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A convergence result for spatially inhomogeneous Preisach operators

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Abstract. Based on the properties of the strongly nonlinear Preisach operator, a convergence result is proved under reasonable assumptions.

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Keywords.

1. Introduction

It is natural to consider materials whose properties are spatially inhomogeneous, i.e. differ at different points.

In this paper we will consider spatially inhomogeneous Preisach hysteresis operators and give a result on continuity of Preisach operators with respect to convergence of density functions.

Hysteresis is a nonlinear phenomena. The basic feature of hysteresis behaviour is a memory effect and irreversibility of the process. A systematic mathematical investigation of hysteresis operators started relatively recently, see e.g. [1], [2], [7], [8], although e.g. the Preisach model itself was introduced much earlier [9].

Our result will be critical for a homogenization argument for a parabolic equation with Preisach hysteresis, analogous to Proposition 2.12 in [6]. This will be published elsewhere.

In [3] a corresponding convergence result is established for the Prandtl-Ishlinskii hysteresis operator, whose nonlinearity is simpler than for the Preisach operator considered here.

The paper is organized as follows. Section 1 is devoted to a description of a simple hysteresis operator, the play, and its basic properties. The Preisach operator is defined in Section 2 and we list also there its important properties. The main result is the subject of Section 3.

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2. The Play operator

Let $r > 0$ be a parameter, $u(t)$ a continuous input function on the time interval $I = [0, T]$ and $w_r^0 \in [-r, r]$ an initial state. We consider a variational inequality

$$\mathcal{G}(t) \in [-r, r], t \in I \tag{1}$$

$$\left(\dot{\mathcal{G}}(t) - \dot{u}(t)\right)(\phi - \mathcal{G}(t)) \geq 0 \text{ for a.e. } t \in I, \text{for all } \phi \in [-r, r] \tag{2}$$

$$\mathcal{G}(0) = w_r^0 \tag{3}$$

for the unknown $\mathcal{G}(t)$. For an input $u \in W^{1,1}(I)$ this problem admits a unique solution $\mathcal{G}_r[u, w_r^0] \in W^{1,1}(I)$. The play operator $\mathcal{E}_r$ with threshold r is defined by the relation

$$\mathcal{E}_r[u, w_r^0] = u - \mathcal{G}_r[u, w_r^0], \tag{4}$$

see Figure 1.

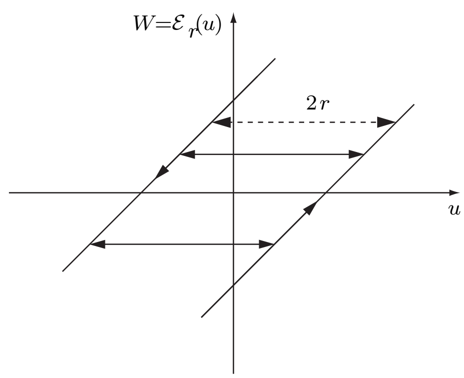

Figure 1

For piecewise monotone inputs, in each interval of monotonicity $[t_0, t_1]$ of the input function $u(t)$, the relation

$$\mathcal{E}_r[u, w_r^0](t) = \max\left\{u(t) - r, \min\{u(t) + r, \mathcal{E}_r[u, w_r^0](t_0)\}\right\} \tag{5}$$

follows from (1) and is often used for an alternative definition of the play operator, see e.g. [8]. In the following we use the "unperturbed" or "virgin" initial state defined by

$$w_r^0 = \min\{r, \max\{-r, u(0)\}\}. \tag{6}$$

and write only $\mathcal{E}_r[u]$.

The play operator has also a geometric "piston-in-cylinder" interpretation. Consider a cylinder of length $2r$ with a piston moving inside. If the input $u(t)$

denotes the position of the piston moving in the cylinder, then the position of the cylinder yields the play operator.

We survey important properties of the play operator.

Theorem 1. *(i) (see e.g. Proposition II.1.1 in* [7]*) The play operator* $\mathcal{E}_r[u]$ *is Lipschitz continuous in* $W^{1,1}(0,T)$.

(ii) (see e.g. Proposition II.4.5 in [7]*) The play operator admits a Lipschitz continuous extension to* $C([0,T])$ *and*

$$|\mathcal{E}_r[u_1](t) - \mathcal{E}_r[u_2](t)| \leq \max_{0\leq s\leq t} |u_1(s) - u_2(s)|, \forall t \in I. \tag{7}$$

(iii) (see e.g. Lemma 2.3.8 in [1]*) The play operator is piecewise monotone in the following sense:*

$$\frac{d\mathcal{E}_r[u]}{dt} \cdot \frac{du}{dt} \geq 0 \text{ a.e. in } (0,T). \tag{8}$$

3. The Preisach operator

The simplest example of a hysteresis nonlinearity is given by a switch or relay with hysteresis, $h_{v,r} : C([0,T]) \times \{-1,1\} \to BV(0,T)$ with input u (magnetic field) and output $h_{v,r}$(magnetization), see Figure 2. The relay is characterized by two parameters $v \in R^1$ (interaction field) and $r > 0$ (critical field of coercivity) and is defined formally as follows: Let R^2_+ denote the set $\{(v,r) \in R^2; r > 0\}$. For given parameters $(v,r) \in R^2_+$, input $u \in C([0,T])$, initial magnetization $\xi \in \{-1,1\}$ and any time $t \in [0,T]$, put

$$X_t := \{\tau \in (0,T]; |u(\tau) - v| = r\}. \tag{9}$$

We then define

$$h_{v,r}[u,\xi](0) = \begin{cases} -1 & \text{if } u(0) \leq v + r \\ \xi & \text{if } v - r < u(0) < v + r \ . \\ 1 & \text{if } u(0) \geq v + r \end{cases} \tag{10}$$

and

$$h_{v,r}[u,\xi](t) = \begin{cases} h_{v,r}[u,\xi](0) & \text{if } X_t = \emptyset \\ -1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = v - r \\ 1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = v + r, \end{cases} \tag{11}$$

see Figure 2.

It is often convenient to use the following representation of the relay by means of the system $\mathcal{E}_r, r > 0$ of play operators.

Lemma 2. *(See e.g. Lemma II.3.6 in Krejčí for more general initial states). Let* $u \in C([0,T])$ *be given. For* $(v,r) \in R^2_+$*, put* $\xi := -1$ *if* $v \geq 0$*,* $\xi = 1$ *if* $v < 0$*.*

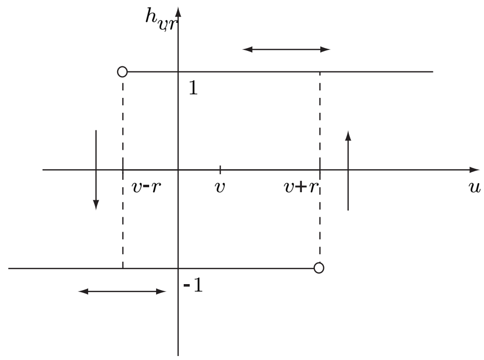

Figure 2

Then for every $t \in [0,T]$ *and* $(v,r) \in R^2_+$, $v \neq \mathcal{E}_r[u](t)$ *we have*

$$h_{v,r}[u,\xi](t) = \begin{cases} -1 & \text{if } v \geq \mathcal{E}_r[u](t) \\ 1 & \text{if } v \leq \mathcal{E}_r[u](t). \end{cases} \tag{12}$$

The output of the Preisach model is formally defined as an average over all elementary switches with a given density function $\psi \in L^1_{loc}(R^2_+)$ by the formula (see e.g. Krasnosel'skii and Pokrovskii [2])

$$\mathcal{P}[u](t) := \int_0^\infty \int_{-\infty}^\infty \psi(v,r) h_{v,r}[u,\xi](t) dv dr, \tag{13}$$

where the initial values of the relays are taken as -1 if $v > 0$ and $+1$ otherwise. Using Lemma 2 on the representation of the relay by a system of plays the output of the Preisach operator can be expressed as

$$\mathcal{P}[u](t) = C + \int_0^\infty g(\mathcal{E}_r[u](t), r) dr, \tag{14}$$

where

$$g(v,r) = \int_0^v \psi(z,r) dz, \tag{15}$$

C is a constant and $\mathcal{E}_r[u](t)$ denotes the play operator.

Remark 1. Notice that the integral in (14) is meaningful if $u \in C([0,T])$ since $\mathcal{E}_r[u](t) = 0$ for r sufficiently large and $g(0,r) = 0$ for all $r > 0$.

In the sequel we will use the following assumptions :
(P1) There exists $\beta \in L^1_{loc}(0,\infty)$, $\beta \geq 0$ a.e. such that

$$0 \leq \psi(z,r) \leq \beta(r) \text{ for a.e. } (z,r) \in R^2_+. \tag{16}$$

For $R > 0$ put $b(R) := \int_0^R \beta(r) dr$.

(P2)

$$\frac{\partial\psi}{\partial z} \in L^{\infty}_{loc}(R^2_+). \tag{17}$$

The next theorem shows conditions under which the Preisach operator is Lipschitz continuous on $C^0([0,T])$. Proof of the extended version of this theorem (for more general initial inputs) can be found in [7], Proposition II.3.11.

Theorem 3. *Let the Assumption (P1) be satisfied and let $R > 0$ be given. Then for every $u, v \in C([0,T])$ such that $\|u\|_{C^0([0,T])}, \|v\|_{C^0([0,T])} \leq R$ the Preisach operator (14) satisfies*

$$\|\mathcal{P}[u] - \mathcal{P}[v]\|_{C^0([0,T])} \leq b(R)\|u - v\|_{C^0([0,T])}. \tag{18}$$

Lemma 4. *(Lemma II.4.1 in* [7]*) Let the Assumptions (P1) and (P2) be satisfied. Then for $u \in W^{1,1}(0,T)$, $r > 0$ and $t \in [0,T]$, we have $\mathcal{P} \in W^{1,1}(0,T)$ and for a.e. $t \in [0,T]$*

$$\dot{\mathcal{P}}[u](t) = \int_0^\infty \dot{\mathcal{E}}_r[u](t)\psi(\mathcal{E}_r[u](t), r)dr. \tag{19}$$

It follows from the previous lemma and from the definition of the play operator that the Preisach operator is piecewise monotone. We have

Theorem 5. *The Preisach operator is under the Assumptions (P1) and (P2) piecewise monotone, this means that for $u \in W^{1,1}(0,T)$*

$$\dot{\mathcal{P}}[u](t)\dot{u}(t) \geq 0 \ a.e. \tag{20}$$

We listed only some properties of the hysteresis operators which will be useful for our purposes. For a full exposition of hysteresis operators and their properties we recommend [1], [2], [7], [8]. In each of these an interested reader can find a different approach.

4. Spatially dependent Preisach operators

We will consider the spatially dependent constitutive relation described by the Preisach operator with a spatially dependent density function $\psi(x, z, r) \in L^1_{loc}(\Omega \times R^2_+)$, $\Omega \subset R^n$ is considered to be a bounded domain with Lipschitz boundary.

For a given input $u(x,t) : \Omega \times I \to R$ we define the output of the spatially dependent Preisach operator as follows:

$$\mathcal{P}[u](x,t) = C + \int_0^\infty g(\mathcal{E}_r[u(x,.)](t), r)dr, \tag{21}$$

where

$$g(x, v, r) = \int_0^v \psi(x, z, r)dr. \tag{22}$$

Theorem 6. *Let $\psi_n \in L^p(\Omega, L^1_{loc}(R^2_+))$, $p \geq 1$, be a sequence of space dependent density functions satisfying the Assumption (P1) for a.e. $x \in \Omega$. Assume that ψ_n converge to ψ in $L^p(\Omega, L^1_{loc}(R^2_+))$. Let $\mathcal{P}_n, \mathcal{P}$ be the Preisach operators corresponding to ψ_n, ψ respectively. Let u_n be a sequence in $L^p(\Omega, C(I))$ such that $u_n(x,.) \in C(I)$ for a.e. $x \in \Omega$ and $\|u_n - u\|_{C(I)} \to 0$ as $n \to \infty$.*

Then $\mathcal{P}_n[u_n](.,t)$ converge to $\mathcal{P}[u](.,t)$ for every $t \in I$ in $L^p(\Omega)$.

Proof. We have for a.e. $x \in \Omega$ and every $t \in I$

$$\int_0^\infty \int_0^{\mathcal{E}_r[u_n](t)} \psi_n(x,z,r)dzdr - \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} \psi(x,z,r)dzdr = \tag{23}$$

$$= \int_0^\infty \int_0^{\mathcal{E}_r[u_n](t)} \psi_n(x,z,r)dzdr - \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} \psi_n(x,z,r)dzdr + \tag{24}$$

$$+ \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} [\psi_n(x,z,r) - \psi(x,z,r)]dzdr = \tag{25}$$

$$\int_0^\infty \int_{\mathcal{E}_r[u](t)}^{\mathcal{E}_r[u_n](t)} \psi_n(x,z,r)dzdr + \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} [\psi_n(x,z,r) - \psi(x,z,r)]dzdr. \tag{26}$$

The first integral on the right hand side of the last expresion can be estimated using the Assumption (P1) as follows

$$\int_0^\infty \int_{\mathcal{E}_r[u](t)}^{\mathcal{E}_r[u_n](t)} \psi_n(x,z,r)dzdr \leq \int_0^\infty \int_{\mathcal{E}_r[u](t)}^{\mathcal{E}_r[u_n](t)} \beta(r)dzdr = \tag{27}$$

$$= \int_0^\infty \beta(r)[\mathcal{E}_r[u_n](t) - \mathcal{E}_r[u](t)]dr \leq \int_0^R \beta(r)|\mathcal{E}_r[u_n](t) - \mathcal{E}_r[u](t)|dr, \tag{28}$$

for some $R > 0$, see Remark 1.

The later term can be futher estimated using the Lipschitz continuity of the play operator in $C([0,T])$ (Theorem 1 (ii)) as follows:

$$\int_0^R \beta(r)|\mathcal{E}_r[u_n](t) - \mathcal{E}_r[u](t)|dr \leq \|u_n - u\|_{C^0([0,T])} b(R), \tag{29}$$

where $b(R)$ is defined in (P1).

The estimates above imply that for every $t \in I$

$$\|\mathcal{P}_n[u_n](t) - \mathcal{P}[u](t)\|_{L^p(\Omega)} \leq b(R)\|u_n - u\|_{L^p(\Omega, C^0([0,T]))} + \tag{30}$$

$$+ \left\| \int_0^\infty \int_0^{\mathcal{E}_r[u](t)} [\psi_n(x,z,r) - \psi(x,z,r)]dzdr \right\|_{L^p(\Omega)}. \tag{31}$$

The first term on the right hand side of the last inequality converges by assumptions to 0. To estimate the second term, again by Remark 1 $\mathcal{E}_r[u](t) = 0$ for r sufficiently large, and if $u(x,.) \in C([0,T])$ for a.e. $x \in \Omega$ by Theorem 1 (ii)

$\mathcal{E}_r[u](x,.) \in C([0,T])$, so the integral over r is through a finite interval, and converges to zero because of the assumption on the convergence of ψ_n. The statement follows. □

Remark 2. It follows from the proof of the previous theorem that the convergence ψ_n to ψ in $L^p(\Omega, L^1_{loc}(R^2_+))$ can be easily replaced by the convergence in $L^\infty(\Omega \times R^2_+))$ weakly star, as is typically the case we get in homogenization arguments, getting the weak star convergence of the Preisach operators in $L^\infty(\Omega)$.

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