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A homogenization result for a parabolic equation with Preisach hysteresis

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In this paper we consider an initial boundary value parabolic problem

$$[cu + \mathcal{P}[u]]_t - div(a{\cdot}\nabla u) = f,$$

with Preisach hysteresis $\mathcal{P}$. The functions c, a, and the density function ψ of the Preisach operator are allowed to depend also on the space variable x. The equation is homogenized by considering a sequence of equations with spatially periodic data c^ϵ, a^ϵ, and ψ^ϵ, where the spatial period ϵ converges to 0. Properties of hysteresis operators and the concept of two-scale convergence are used to show the convergence of the corresponding solutions to the solution of the homogenized problem.

1 Introduction

Let $I = [0, T]$ be a time interval and $\Omega \in R^n$ a bounded domain with Lipschitz boundary. We study the parabolic equation

$$[c(x)u + \mathcal{P}[u]]_t - div(a(x){\cdot}\nabla u) = f, \tag{1}$$

where $\mathcal{P}$ represents a hysteresis operator of Preisach type, characterized by a density function ψ. The functions c, a, and ψ are spatially dependent.

A systematic mathematical investigation of Preisach operators, as an important example of hysteresis operators, started relatively recently, see e.g. [1,7,8] or [13]. The model itself was introduced much earlier [14].

Eq. (1) describes unsaturated flow of a compressible fluid through a porous medium with hysteresis effect taken into account, when gravitational effect is neglected. Referring to measurements carried out in [6], D. Flynn [3] recently proposed a Preisach formulation to model soil-moisture hysteresis for particular soils. A porous medium can be represented as a large system of micro-tubes, each of them behaving like a "nonlinear play" because of capilarity effects. The "play" representation of the Preisach operator makes it possible to interpret the resulting Preisach hysteresis as a homogenized limit of spatially distributed nonlinear plays. Since saturation takes place, the correct form of the term under time derivative ought to be $b(x, u) + \mathcal{P}[u]$ with b bounded. By the "hysteresis-parabolic" maximum principle derived in [5], we restrict ourselves to a subdomain, where $b(x, u)$ can be approximated by a linear term $c(x)u$.

We assume that the data c, a and ψ are ϵ-periodic and consider a sequence u_ϵ of solutions of (1) for each $\epsilon > 0$. The field of mathematics which treats the asymptotic behavior of the sequence of solutions u_ϵ is known as homogenization. It is used in modelling composite materials with periodic structure. If the space period is too small, t.m. the space microstructure is too fine, we want to reduce the computational complexity by replacing the quickly changing coefficients by constant ones, corresponding to the idealized homogeneous material, which on the macroscopic level has the same qualitative and quantitative properties. The theory started in the seventies, see e.g. [4]. The main result of this paper is a homogenization result for (1).

A parabolic homogenization problem with a space dependent Preisach hysteresis operator, defined as a weighted superposition of relays was considered in [13] (Sect. XI.7) in a setting corresponding to (1) with $c(x) = 1$ and $a(x) = 1$. The novelty with respect to [13] consists in an additional spatial homogenization of the coefficients, using the concept of two-scale convergence. The variational formulation of the Preisach model enables us to simplify considerably the proofs, which become straightforward and self-contained.

The paper is organized as follows. Sect. 1 is devoted to a survey of relevant results for Preisach hysteresis. For more details as well as exposure to other hysteresis operators see e.g. [1,7,8] or [13]. In Sect. 2 we briefly introduce the problem, give assumptions and state an existence and uniqueness result proved e.g. in [1]. In order to pass to the limit in the elliptic

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term we will use the concept of two-scale convergence, introduced by Nguetseng [11] and further extended by Allaire [2]. Basic definitions and properties are recalled in Sect. 3. Sect. 4 contains the main homogenization result for (1) and its proof.

2 Hysteresis operators

2.1 The Play operator

Let $r > 0$ be a parameter, $u(t)$ an absolutely continuous input function on the time interval I and $w_r^0 \in [-r, r]$ an initial state. We consider a variational inequality

$$\mathcal{G}(t) \in [-r, r], \quad t \in I \;, \tag{2}$$

$$\left(\dot{\mathcal{G}}(t) - \dot{u}(t)\right)(\phi - \mathcal{G}(t)) \geq 0 \text{ for a.e. } t \in I, \text{for all } \phi \in [-r, r] \;, \tag{3}$$

$$\mathcal{G}(0) = w_r^0, \tag{4}$$

for the unknown $\mathcal{G}(t)$. For an input $u \in W^{1,1}(I)$ this problem admits a unique solution $\mathcal{G}_r[u, w_r^0] \in W^{1,1}(I)$. The play operator $\mathcal{E}_r$ with threshold r is defined by the relation

$$\mathcal{E}_r[u, w_r^0] = u - \mathcal{G}_r[u, w_r^0], \tag{5}$$

see Fig. 1.

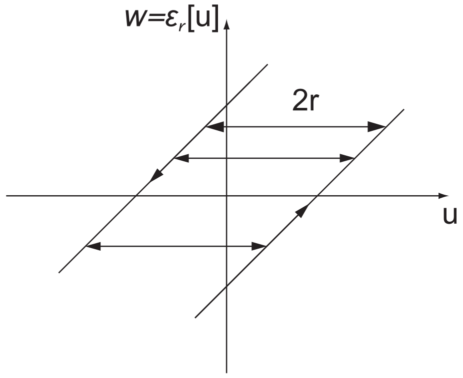

Fig. 1 The play operator

For piecewise monotone inputs, in each interval of monotonicity $[t_0, t_1]$ of the input function $u(t)$, the relation

$$\mathcal{E}_r[u, w_r^0](t) = \max\left\{u(t) - r, \min\{u(t) + r, \mathcal{E}_r[u, w_r^0](t_0)\right\} \tag{6}$$

follows from (1) and is often used for an alternative definition of the play operator, see e.g. [13]. In the following we use the "unperturbed" or "virgin" initial state defined by

$$w_r^0 = \min\left\{r, \max\{-r, u(0)\}\right\}. \tag{7}$$

and write only $\mathcal{E}_r[u]$.

We survey important properties of the play operator.

Theorem 1. *(i) The play operator $\mathcal{E}_r[u]$ is continuous in $W^{1,p}(0,T)$ for $p \in [1, \infty)$ and Lipschitz continuous for $p = 1$. (ii) The play operator admits a Lipschitz continuous extension to $C([0,T])$ and*

$$|\mathcal{E}_r[u_1](t) - \mathcal{E}_r[u_2](t)| \leq \max_{0 \leq s \leq t} |u_1(s) - u_2(s)|, \forall t \in I. \tag{8}$$

(iii) The play operator is piecewise monotone in the following sense:

$$\frac{d\mathcal{E}_r[u]}{dt} \cdot \frac{du}{dt} \geq 0 \textit{ a.e. in } I, \textit{ for all } u \textit{ absolutely continuous}. \tag{9}$$

2.2 The Preisach operator

The simplest example of a hysteresis nonlinearity is given by a switch or relay with hysteresis, $h_{v,r} : C([0,T]) \times \{-1,1\} \to BV(0,T)$ with input u (magnetic field) and output $h_{v,r}$(magnetization), see Fig. 2. The relay is characterized by two parameters $v \in R^1$ (interaction field) and $r > 0$ (critical field of coercivity) and is defined formally as follows: Let R^2_+ denote the set $\{(v,r) \in R^2; r > 0\}$. For given parameters $(v,r) \in R^2_+$, input $u \in C([0,T])$, initial magnetization $\theta \in \{-1,1\}$ and any time $t \in I$, put

$$X_t := \{\tau \in (0,T]; |u(\tau) - v| = r\}. \tag{10}$$

We then define

$$h_{v,r}(u,\theta)(0) = \begin{cases} -1 & \text{if } u(0) \le v + r \ , \\ \theta & \text{if } v - r < u(0) < v + r \ , \\ 1 & \text{if } u(0) \ge v + r \ , \end{cases} \tag{11}$$

and

$$h_{v,r}(u,\theta)(t) = \begin{cases} h_{v,r}(u,\theta)(0) & \text{if } X_t = \emptyset \ , \\ -1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = v - r \ , \\ 1 & \text{if } X_t \neq \emptyset \text{ and } u(\max X_t) = v + r, \end{cases} \tag{12}$$

see Fig. 2.

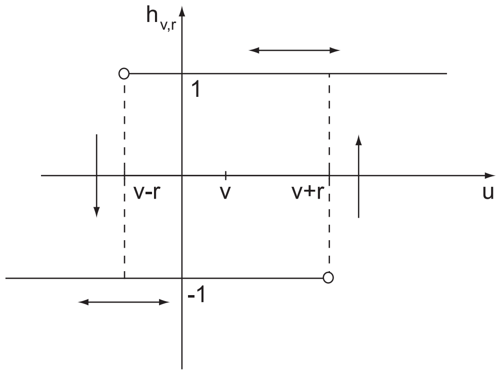

Fig. 2 The relay operator

It is often convenient to use the following representation of the relay by means of the system $\mathcal{E}_r, r > 0$ of play operators.

Lemma 2. *(see e.g. Lemma 3.6 in [8] for more general initial states). Let $u \in C([0,T])$ be given. For $(v,r) \in R^2_+$, put $\theta := -1$ if $v \ge 0$, $\theta = 1$ if $v < 0$. Then for every $t \in I$ and $(v,r) \in R^2_+$, $v \neq \mathcal{E}_r[u](t)$ we have*

$$h_{v,r}(u,\theta)(t) = \begin{cases} -1 & \textit{if } v > \mathcal{E}_r[u](t) \\ 1 & \textit{if } v < \mathcal{E}_r(u(t)). \end{cases} \tag{13}$$

The output of the Preisach model is formally defined as an average over all elementary switches with a given density function $\psi \in L^1_{loc}(R^2_+)$ by the formula (see e.g. Krasnosel'skii and Pokrovskii [7])

$$\mathcal{P}(t) := \int_0^\infty \int_{-\infty}^\infty \psi(v,r) h_{v,r}(u,\theta)(t) dv dr, \tag{14}$$

where the initial values of the relays are taken as -1 if $v > 0$ and $+1$ otherwise. To justify the integration in (14) we need to assume that the antisymmetric part $\psi_a(v,r) := \frac{1}{2}(\psi(v,r) - \psi(-v,r))$ of ψ satisfies $\psi_a \in L^1(R^2_+)$ and we consider the integral in the sense of principal value. Using Lemma 2 on the representation of the relay by a system of plays the output of the Preisach operator can be expressed as

$$\mathcal{P}(t) = C + \int_0^\infty g(\mathcal{E}_r[u](t), r) dr, \tag{15}$$

where

$$g(v,r) = \int_0^v \psi(z,r)dz, \tag{16}$$

C is a constant and $\mathcal{E}_r[u](t)$ denotes the play operator.

In the sequel we will use the following assumptions :

(P1) There exists $\beta \in L^1_{loc}(0,\infty)$, $\beta \geq 0$ a.e. such that

$$0 \leq \psi(z,r) \leq \beta(r) \text{ for a.e. } (z,r) \in R^2_+. \tag{17}$$

For $R > 0$ put $b(R) := \int_0^R \beta(r)dr$.

(P2)

$$\frac{\partial \psi}{\partial z} \in L^\infty_{loc}(R^2_+). \tag{18}$$

The next theorem shows conditions under which the Preisach operator is Lipschitz continuous on $C^0([0,T])$. Proof of the extended version of this theorem (for more general initial inputs) can be found in [8].

Theorem 3. *Let the Assumption (P1) be satisfied and let $R > 0$ be given. Then for every $u, v \in C([0,T])$ such that $\|u\|_{C^0([0,T])}, \|v\|_{C^0([0,T])} \leq R$ the Preisach operator (15) maps $C^0([0,T]) \to C^0([0,T])$ and satisfies*

$$\|\mathcal{P}[u] - \mathcal{P}[v]\|_{C^0([0,T])} \leq b_1(R)\|u - v\|_{C^0([0,T])}. \tag{19}$$

Lemma 4. *Let the Assumptions (P1) and (P2) be satisfied. Then for $u \in W^{1,1}(0,T)$, $r > 0$ and $t \in I$, we have $\mathcal{P} \in W^{1,1}(0,T)$ and for a.e. $t \in I$*

$$\dot{\mathcal{P}}[u](t) = \int_0^\infty \dot{\mathcal{E}}_r[u](t)\psi(\mathcal{E}_r[u](t), r)dr. \tag{20}$$

It follows from the previous lemma and from the definition of the play operator that the Preisach operator is piecewise monotone. We have

Theorem 5. *The Preisach operator is under the Assumptions (P1) and (P2) piecewise monotone, this means that for $u \in W^{1,1}(0,T)$*

$$\dot{\mathcal{P}}[u](t)\dot{u}(t) \geq 0 \text{ a.e.} \tag{21}$$

2.3 Spatially dependent Preisach operators

We will consider the spatially dependent constitutive relation described by the Preisach operator with a spatially dependent density function $\psi(x,z,r) \in L^1_{loc}(\Omega \times R^2_+)$.

Theorem 6. *Let ψ_n be a sequence of space dependent density functions in $L^\infty(\Omega \times R^2_+)$, satisfying the Assumption (P1) for a.e. $x \in \Omega$. Assume that ψ_n converge to ψ in $L^\infty(\Omega \times R^2_+)$ weakly star. Let $\mathcal{P}_n, \mathcal{P}$ be the Preisach operators corresponding to ψ_n, ψ respectively. Let u_n be a sequence in $L^2(\Omega, C(I))$ and $\|u_n - u\|_{L^2(\Omega, C(I))} \to 0$ as $n \to \infty$.*

Then $\mathcal{P}_n[u_n](.,t)$ converge to $\mathcal{P}[u](.,t)$ for every $t \in I$ in $L^\infty(\Omega)$ weakly star.

Proof of this Theorem can be found in [9].

3 The heat equation with hysteresis

3.1 Statement of the problem

We set $Q := \Omega \times [0,T]$. We will consider the diffusion equation with hysteresis in the form

$$[c(x)u + \mathcal{P}[u]]_t - div(a(x)\cdot\nabla u) = f \text{ in } Q, \tag{22}$$

with homogeneous Dirichlet boundary conditions

$$u(x,t) = 0 \text{ on } \partial\Omega \times I \tag{23}$$

and with an initial condition

$$u(x,0) = u_0(x) \text{ for } x \in \Omega. \tag{24}$$

Here $\mathcal{P}$ denotes a Preisach operator which we had defined and studied in the previous section.

The data are assumed to satisfy the following requirements:

(A1) $c \in L^\infty(\Omega)$ and there exist constants $c_m, c_M > 0$ such that

$$0 < c_m \le c(x) \le c_M \text{ for a.e. } x \in \Omega. \tag{25}$$

(A2) The operator

$$Au = -div(a \cdot \nabla u) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) \tag{26}$$

is elliptic, i.e.

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2 \text{ for all } \xi \in R^n \text{ and for a.e. } x \in \Omega, \quad \text{and} \quad a_{ij} \in L^\infty(\Omega)\ . \tag{27}$$

(A3)$f \in L^2(I, L^2(\Omega))$.

(A4) $u_0 \in H_0^1(\Omega)$.

(A5) $\mathcal{P}[u]$ is a Preisach operator with density function $\psi(x,z,r)$ which satisfies the assumption (P1) for a.e. $x \in \Omega$.

3.2 Existence result for the parabolic equation with Preisach hysteresis

Theorem 7. *Let the assumptions (A1)-(A5) hold. Then there exists a unique weak solution of the system (22)-(24) satisfying* $u \in Z := L^\infty(0,T;H_0^1(\Omega)) \bigcap H^1(0,T;L^2(\Omega))$, $w \in L^2(\Omega; C[0,T])$, *(24) holds for a.e.* $x \in \Omega$ *and such that*

$$\int_0^T \int_\Omega [c(x)u(x,t) + \mathcal{P}[u]]\phi(x,t)_t dxdt + \int_0^T \int_\Omega a(x)\nabla u(x,t)\cdot\nabla\phi(x,t)dxdt = \int_0^T \int_\Omega f(x,t)\phi(x,t)dxdt \tag{28}$$

holds for all $\phi \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$. *The solution* u *satisfies the estimate*

$$\max_{t\in(0,T)} \int_\Omega (\|\nabla u\|^2 + \|u\|^2)dx + \int_0^T \int_\Omega \|u_t\|^2 dxdt \le c, \tag{29}$$

where the constant c *depends on the domain* Ω*, the norm of* f *in* $L^2(I, L^2(\Omega))$ *and* u_0 *in* $H_0^1(\Omega)$ *and also on the constants* c_m, c_M, *and* α.

Detailed proof of the existence part of this Theorem can be found in [13] or alternatively in [1], where also uniqueness result for the Preisach operator is given.

4 Two-scale convergence

Let $Y = [0,1]^n$ be the closed unit cube. We use the subscript # to denote the space of Y-periodic functions. In particular, $H^1_\#(Y)$ is the space of all functions in $H^1_{\text{loc}}(R^n)$ which are $Y-$periodic. For more details see e.g. [12].

Let us consider a sequence of functions $u_\epsilon(x)$ in $L^p(\Omega)$ (by ϵ we denote for simplicity arbitrary, but prescribed sequence of positive numbers converging to 0). The classical definition of two-scale convergence was first introduced by Nguetseng [11] and then expanded by Allaire [2].

Definition 1. A sequence of functions u_ϵ in $L^p(\Omega)$ is said to two-scale converge to a limit $u_0(x,y)$ belonging to $L^p(\Omega \times Y)$ if for any function $\psi(x,y)$ in $C_0^\infty[\Omega; C_\#^\infty(Y)]$ (a space of compactly supported infinitely differentiable functions with Y-periodic values in $C^\infty(R^n)$) we have

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon(x)\psi\left(x, \frac{x}{\epsilon}\right)dx = \int_\Omega \int_Y u_0(x,y)\psi(x,y)dydx. \tag{30}$$

Remark 1. Notice that although u_ϵ is a sequence of functions of n variables, the two-scale limit is a function of $2n$ variables. It enables to describe the periodic behavior of u_ϵ better.

Two-scale strong convergence is defined analogously:

Definition 2. We say that a sequence $u_\epsilon \subset L^p(\Omega)$ converges two-scale strongly to u_0 if u_ϵ two-scale converges to this limit and moreover $\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^p(\Omega)} = \|u_0\|_{\Omega\times Y}$.

The two-scale convergence as the usual weak convergence makes bounded sets in $L^2(\Omega)$ relatively sequentially compact as stated in the next Theorem.

Theorem 8. *From each bounded sequence u_ϵ in $L^2(\Omega)$ we can extract a subsequence two-scale converging to a limit function $u_0(x,y) \in L^2(\Omega \times Y)$. Moreover, u_ϵ converges weakly in $L^2(\Omega)$ to $u(x) = \int_Y u_0(x,y)dy$.*

The last result deals with the case when we have additional bounds on sequences of derivatives.

Theorem 9. *Let u_ϵ be a bounded sequence in $H^1(\Omega)$ that converges weakly to a limit $u(x)$ in $H^1(\Omega)$. Then u_ϵ two-scale converges to $u(x)$, and there exists a function $u_1(x,y)$ in $L^2[\Omega; H^1_\#(Y)]$ such that, up to a subsequence, ∇u_ϵ two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$.*

For proofs of the above theorems as well as for a very nice overview of results about two-scale convergence we refer to [10].

5 Homogenization

Let ϵ be a sequence of positive numbers which tend to zero. For each ϵ we consider the so called periodic problem - the problem of the form (22)-(24) with ϵ-periodic data in the constitutive relations, namely

$$(c^\epsilon(x)u^\epsilon + \mathcal{P}^\epsilon[u^\epsilon])_t - div(a^\epsilon(x)\cdot\nabla u^\epsilon) = f^\epsilon \qquad \text{in } Q, \tag{31}$$

$$u^\epsilon(x,t) = 0 \qquad \text{on } \partial\Omega \times I\ , \tag{32}$$

$$u^\epsilon(x,0) = u_0(x) \text{ for } x \in \Omega. \tag{33}$$

The coefficients and operators denoted by superscript ϵ have the special form

$$c^\epsilon(x) = c\left(\frac{x}{\epsilon}\right), \tag{34}$$

$$a^\epsilon_{ij}(x) = a_{ij}\left(\frac{x}{\epsilon}\right), \tag{35}$$

$$\mathcal{P}^\epsilon[u](x,t) = \int_0^\infty \int_0^{\mathcal{E}_r[u](x,t)} \psi_\epsilon(x,z,r)dzdr\ , \tag{36}$$

$$\text{and } \psi_\epsilon(x,z,r) = \psi\left(\frac{x}{\epsilon}, z, r\right). \tag{37}$$

We will need the following assumptions:
$(A^\epsilon 1)$ $c \in L^\infty_\#(Y)$ and there exist a constants $c_m, c_M > 0$ such that

$$0 < c_m \le c(y) \le c_M \text{ for a.e. } y \in Y. \tag{38}$$

$(A^\epsilon 2)$ $a_{ij} \in L^\infty_\#(Y)$,

$$\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \ge \alpha|\xi|^2 \text{ holds for all } \xi \in R^n \text{ and for a.e. } y \in Y. \tag{39}$$

$(A^\epsilon 3)$ $f^\epsilon \in L^2(I, L^2(\Omega))$ and f^ϵ converge to $f^\star$ weakly in $L^2(I, L^2(\Omega))$.
$(A^\epsilon 4)$ $u_0 \in H^1_0(\Omega)$.
$(A^\epsilon 5)$ $\mathcal{P}[u]$ is a Preisach operator with density function $\psi(y,z,r)$ satisfying the assumption (P1) for a.e. $y \in Y$, $\psi(y,z,r)$ is Y-periodic in y.

It follows from Theorem 7 that under the above assumptions the problem (31)-(33) has a unique solution u^ϵ for each $\epsilon > 0$.

Notice that since the data are periodic, they will oscillate very rapidly for small values of ϵ.

We will show that as ϵ tends to zero, the solutions u^ϵ of the periodic problem (31)-(33) converge to a function $u^\star$ - the solution of the so-called homogenized problem, which not any longer involves rapidly oscillating functions.

The homogenized problem consists of the equation

$$(c^\star u^\star + \mathcal{P}^\star[u^\star])_t - div(a^\star \cdot \nabla u^\star) = f^\star \qquad \text{in } Q, \tag{40}$$

$$u^\star(x,t) = 0 \qquad \text{on } \partial\Omega \times I\ , \tag{41}$$

$$u^\star(x,0) = u_0(x) \text{ for } x \in \Omega. \tag{42}$$

where the homogenized coefficient $c^\star$ is the weak limit of c^ϵ defined by

$$c^\star = \int_Y c(y)dy, \tag{43}$$

the limit hysteresis operator

$$\mathcal{P}^\star[u](x,t) = \int_0^\infty \int_0^{\mathcal{E}_r[u](x,.)} \psi^\star(z,r)dzdr \tag{44}$$

is determined by the function $\psi^\star(z,r)$ being also the weak limit of $\psi_\epsilon(y,z,r)$ i.e.

$$\psi_\star(z,r) = \int_Y \psi(y,z,r)dy. \tag{45}$$

The matrix of coefficients is defined by

$$a^\star = \int_Y (a + a \cdot \nabla_y \chi)dy, \text{ i.e. } a^\star_{ij} = \int_Y \left(a_{ij} + \sum_{k=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_k} \right) dy, \tag{46}$$

where the auxiliary functions χ_k are the unique weak solutions of the following elliptic, so-called cell problem, see e.g. [4]:

Find $\chi = (\chi_1, ..., \chi_n) \in H^1_\#(Y)^n$ such that

$$-div_y(a \cdot \nabla_y \chi + a) = 0, \text{ i.e. } -\sum_{i=1}^n \frac{\partial}{\partial y_i} \left(\sum_{k=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_k} + a_{ij} \right) = 0, \tag{47}$$

$$\int_Y \chi_j dy = 0. \tag{48}$$

Note that, in general, the matrix of coefficients $a^\star$ differs from the weak limit of a^ϵ.

The main homogenization result of the paper reads as follows:

Theorem 10. *Let the Assumptions* $(A^\epsilon 1)$-$(A^\epsilon 5)$ *be satisfied. Then as* $\epsilon \to 0$*, the solutions* u^ϵ *of the periodic problem* (31)*–*(33) *converge weakly star in* $L^\infty(0,T;H^1_0(\Omega)) \bigcap H^1(0,T;L^2(\Omega))$ *to the solution* $u^\star$ *of the homogenized problem* (40)*–*(42) *defined by* (43)*–*(46)*.*

Proof. As mentioned earlier, it follows from Theorem 7 and the Assumptions $(A^\epsilon 1)$-$(A^\epsilon 5)$ that the periodic problem for each $\epsilon > 0$ and also the homogenized problem admit unique solutions. It follows from classical arguments, see e.g. [4] that since the sequence c^ϵ is bounded in $L^\infty(\Omega)$, $c^\epsilon \to c^\star$ weakly star in $L^\infty(\Omega)$ and similarly $\psi^\epsilon \to \psi^\star$ weakly star in $L^\infty(\Omega \times R^2_+)$.

The estimate (29) together with the fact that the sequence f^ϵ weakly converges and therefore is bounded in $L^2(I, L^2)$, implies that the sequence of solutions u^ϵ is bounded in Z. This implies that there exists $u^* \in Z$ and $u_1 \in L^\infty(0,T;H^1_0(\Omega))$ such that possibly after selecting a subsequence

$u^\epsilon \to u^*$ weakly star in Z,

$\nabla u^\epsilon(x,t) \to \nabla u^*(x,t) + \nabla_y u_1(x,y,t)$ two-scale in $L^\infty(0,T;L^2(\Omega))^n$, in L^∞ the convergence is weak-star only.

The compact imbedding $Z \hookrightarrow L^2(\Omega; C([0,T]))$ yields that $u^\epsilon \to u^*$ strongly in $L^2(\Omega; C([0,T]))$ and $\mathcal{P}^\epsilon[u^\epsilon] \to \mathcal{P}[u^*]$ for every $t \in (0,T)$ in $L^\infty(\Omega)$ by Theorem 6.

We have to prove that u^* solves the problem (40). We choose in (31) a sequence of test functions $v^\epsilon(x,t) = \psi_\epsilon(x)\phi(t) = [v_0(x) + \epsilon v_1(x, \frac{x}{\epsilon})]\phi(t)$, where $v_0 \in H_0^1(\Omega)$, $v_1 \in C_0^\infty(\bar\Omega; H^1_\#(Y))$ and $\phi \in C_0^\infty(0,T)$. Clearly v^ϵ converges strongly to $v_0(x)$ and its gradient ∇v^ϵ strongly two-scale converges to $\nabla v_0 + \nabla_y v_1(x,y)$. We obtain

$$\int_0^T\!\!\int_\Omega - \left(c^\epsilon(x)u^\epsilon + \mathcal{P}^\epsilon[u^\epsilon]\right)\psi_\epsilon\phi_t$$
$$+\, a^\epsilon(x)\cdot\nabla u^\epsilon\left[\nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\epsilon}\right) + \epsilon\nabla_x v_1\left(x, \frac{x}{\epsilon}\right)\right]dxdt$$
$$= \int_0^T\int_\Omega f^\epsilon v^\epsilon dxdt \tag{49}$$

We can now pass to the limit as $\epsilon \to 0$ since each term contains a product of at most one weakly or two-scale converging sequence and obtain, using similar arguments as in the proof of Theorem 14 in [10]:

$$\int_0^T\int_\Omega\left\{-\left(c^\star u^* + \mathcal{P}^\star[u^*]\right)v_0\phi_t + \int_Y a(y)\cdot(\nabla u^* + \nabla_y u_1)[\nabla v_0(x) + \nabla_y v_1]dy\right\}dxdt$$
$$= \int_0^T\int_\Omega f^\star v_0 dxdt. \tag{50}$$

Choosing different v_0 and $\phi(t)$ with $v_1 = 0$ we get

$$\left(c^\star u^* + \mathcal{P}^\star[u^*]\right)_t - \mathrm{div}_x\left[\int_Y a(y)(\nabla u^*(x) + \nabla_y u_1(x,y))dy\right] = f^\star \tag{51}$$

and choosing different v_1 and $\phi(t)$ with $v_0 = 0$

$$-div_y[a(y)(\nabla u^*(x) + \nabla_y u_1(x,y))] = 0. \tag{52}$$

Comparing the last equation with (47), we conclude that $u_1(x,y)$ must have the form

$$u_1(x,y) = \chi(y)\nabla u^*(x), \tag{53}$$

and using (53) in (51) we get the limit equation

$$\left(c^\star u^* + \mathcal{P}^\star[u^*]\right)_t - div_x(a^\star\cdot\nabla u^*) = f^\star. \tag{54}$$

It follows that u^* is the solution of the homogenized problem (40), and since such a solution is unique, the whole sequence u^ϵ converges. □

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