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NONLINEAR SEMIGROUP METHODS IN PROBLEMS WITH HYSTERESIS

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Abstract. Results from a nonlinear semigroup theory are applied to get existence and uniqueness for PDEs with hysteresis. The hysteresis nonlinearity considered is of the generalized play operator type, but can be easily extended to a generalized Prandtl-Ishlinskii operator of play type, both possibly discontinuous.

1. **Introduction.** We show in this paper how nonlinear semigroup theory can be used to solve some partial differential equations with hysteresis nonlinearities.

Hysteresis is a nonlinear phenomena, its mathematical studies started about 35 years ago by Russian scientists, see [9]. Hysteresis arise in plasticity, friction, ferromagnetism, ferroelectricity, superconductivity, adsorption and desorption, examples can also be found in biology, chemistry and economics and its coupling to PDEs is natural from a physical point of view.

We will introduce in Section 2 the generalized play operator as defined by A.Visintin, classical play operator being a special example. A generalized Prandtl-Ishlinskii operator is also defined in the sense of A.Visintin. We list some of the basic properties, emphasizing especially the so-called Hilpert inequality, which is the basic property needed for application of the nonlinear semigroup theory. For an introduction to hysteresis operators and more details on its properties and applications, see also [2], [10], [17].

Nonlinear semigroup theory started in 1967 by Komura, when he announced the theory of generation of nonlinear semigroup in a Hilbert space, was extended to Banach space by Crandall and Liggett and it is a widely used tool for solving nonlinear PDEs. Section 3 is devoted to a survey of basic relevant results from a nonlinear semigroup theory, formulated generally in a Banach space. However, some results which hold only in a Hilbert space are mentioned as well.

We illustrate the use of nonlinear semigroup theory for PDEs with hysteresis on two examples. In both cases we couple the PDE with a generalized play operator, although generalization to the generalized Prandtl-Ishlinskii operator of play type is straightforward. Let us note that either generalized play operator or a generalized Prandtl-Ishlinskii operator of play type defined by A.Visintin include discontinuous

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hysteresis operators of a relay type or Preisach type resp. and the combination of such operators with a PDE is a very nontrivial problem.

In section 4 we reformulate the PDEs as Cauchy problems in L^1 and prove the existence and uniqueness of solutions by applying the results of nonlinear semigroups theory. The use of nonlinear semigroup theory in L^1 in problems with hysteresis was introduced by A. Visintin, [16], [17] and is motivated by results of Hilpert, [4]. Asymptotic results for Problem (1) and a Problem (2) (in one space dimension), were obtained from the nonlinear semigroup theory in [7] and [8] resp.

R.Showalter [12] used a slightly different approach, he also applied the results of a nonlinear semigroup theory in a Hilbert space getting much stronger results, but this approach is limited to the coupling of the PDE with a classical play operator (or Prandtl-Ishlinskii operators defined as superpositions of such classical play operators) whose simple structure allows us to get m-accretivity of the corresponding operator in L^2. We present a different formulation at the end of the last section. We show that this formulation allows us to use results from nonlinear semigroup theory in Hilbert spaces, to get a convergence of the solution of the corresponding Cauchy problem as $t \to \infty$. This improves the results from [8] for the classical play operator.

2. **Hysteresis operators.**

2.1. **Generalized play operator.** Let $\gamma_l, \gamma_r : \mathbb{R} \to \mathbb{R}$ be maximal monotone (possibly multivalued) functions with

$$\inf \gamma_r(u) \leq \sup \gamma_l(u) \qquad \forall u \in \mathbb{R}. \tag{1}$$

Now, given $w^0 \in \mathbb{R}$, we construct the hysteresis operator $\mathcal{E}(\cdot, w^0)$ as follows. Let u be any continuous, piecewise linear function on $\mathbb{R}^+$ such that u is linear on $[t_{i-1}, t_i]$ for $i = 1, 2, \ldots$. We then define $w := \mathcal{E}(u, w^0) : \mathbb{R}^+ \to \mathbb{R}$ by

$$w(t) : \begin{cases} \min\{\gamma_l(u(0)), \max\{\gamma_r(u(0)), w^0\}\} & \text{if } t = 0, \\ \min\{\gamma_l(u(t)), \max\{\gamma_r(u(t)), w(t_{i-1})\}\} & \text{if } t \in (t_{i-1}, t_i], i = 1, 2, \ldots \end{cases}$$

Note that $w(0) = w^0$ only if $\gamma_r(u(0)) \leq w^0 \leq \gamma_l(u(0))$. This operator is called a generalized play, see Figure 1.

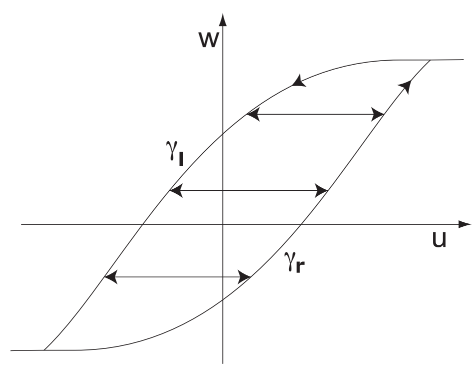

FIGURE 1.

If γ_l, γ_r are moreover continuous, it was proved in Visintin [17], that for any continuous piecewise linear functions u_1, u_2 on $\mathbb{R}^+$, with the notation $\epsilon_i : \mathcal{E}(u_i, w_i^0)$, $i = 1, 2$, we have the following inequality:

$$\max_{[t_1,t_2]} |\epsilon_1 - \epsilon_2| \leq \max\left\{ |\epsilon_1(t_1) - \epsilon_2(t_2)|, m_M\left(\max_{[t_1,t_2]} |u_1 - u_2|\right)\right\} \qquad \forall [t_1, t_2] \subset [0, T], T \in \mathbb{R}^+, \tag{2}$$

where for any continuous function $f : \mathbb{R} \to \mathbb{R}$ and any constant $M > 0$, $|f|_M(h)$ denotes its local modulus of continuity:

$$|f|_M(h) := \sup\{|f(y_1) - f(y_2)| : y_1, y_2 \in [-M, M], |y_1 - y_2| \leq h\} \qquad \forall h > 0, \tag{3}$$

$$m_M(h) := \max\{|\gamma_l|_M(h), |\gamma_r|_M(h)\} \qquad \forall h, M > 0, \tag{4}$$

and

$$M := \max\{|u_i(t)| : t \in [0, T], i = 1, 2\}. \tag{5}$$

Hence $\mathcal{E}(\cdot, w^0)$ has a unique continuous extension, denoted by $\mathcal{E}(\cdot, w^0)$ again, to an operator

$$\mathcal{E} : C(\mathbb{R}^+) \times \mathbb{R} \to C(\mathbb{R}^+). \tag{6}$$

The inequality (2) holds also for this extended operator, which is then uniformly continuous on bounded sets. If γ_l, γ_r are Lipschitz continuous, then $\mathcal{E}$ is also Lipschitz continuous and operates and is bounded from $W^{1,p}(0, T)$ to $W^{1,p}(0, T)$, for any $p \in [+, \infty]$.

The generalized play operator can be also equivalently defined as a solution $w \in W^{1,1}(0, T)$ of a variational inclusion of the following type

$$\frac{dw}{dt} \in \phi(u, w) \text{ a.e. in } (0, T) \tag{7}$$

$$w(0) = w^0, \tag{8}$$

where

$$\phi(u, w) = \begin{cases} \{+\infty\} & \text{if} \quad w < \inf \gamma_r(u) \\ \widetilde{\mathbb{R}}^+ & \text{if} \quad w \in \gamma_r(u) \setminus \gamma_l(u) \\ \{0\} & \text{if} \quad \sup \gamma_r(u) < w < \inf \gamma_l(u) \\ \widetilde{\mathbb{R}}^- & \text{if} \quad w \in \gamma_l(u) \setminus \gamma_r(u) \\ \{-\infty\} & \text{if} \quad w > \sup \gamma_l(u) \\ \widetilde{\mathbb{R}} & \text{if} \quad w \in \gamma_l(u) \cap \gamma_r(u) \end{cases} \tag{9}$$

Here $\widetilde{\mathbb{R}} := [-\infty, +\infty]$, $\widetilde{\mathbb{R}}^+ := [0, +\infty]$, $\widetilde{\mathbb{R}}^- := [-\infty, 0]$. The generalized play operator satisfies the Hilpert inequality [4]:

Theorem 2.1. *Let $(u_i, w_i^0) \in W^{1,1}(0, T) \times R$ $(i = 1, 2)$ and $h : [0, T] \to R$ be a measurable function such that $h \in H(u_1 - u_2)$ a.e. in $(0, T)$, H denotes the Heaviside graph. Then*

$$\frac{d}{dt}\left[\mathcal{E}(u_1, w_1^0) - \mathcal{E}(u_2, w_2^0)\right] h \geq \frac{d}{dt}\left\{\left[\mathcal{E}(u_1, w_1^0) - \mathcal{E}(u_2, w_2^0)\right]^+\right\} \text{ a.e. in } (0, T). \tag{10}$$

Proof. For the simplicity of the proof set $\epsilon_i = \mathcal{E}(u_i, w_i^0), \tilde{\epsilon} = \epsilon_1 - \epsilon_2$. We have

$$\frac{d}{dt}(\tilde{\epsilon}) = \frac{d\tilde{\epsilon}}{dt} k \text{ a.e. in } (0, T), \tag{11}$$

where k is any measurable function such that $k(t) \in H(\tilde{\epsilon}(t))$ in $(0,T)$. So it suffices to show

$$\frac{d\tilde{\epsilon}}{dt}(h-k) \geq 0, \text{ a.e. in } (0,T). \tag{12}$$

Let us consider the different possibilities which can occur at a generic $t \in (0,T)$.

1. If $\epsilon_1 = \epsilon_2$ we can take $k = h$.
2. If either $\epsilon_1 > \epsilon_2$ and $u_1 > u_2$ or $\epsilon_1 < \epsilon_2$ and $u_1 < u_2$ then $k = h$.
3. $\epsilon_1 > \epsilon_2$ and $u_1 \leq u_2$ or $\epsilon_1 < \epsilon_2$ and $u_1 \geq u_2$. Let us consider e.g. the first case, the other one is analogous. It follows from the properties of the play operator that $(u_1, \epsilon_1) \notin$ graph (γ_r), this means that, by construction of the play operator, $\frac{d\epsilon_1}{dt} \leq 0$ and $(u_2, \epsilon_2) \notin$ graph (γ_l) so $\frac{d\epsilon_2}{dt} \geq 0$.

Therefore $\frac{d\tilde{\epsilon}}{dt} \leq 0$. Moreover as $\tilde{\epsilon} > 0$, $k = 1 \geq h$. The result follows. □

2.2. **Classical play operator.** Special example of the generalized play operator as defined in [17] is the classical play operator. It can be obtained from the general definition as above with the choice of hysteresis boundary curves as

$$\gamma_l(u) = u + r, \gamma_r(u) = u - r, r \geq 0 \text{ is a parameter }, \tag{13}$$

or can be defined equivalently as follows.

Let $u(t)$ be a continuous input function on the time interval $I = [0,T]$ and $w_r^0 \in [-r, r]$ an initial state. We consider a variational inequality

$$\mathcal{G}(t) \in [-r, r], t \in I \tag{14}$$

$$\left(\dot{\mathcal{G}}(t) - \dot{u}(t)\right)(\phi - \mathcal{G}(t)) \geq 0 \text{ for a.e. } t \in I, \text{for all } \phi \in [-r, r] \tag{15}$$

$$\mathcal{G}(0) = w_r^0 \tag{16}$$

for the unknown $\mathcal{G}(t)$. For an input $u \in W^{1,1}(I)$ this problem admits a unique solution $\mathcal{G}_r[u, w_r^0] \in W^{1,1}(I)$. The play operator $\mathcal{E}_r$ with threshold r is defined by the relation

$$\mathcal{E}_r[u, w_r^0] = u - \mathcal{G}_r[u, w_r^0], \tag{17}$$

see Figure 2.

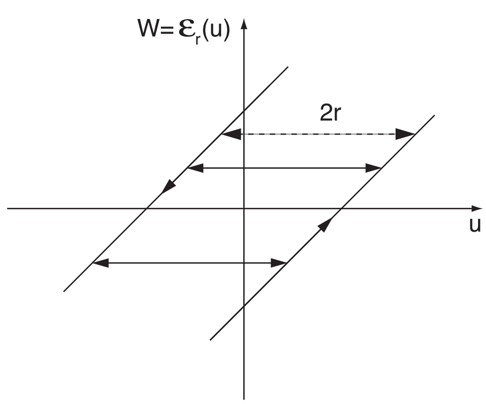

FIGURE 2.

2.3. **Generalized Prandtl-Ishlinskii operator of play-type.** To define the generalized Prandtl-Ishlinskii operator of play-type, let us assume that we are given a measure space $(\mathcal{P}, \mathcal{A}, \mu)$, where μ is a finite Borel measure. For μ-almost any $\rho \in \mathcal{P}$, let $(\gamma_{\rho l},\ \gamma_{\rho r})$ be a pair of functions $\mathbb{R} \to \mathbb{R}$, satisfying (1), and for each $\rho \in \mathcal{P}$ let $w^0_\rho \in \mathbb{R}$, be a given initial value. Let $\mathcal{E}_\rho(\cdot, w^0_\rho)$ be the corresponding generalized play operator corresponding to the couple $(\gamma_{\rho l}, \gamma_{\rho r})$. Then the operator defined as

$$\tilde{\mathcal{E}}_\mu\left(\tilde{u}, \{w^0_\rho\}_{\rho\in\mathcal{P}}\right) = \int_{\mathcal{P}} \mathcal{E}_\rho\left(\tilde{u}, w^0_\rho\right) d\mu(\rho)$$

is a generalized Prandtl-Ishlinskii operator of play-type. Intuitively, this operator is a weighted superposition of generalized plays with boundary curves $\gamma_{\rho l},\ \gamma_{\rho r}$.

Let us denote by $M(\mathcal{P})$ the set of measurable functions $\mathcal{P} \to \mathbb{R}$. If the families of curves $\gamma_{\rho l},\ \gamma_{\rho r}$ for any ρ in $\mathcal{P}$ are equicontinuous, then by the estimate (2), also $\tilde{\mathcal{E}}_\mu$ is strongly continuous from $C(\mathbb{R}^+) \times M(\mathcal{P})$ to $C(\mathbb{R}^+)$.

The generalized Prandtl-Ishlinskii operator satisfies also the Hilpert inequality:

Theorem 2.2. *Let $(u_i, \{w^0_{i\rho}\}_{\rho\in\mathcal{P}}) \in W^{1,1}(0,T) \times M(\mathcal{P})$ $(i = 1, 2)$ and $h : [0,T] \to R$ be a measurable function such that $h \in H(u_1 - u_2)$ a.e. in $(0,T)$, H denotes the Heaviside graph. Then*

$$\frac{d}{dt}\left[\tilde{\mathcal{E}}\left(u_1, \{w^0_{1\rho}\}_{\rho\in\mathcal{P}}\right) - \tilde{\mathcal{E}}\left(u_2, \{w^0_{2\rho}\}_{\rho\in\mathcal{P}}\right)\right] h \geq$$
$$\frac{d}{dt}\left\{[\tilde{\mathcal{E}}\left(u_1, \{w^0_{1\rho}\}_{\rho\in\mathcal{P}}\right) - \tilde{\mathcal{E}}\left(u_2, \{w^0_{2\rho}\}_{\rho\in\mathcal{P}}\right)]^+\right\} \text{ a.e. in } (0,T).$$

The proof is a straightforward consequence of Theorem 2.1 and the definition of generalized Prandtl-Ishlinskii operator.

The hysteresis relation will be assumed to hold pointwise in space:

$$w(x,t) = [\mathcal{E}(u(x,.), w^0(x))](t) \qquad \text{in } [0,T], \text{ a.e. in } \Omega. \tag{18}$$

3. **Nonlinear semigroup theory.** In this section we recall some standard results of the nonlinear semigroup theory. For more details and for the proofs of the results of this section, see e.g. citeBarbu.

Definition 3.1. *Let B be a (real) Banach space, A (possibly nonlinear and multivalued) operator $A : D(A) \subset B \to B$ is accretive if*

$$\forall u_i \in D(A), \forall v_i \in A(u_i)(i = 1, 2), \forall\lambda > 0,$$
$$||u_1 - u_2||_B \leq ||u_1 - u_2 + \lambda(v_1 - v_2)||_B.$$

It is equivalent to requiring that $(I + \lambda A)^{-1}$ is a contraction on $Rg(I + \lambda A)$, $\forall\lambda > 0$.

Definition 3.2. *If in addition $Rg(I + \lambda A) = B$ for some $\lambda > 0$, then A is called m-accretive.*

For A m-accretive let us consider the approximate problem

$$\frac{u^h_k - u^h_{k-1}}{h} + A(u^h_k) \ni f^h_k, k = 1, 2, ..., \tag{19}$$
$$u^h_0 = u^0$$

(the derivative in the evolution equation is approximated by a backward-difference quotient of step size $h > 0$ and f by a step function f^h_k).

Let us define the step function

$$u_k^h(t) = u_k^h \text{ for } kh \le t < (k+1)h. \tag{20}$$

The m-accretivity of A implies that the scheme (19) is uniquely solved recursively and the famous Crandall-Ligget Theorem holds:

Theorem 3.1. *(Crandall-Liggett)* [3]*: If A is m-accretive, $f \in L^1(0,T,B)$ and $u^0 \in \overline{D(A)}$ and $f^h \to f$ in $L^1(0,T,B)$, then $u^h \to u(\cdot)$ uniformly as $h \to 0$ and $u(\cdot) \in C(0,T,B)$.*

Theorem 3.2. *If A is m-accretive, $f \in L^1(0,T,B)$ and $u^0 \in \overline{D(A)}$, then the Cauchy problem*

$$\frac{du}{dt} + A(u(t)) \ni f \tag{21}$$

$$u(0) = u^0 \tag{22}$$

has one (and only one) integral solutionu. In the case $f = 0$, $u = S(t)u_0$, where $S(t)$ is a nonlinear semigroup of contractions generated by the operator A. If f has bounded variation in $[0,T]$ and $u^0 \in D(A)$, then the integral solution is Lipschitz continuous.

Definition 3.3. *u is an integral solution of (21) (in the sense of Benilan) if*
(i) $u : [0,T] \to B$ is continuous
(ii) $u(t) \in D(\bar{A})$ for any $t \in [0,T]$,
(iii) $u(0) = u^0$ and

$$||u(t_2 - v)||_B^2 \le ||u(t_1 - v)||_B^2 +$$
$$2\int_{t_1}^{t_2} \lim_{\lambda \to 0} \frac{||u(\tau - v + \lambda(f(\tau) - z))||_B^2 - ||u(\tau - v)||_B^2}{2\lambda} d\tau.$$

In the case B is a Hilbert space, stronger results can be obtained.

Theorem 3.3. *Accretivity of A in a Hilbert space B is equivalent to requiring that*

$$\langle v_1 - v_2, u_1 - u_2 \rangle_B \ge 0 \qquad \forall u_1, u_2 \in D(A), \forall v_1 \in A(u_1), v_2 \in A(u_2). \tag{23}$$

Theorem 3.4. *If B is a Hilbert space, A an m-accretive operator, $f \in W^{1,1}(0,T,B)$ and $u^0 \in D(A)$, then there exists a unique solution $u \in W^{1,\infty}(0,T,B)$ of the Cauchy problem (21)-(22) with $u(t) \in D(A)$.*

Theorem 3.5. [14] *Let A be an m-accretive operator in Hilbert space B and let $S(t)$ be the semigroup generated by A. If A is such that $F := A^{-1}0 \neq \emptyset$, for every $[x,y] \in A$, $x \notin F$, $(y, x - Px) > 0$, where P denotes the projection on F and $(I + A)^{-1}$ is a compact operator, then for every $x \in \overline{D(A)}$, $S(t)x$ converges strongly as $t \to \infty$ to a fixed point of $S(t)$.*

4. **Problems with hysteresis.** In this section we transform some partial differential equations with hysteresis into systems of differential inclusions. We assume in the whole section that Ω is a bounded subset of $\mathbb{R}^N$, $Q = \Omega \times [0,T]$ and the hysteresis relation is assumed to hold pointwise in space (18).

Problem (1):

$$\frac{\partial}{\partial t}(u + w) - \triangle u = f \qquad \text{in } \Omega \times (0,T) = Q, \tag{24}$$

where the hysteresis relation $w = \mathcal{E}(u, w^0)$ represents a generalized play,

As pointed out by Visintin [17], the equation (24) is formally equivalent to

$$\begin{aligned} \frac{\partial u}{\partial t} + \xi - \triangle u &= f && \text{in } Q \\ \frac{\partial w}{\partial t} - \xi &= 0 && \text{in } Q \\ \xi &\in \phi(u, w) && \text{in } Q, \end{aligned} \tag{25}$$

where ϕ was defined in (9). We can write the Cauchy problem for (25) coupled with homogeneous Dirichlet boundary conditions as

$$\frac{\partial U}{\partial t} + A_1 U \ni F \qquad \text{in } Q \tag{26}$$

$$U(0) = U_0 \qquad \text{in } \Omega \tag{27}$$

$$\text{where} \quad U = \begin{pmatrix} u \\ w \end{pmatrix}, \qquad F = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

$$A_1 \begin{pmatrix} u \\ w \end{pmatrix} = A_1 U = \left\{ \begin{pmatrix} \xi - \triangle u \\ -\xi \end{pmatrix}, \xi \in \phi(U) \cap \mathbb{R} \right\}$$

and

$$D(A_1) = \Big\{ U = \begin{pmatrix} u \\ w \end{pmatrix}; \inf \gamma_r(u) \le w \le \sup \gamma_l(u) \text{ a.e. on } \Omega, U \in L^1(\Omega, \mathbb{R}^2), \\ u \in W_0^{1,1}(\Omega), -\triangle u \in L^1(\Omega) \Big\} \tag{28}$$

Problem (2):

$$\frac{\partial}{\partial t}(u + w) + \frac{\partial u}{\partial x} = f \qquad \text{in } (0, L) \times (0, T) = Q, \tag{29}$$

where the hysteresis relation $w = \mathcal{E}(u, w^0)$ represents a generalized play,

The problem can be reformulated similarly as Problem (1) as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} + \xi + \frac{\partial u}{\partial x} &= f && \text{in } Q \\ \frac{\partial w}{\partial t} - \xi &= 0 && \text{in } Q \\ \xi &\in \phi(u, w) && \text{in } Q, \end{aligned} \tag{30}$$

where ϕ was defined in (9). We can write the Cauchy problem for (30) as

$$\frac{\partial U}{\partial t} + A_2 U \ni F \qquad \text{in } Q \tag{31}$$

$$U(0) = U_0 \qquad \text{in } \Omega \tag{32}$$

$$\text{where} \quad U = \begin{pmatrix} u \\ w \end{pmatrix}, \qquad F = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

$$A_2 \begin{pmatrix} u \\ w \end{pmatrix} = A_2 U = \left\{ \begin{pmatrix} \xi \\ -\xi \end{pmatrix} + RU, \xi \in \phi(U) \cap \mathbb{R} \right\}, B(u) := \frac{\partial u}{\partial x} \\ R(U) := (B(u), 0) \tag{33}$$

and

$$D(A_2) = \Big\{ U = \begin{pmatrix} u \\ w \end{pmatrix}; \inf \gamma_r(u) \le w \le \sup \gamma_l(u) \text{ a.e. on } \Omega,$$
$$U \in L^1(\Omega, \mathbb{R}^2), Bu \in L^1(\Omega), u(0) = 0 \Big\}. \quad (34)$$

We have the following theorem, see [17], p.234 :

Theorem 4.1. *Assume that γ_l, γ_r are maximal monotone, satisfy (1), and are affinely bounded, that is, there exist constants $C_1, C_2 > 0$, such that $\forall v \in \mathbb{R}, \forall z \in \gamma_h(v)$*

$$\|z\| \le C_1 \|v\| + C_2 \quad (h = l, r) \tag{35}$$

Then the operators A_1, A_2 resp. defined above are m- and T-accretive in $L^1(\Omega, \mathbb{R}^2)$. If $f \in L^1(\Omega \times (0,T))$, the Cauchy problems (26)-(27) and (31)-(32) have one and only one integral solution $U : [0,T] \to L^1(\Omega, \mathbb{R}^2)$, which depend continuously on the data u_0, w_0, f. Moreover, if $f \in BV(0,T; L^1(\Omega))$ and $-\triangle u_0 \in L^1(\Omega)$, then U is Lipschitz continuous.

Proof. : We outline the proof of the accretivity of the operator A_2. Consider the generalized play operator. Let $\gamma_{rn} := n[I - (I + \frac{1}{n}\gamma_r)^{-1}]$, $\gamma_{ln} := n[I - (I + \frac{1}{n}\gamma_l)^{-1}]$ $\forall n \in N$, be Yosida approximations of γ_l and γ_r respectively and define the corresponding $\mathcal{A}_n$ and ϕ_n as in (33), (9).

We claim that for any $n \in N$ and for any $F_1, F_2 \in L^2(\Omega, \mathbb{R}^2_1)$, setting $U_i := (I + a\mathcal{A}_n)^{-1}(F_i) \quad (i = 1, 2)$,

$$\|(U_1 - U_2)^+\|_{L^1(\Omega, \mathbb{R}^2_1)} \le \|(F_1 - F_2)^+\|_{L^1(\Omega, \mathbb{R}^2_1)}. \tag{36}$$

Then the analogous inequality holds for the negative parts, whence

$$\|(U_1 - U_2)\|_{L^1(\Omega, \mathbb{R}^2_1)} \le \|(F_1 - F_2)\|_{L^1(\Omega, \mathbb{R}^2_1)}. \tag{37}$$

In order to prove (36), first let us take any $\delta > 0$, set

$$H_\delta(\bar{u}) = \begin{cases} 0 & \text{if} \quad \bar{u} < 0, \\ \frac{\bar{u}}{\delta} & \text{if} \quad 0 \le \bar{u} \le \delta, \quad \forall \bar{u} \in \mathbb{R}, \\ 1 & \text{if} \quad \bar{u} > \delta, \end{cases}$$

and denote by H the Heaviside graph. In this argument we omit the (fixed) index n.

Let us set $\tilde{U} = (\tilde{u}, \tilde{v}) := U_1 - U_2$, $\tilde{\xi} := \xi_1 - \xi_2$. For any measurable function h_1 such that $h_1 \in H(\tilde{v})$ a.e. in Ω, we have

$$\begin{aligned} \|(F_1 - F_2)^+\|_{L^1(\Omega, \mathbb{R}^2_1)} &= \int [(\tilde{u} + a\tilde{\xi} + a\mathcal{B}\tilde{u})^+ + (\tilde{v} - a\tilde{\xi})^+] \,\mathrm{d}x \\ &\ge \int_\Omega [(\tilde{u} + a\tilde{\xi} + a\mathcal{B}\tilde{u})H_\delta(\tilde{u}) + (\tilde{v} - a\tilde{\xi})h_1] \,\mathrm{d}x \\ &\ge \int_\Omega \{\tilde{u}H_\delta(\tilde{u}) + \tilde{v}h_1 + a\tilde{\xi}[H_\delta(\tilde{u}) - h_1]\} \,\mathrm{d}x \\ &\quad + a \int_\Omega \mathcal{B}\tilde{u}H_\delta(\tilde{u}) \,\mathrm{d}x. \end{aligned} \tag{38}$$

Note that

$$H_\delta(\tilde{u}) \to h_0 = \begin{cases} 0 & \text{if} \quad \tilde{u} \le 0 \\ 1 & \text{if} \quad \tilde{u} > 0 \end{cases} \qquad a.e. \text{ in } \Omega.$$

Hence

$$\begin{aligned} \int_\Omega [\tilde{u} H_\delta(\tilde{u}) + \tilde{v} h_1] \, \mathrm{d}x \quad &\to \quad \int_\Omega (\tilde{u}^+ + \tilde{v}^+) \, \mathrm{d}x = \|\tilde{U}^+\|_{L^1(\Omega, \mathbb{R}_1^2)}, \\ \int_\Omega \tilde{\xi}[H_\delta(\tilde{u}) - h_1] \, \mathrm{d}x \quad &\to \quad \int_\Omega \tilde{\xi}(h_0 - h_1) \, \mathrm{d}x. \end{aligned}$$

Here keeping in mind that γ_{rn} and γ_{ln} are single valued, the last term is nonnegative, because of the properties of the generalized play operator which are formulated in the Hilpert inequality, see (10) and the last term on the right hand side of (38) is nonnegative as well as can be easily checked. □

Let us consider Problems (1) and (2) coupled with a classical play operator. In this case, because of the properties of the classical play operator it can be shown that the operator A_1 or A_2 resp. are m-accretive in a Hilbert space $L^2(\Omega) \times L^2(\Omega)$ and then Theorem 3.4 can be applied getting much stronger results. To show accretivity e.g. for A_1 we need to show (23) which in our case takes the form:

$$\int_\Omega (\xi_1 - \xi_2)(u_1 - u_2) - \triangle(u_1 - u_2)(u_1 - u_2) - (\xi_1 - \xi_2)(w_1 - w_2) dx \ge 0, \quad (39)$$

which is equivalent to

$$\int_\Omega (\xi_1 - \xi_2)(u_1 - u_2 - w_1 + w_2) + [\nabla(u_1 - u_2)]^2 dx \ge 0. \quad (40)$$

Using the definition of the classical play operator, it follows that the first term is nonnegative, and the result follows.

Applying the above result, together with Theorem 3.5, we get the following Theorem:

Theorem 4.2. *The operators A_1 and A_2 (when the hysteresis relation is the classical play operator) are m-accretive in $L^2(\Omega) \times L^2(\Omega)$, therefore the corresponding Problems (24) and (29) have for $u^0 \in D(A_1)$ or $D(A_2)$ (resp.) a unique solution $u \in W^{1,\infty}(0, T, L^2(\Omega)$. If $f = 0$ this solution converges strongly as $t \to \infty$ to 0.*

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