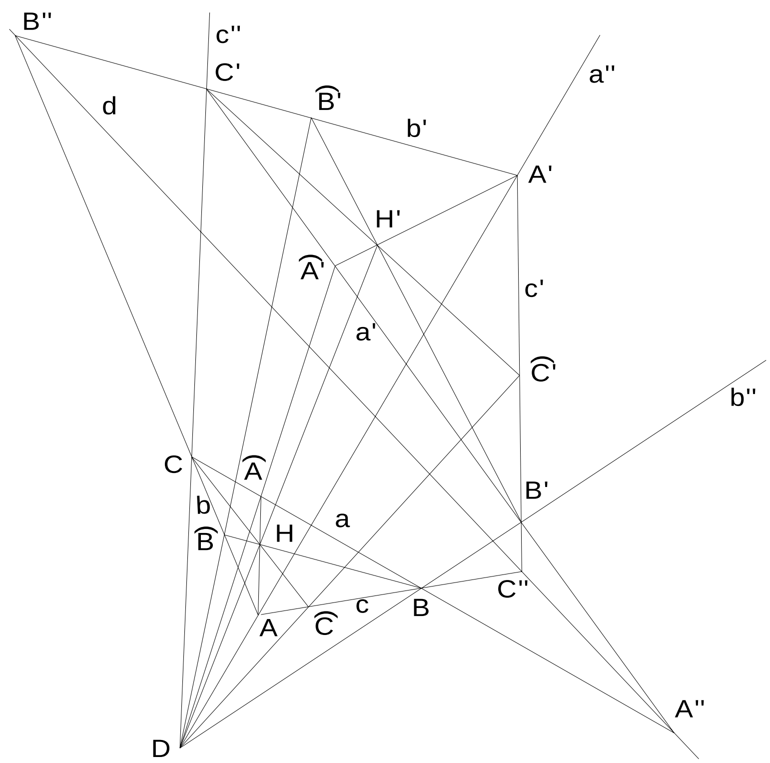

Figure 2.

conic k with respect to $\{A',B',C',d\}$ will be called the *companion conic* of $\mathcal{D}$. The prolonged accompanying figure is described in Fig. 2.

A Desarguesian configuration with all points lying in the same plane will be called *planar*.

A Desarguesian configuration $\mathcal{D} = (A,B,C,A',B',C')$ in $\overline{\mathbb{E}}^3$ will be called *orthonormed* if the points A,B,C,D are finite with mutually perpendicular line segments DA, DB, DC of the same length, and if A',B' C' are points at infinity.

For a given Desarguesian configuration $\mathcal{D} = (A,B,C,A',B',C')$, the notation of Fig. 2 will be used frequently.

If alower index is used in the denotation of a Desarguesian configuration then the index will be placed at all symbols of the accompanying figure as well.

In the following considerations all Desarguesian configurations are situated in $\overline{\mathbb{E}}^3$.

§3 Theorem of Kruppa

We restate the Theorem of Erwin Kruppa as follows (cf. [1], [2] and [3], p. 81).

Theorem: *A planar Desarguesian configuration $\mathcal{D}$ of finite points, with finite centre and finite axis is the image of an orthonormed Desarguesian configuration under a central projection iff the companion conic of $\mathcal{D}$ is an imaginary circle.*

In the sequel we outline the proof of this Theorem using a point of view of N.F. Chetverukhin (see e.g. [3], pp. 88–91). We use a shorter phrase “to be admissible” instead of “to be an image of an orthonormed Desarguesian configuration under a central projection”.

$\Rightarrow$Let $\mathcal{D} = (A,B,C,A',B',C')$ be a planar admissible Desarguesian configuration with finite points A,B,C,A',B',C', and with finite centre and finite axis.

Let $\mathcal{D}$ be the image of an orthonormed Desarguesian configuration $\mathcal{D}_o = (A_o,B_o,C_o,A_o',B_o',C_o')$ under the projection $\mathbf{\Pi}$ from a finite point S to the plane $\sigma=ABC$ (of course $S\notin \sigma$). We need to show that the companion conic k of $\mathcal{D}$ is an imaginary circle. As $\mathcal{D}_\mathbf{o}$ is orthonormed, its harmonic pole H_o is an orthocentre of the equilateral triangle $A_oB_oC_o$ and for the line D_oH_o (perpendicular to $A_0B_0C_0$) the harmonic pole H' of $\mathcal{D}$ is the vanishing point under $\mathbf{\Pi}$.

Similarly, the axis d of $\mathcal{D}$ is the vanishing line of the plane $A_oB_oC_o$ under $\mathbf{\Pi}$. The polarity $\mathcal{P}$ in the bundle of points and lines through S, under which $SA'\mapsto$SB′C′, $SB'\mapsto$SA′C′, $SC'\mapsto$SA′B′, $SH'\mapsto$Sd, is orthogonal and intersects the plane σ in the polarity under which $A'\mapsto B'C',\ B'\mapsto A'C',\ C'\mapsto A'B',\ H'\mapsto d$, i.e. in the polarity with fundamental conic k. As $\mathcal{P}$ is orthogonal, k must be an imaginary circle and its real representative is circumscribed around the orthocentre of $(A'B'C')$ with radius equal to the distance from S of σ.

$\Leftarrow$ Let $\mathcal{D} = (A,B,C,A',B',C')$ be a planar Desarguesian configuration such that A,B,C, A', B', D',d are finite and k is an imaginary circle. We take one of the Laguerre points of the polarity on k as the central projection centre S. From S this polarity is projected by the orthogonal polarity $\mathcal{P}$ in the bundle of lines and planes through S. We construct an orthonormed Desarguesian configuration $\boldsymbol{\mathcal{D}_o} = (A_o,B_o,C_o,A_o',B_o',C_o')$ as follows: Put D_0 as the point D, A_o',B_o',C_o' as points at infinity on SA', SB', SC', respectively and A_o, B_o, C_o as the points $SA \sqcap D_oA_o'$, $SB \sqcap D_oB_o'$, $SC \sqcap D_oC_o'$, respectively. Since the line D_0H' is perpendicular to the plane $A_oB_oC_o$ (because d is the vanishing line of the plane $A_oB_oC_o$) and H_0 is the orthocentre of $\{A_o,B_o,C_o\}$, the points A_o,B_o,C_o must be equally distant from the point D_0. Since $\mathcal{P}$ is orthogonal, the lines D_oA_o, D_oB_o, D_oC_o must be mutually perpendicular. Thus $\mathcal{D}$ is admissible. □

§4 Tests of admissibility

Be $\mathcal{D} = (A,B,C,A',B',C')$ a planar Desarguesian configuration with A,B,C,A',B',C',D,d finite. The points $A'B'C'$ are supposed to be vertices of an acute-angled triangle. The Theorem of Kruppa leads to a simple test for $\mathcal{D}$ to be admissible (for all $\mathcal{D}$ having a fixed fragment $\{A',B',C',H',d\}$). First, one finds the point S from which the segments $A'B'$, $A'C'$, $B'C'$ are visible under the right angle. There are two possible choices for S, see Fig. 3. *The circle in* $A'B'C'$ *circumscribed around the orthocentre of* (A',B',C') *with the radius equal to the distance of the point* S *from* $A'B'C'$ *(the distance circle for the corresponding central projection) is the real representative of the companion conic of* $\mathcal{D}$ *iff* H' *is the antipole of* d *according to the distance circle.*

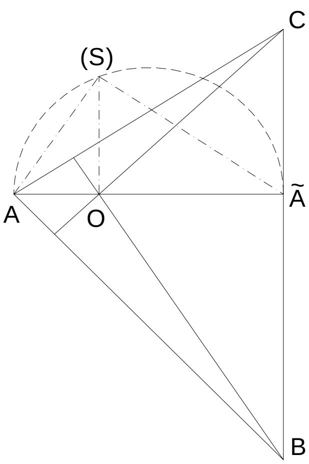

Figure 3.

If one prolongs the accompanying figure of $\mathcal{D}$ as pointed out in Fig. 4, one can utilize just the point $\widetilde{D}$ (diagonally opposite to D). Let $\boldsymbol{\mathcal{D}_o} = (A_o,B_o,C_o,A_o',B_o',C_o')$ be an auxiliary orthonormed Desarguesian configuration. It is known that there exists a singular collinear transformation of $\overline{\mathbb{E}}^3$ into itself carrying $\boldsymbol{\mathcal{D}_o}$ onto $\mathcal{D}$ (with $A_0 \mapsto A$, $B_0 \mapsto B$ etc.). Then, one can construct the vanishing point IV of the line $D_0\widetilde{D_0}$ under this transformation as described on Fig. 4. The construction of the vanishing line l of the plane going through the midpoint of the line segment $D_0\widetilde{D}$ perpendicularly to $D_0\widetilde{D}$ is more complicated and omitted here. Then, it can be shown that $\mathcal{D}$ is admissible iff IV is antipole of l according to the distance circle.

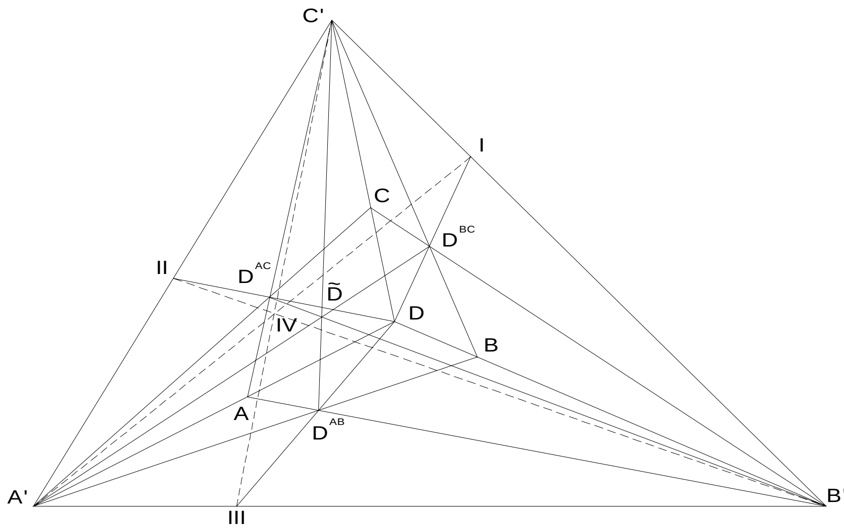

Figure 4.

Second prolongation of the accompanying figure: The new points are:

$$D^{AB} = AB' \sqcap BA', \qquad \widetilde{D} = A'D^{BC} \sqcap B'D^{AC},$$
$$D^{AC} = AC' \sqcap CA', \qquad \widetilde{D} = B'D^{AC} \sqcap C'D^{AB},$$
$$D^{BC} = BC' \sqcap CB', \qquad \widetilde{D} = C'D^{AB} \sqcap A'D^{BC},$$
$$I = DD^{BC} \sqcap B'C',\ II = DD^{AC} \sqcap A'C',\ III = DD^{AB} \sqcap A'B',$$
$$IV = A'I \sqcap B'II = A'I \sqcap C'III = B'II \sqcap C'III.$$

A simpler criterion was obtained in a recent work of Miklós Hoffmann and Paul Yiu, see [5]: *A Desarguesian configuration* $\mathcal{D}$ *is admissible iff the vanishing point IV coincides with the square root point of the orthocentre* of (A',B',C').

The construction of the square root point is outlined at Fig. 5.

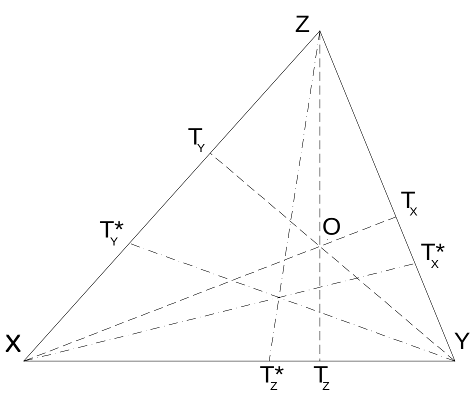

Figure 5

$T_XT_Y \sqcap XY = Q_Z$, $T_XT_Z \sqcap XZ = Q_Y$,
$T_YT_Z \sqcap YZ = Q_X$,
T_X^* between Y,Z, T_Y^* between X,Z,
T_Z^* between X,Y,

$$\overline{Q_ZT_X} \bullet \overline{Q_ZT_Y} = \left(\overline{Q_ZT_Z^*}\right)^2,$$
$$\overline{Q_YT_X} \bullet \overline{Q_YT_Z} = \left(\overline{Q_YT_Y^*}\right)^2,$$
$$\overline{Q_XT_Y} \bullet \overline{Q_XT_Z} = \left(\overline{Q_XT_X^*}\right)^2,$$

The square root point of the orthocentre O is the common point of the lines XT_X^*, YT_Y^*, ZT_Z^*.

We add a criterion for the admissibility of a given planar Desarguesian configuration $\mathcal{D} = (A,B,C,A',B',C')$ with A,B,C,A',B',C',D,d finite, using only simple properties of central projections. This criterion concerns all $\mathcal{D}$ having a fixed fragment $\{A',B',C',d\}$. One constructs again a point S from which the segments $A'B',A'C',B'C'$ are visible under the right angle. On the lines SA', SB',SC' one finds the points A_o,B_o,C_o with unit distance from S (eight solutions) and consequently the vanishing line of the plane $A_oB_oC_o$ under central projection from S. *The Desarguesian configuration* $\mathcal{D}$ *is admissible iff this vanishing line coincides with the axis d of* $\mathcal{D}$.

§5 Analytical conditions of Beskin

Let $\mathcal{D} = (A,B,C,A',B',C')$ be a planar Desarguesian configuration with A,B,C,A',B',C',D,d finite. First, we add the points $A''' = BC \sqcap a''$, $B''' = AC \sqcap b''$, $C''' = AB \sqcap c''$ to the accompanying figure of $\mathcal{D}$. Then, we denote by a_1, a_2, a_3 the lengths of the line segments DA, DB, DC respectively. Further, we denote by p, q, r, μ_1, μ_2, μ_3, λ_1, λ_2, λ_3 the modified ratios $(DA'A)$, $(DB'B)$, $(DC'C)$, $(DA'''A)$, $(DB'''B)$, $(DC'''C)$, $(BA'''A)$, $(CB'''B)$, $(AC'''C)$, respectively. (The modified ratio (XYZ) is defined as the ratio of signed distance $\overline{XY}$ to the signed distance $\overline{YZ}$, where X,Y,Z are mutually distinct collinear finite points.) The conditions of N.M. Beskin for $\mathcal{D}$ to be admissible are then

$$\frac{1}{p^2}\left(1+\frac{\mu^2}{\delta^2}\left(1+p^2\right)+\frac{2}{\delta}\frac{\mu_1\left(1+p\right)}{p}\right):\frac{1}{q^2}\left(1+\frac{\mu^2}{\delta^2}\left(1+q^2\right)+\frac{2}{\delta}\frac{\mu_2\left(1+q\right)}{q}\right):$$

$$:\frac{1}{r^2}\left(1+\frac{\mu^2}{\delta^2}\left(1+r^2\right)+\frac{2}{\delta}\frac{\mu_3\left(1+r\right)}{r}\right)=a_1^2:a_2^2:a_3^2,$$

where $\mu^2=\frac{\mu_1^2}{p^2}+\frac{\mu_2^2}{q^2}+\frac{\mu_3^2}{r^2}$, $\delta=\frac{\mu_1}{p}+\frac{\mu_2}{q}+\frac{\mu_3}{r}=-\frac{\mu_1\left(1+p\right)}{p}-\frac{\mu_2\left(1+q\right)}{q}-\frac{\mu_3\left(1+r\right)}{r}$.

(see [3], pp. 94–109).

If the metric description of $\mathcal{D}$ is chosen more conveniently, the analytical conditions for $\mathcal{D}$ to be admissible can result more simply, as e.g. in the beautiful condition of Szabó–Stachel–Vogel (see [4], p. 5):

If for a planar Desarguesian configuration $\mathcal{D} = (A,B,C,A',B',C')$ *with* $A,B,C,A',B'\ C',D$ *finite there are known acute angles* α, β, γ *of the triangle* (A',B',C') *and the signed distances* $e=\overline{DA}$, $f=\overline{AA'}$, $g=\overline{DB}$, $h=\overline{BB'}$, $i=\overline{DC}$, $j=\overline{CC'}$ *, then* $\mathcal{D}$ *is* *admissible iff* $\left(\frac{e}{f}\right)^2 : \left(\frac{g}{h}\right)^2 : \left(\frac{i}{j}\right)^2 = \tan\alpha : \tan\beta : \tan\gamma$.

We outline the construction, given $\alpha, \beta, \gamma, e, f, g, h, i, j$:

First, we take an arbitrary triangle $\left(\hat{A}, \hat{B}, \hat{C}\right)$ with prescribed acute angles α, β, γ. Then, we find a point $\hat{D}$ having prescribed ratios of distances relatively to the vertices $\hat{A}, \hat{B}, \hat{C}$, namely $|e+f| : |g+h| : |i+j|$. The point $\hat{D}$ must lie on two circles; the points of each of them have prescribed ratios of distances from $\hat{A}, \hat{B}$, respectively from $\hat{A}, \hat{C}$. Finally, we use a similarity under which the triangle $\left(\hat{A}, \hat{B}, \hat{C}\right)$ goes over onto the triangle (A', B', C') and the point $\hat{D}$ onto the point D with prescribed distances from A', B', C'. The construction of the remaining points A, B, C is easy.

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