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ENTROPY CONDITION FOR A QUASILINEAR HYPERBOLIC EQUATION WITH HYSTERESIS

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Abstract. A quasilinear hyperbolic equation with hysteresis is studied. For the integral solution of this equation we derive an entropy condition of the type introduced by Kružkov.

1. Introduction

In this paper we study a hyperbolic equation of first order of the form

$$u_t + [\phi(u)]_x = 0, \qquad u(0) = u_0 \tag{1.1}$$

and the corresponding quasilinear hyperbolic equation with hysteresis

$$\frac{\partial}{\partial t}(u+w) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j u) + cu = f, \tag{1.2}$$

where $w = \mathcal{F}(u)$ represents hysteresis.

It is well known that even for ϕ and u_0 smooth (1.1) exhibits singularities in a finite time. To be able to continue the solution, one has to pass to a generalized concept of weak solutions where discontinuities are allowed. Weak solutions are in general not uniquely determined by the data, and further physically motivated conditions have to be prescribed. The simplest one is an entropy condition stating that the entropy of the system must be decreasing, generalized by Olejnik [5]. A different condition was derived by Kružkov [4] and there are many others. Inspired by Kružkov's work, Crandall [1] shows that the unique integral solution of (1.1), constructed by the method of nonlinear semigroups, satisfies an entropy condition derived by Kružkov. In the first section we give a brief overview of their results.

We consider Equation (1.2) coupled with a generalized play or Prandtl-Ishlinskii operator of play type. It was expected (see [9]) that the integral

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solution of (1.2), for which existence was proved in [9] using the semigroup approach, and which is unique by construction, fulfills a condition of the type introduced by Kružkov. To derive such an entropy condition for the integral solution of (1.2) with hysteresis was posed as an open problem in Visintin's book and we present a solution to this problem in the second section. The method enables us to deal with continuous and discontinuous hysteresis as well.

M. Peszyńska and R.E. Showalter, [6], study a special case of (1.2), namely

$$\frac{\partial}{\partial t}(u+w) + \frac{\partial u}{\partial x} = 0. \tag{1.3}$$

Equation (1.3) arises in applications in chemical and geological engineering as a generic model for the transport and adsorption of a chemical concentration; for a general study see, e.g., [7]. In [6] differentiable solutions of (1.3) are obtained using the theory of m-accretive operators in L^2 and switching the variables x and t. The result is proved for a linear play operator and a Prandtl-Ishlinskii operator of play type as well. So, like the results of Krejčí [2], for the solutions to one-dimensional quasilinear wave equations with and without hysteresis, the presence of hysteresis terms in the equation prevents there the formation of shocks. These results hold for special initial conditions which guarantee that the solution is initially inside the hysteresis loop and for continuous and symmetric hysteresis boundary curves. To study the presence of shocks in the equation (1.3) with discontinuous hysteresis or in the more general case of (1.2) would be an interesting open problem. We would like to initiate such research with a few comments in the conclusion.

2. Entropy conditions and uniqueness of solutions for a hyperbolic equation of first order

We will study the equation

$$u_t + \sum_{i}^{N} \left(\phi_i(u)\right)_{x_i} = 0, \qquad \text{for } t > 0, x \in \mathbb{R}^N, \tag{2.1}$$

where $u = u(x,t)$, $x \in \mathbb{R}^N$ and we denote by $\phi = (\phi_1, ..., \phi_N) : \mathbb{R} \to \mathbb{R}^N$ a continuous function with $\phi(0) = 0$.

Consider first the case $N = 1$. If ϕ is a smooth function, we can rewrite (2.1) as

$$u_t + \phi'(u)u_x = 0$$

Characteristics are defined by the following equation: (for simplicity, the projections of characteristics on the (x, t) plane are still called characteristics)

$$\frac{dx}{dt} = \phi^{'}(u).$$

If $u(x, t)$ solves (2.1), then along a characteristic

$$\frac{d}{dt}u(x(t), t) = u_x \frac{dx}{dt} + u_t = u_x \phi^{'}(u) + u_t = 0,$$

so u is constant along characteristics and it follows that characteristics have constant slope. In other words, the characteristics are straight lines with parametric velocity $\phi^{'}(u)$ along these lines.

Assume now for convenience that $\phi''(u) > 0$. If $u(x, 0) = u_0(x)$ and $u_0(x)$ is decreasing, then there are points $x_1 < x_2$ with $\phi^{'}(u_0(x_1)) > \phi^{'}(u_0(x_2))$, and the characteristics starting at $(x_1, 0)$ and $(x_2, 0)$ will intersect at a point P for $t > 0$; see Figure 1.

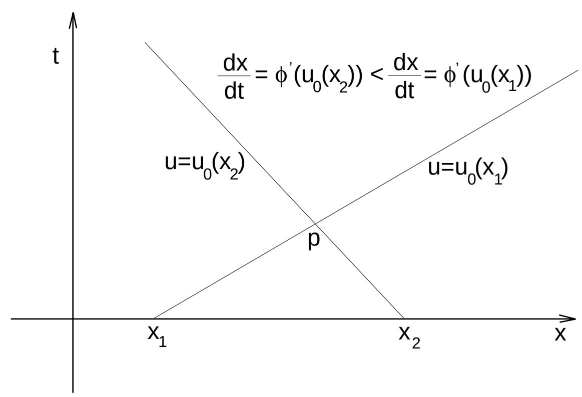

FIGURE 1. Characteristics intersect.

At the point P a continuous solution is overdetermined, since different characteristics meet there and each carries a different value of u. It turns out that the solution must be discontinuous. (We also can easily see that when $\phi''(u) > 0$, $u(x, t)$ is globally defined and continuous if and only if $u_0(x)$ is nondecreasing and continuous.)

The above conclusion is independent of the smoothness properties of ϕ and $u_0(x)$. No matter how smooth the initial data, the solution may still have discontinuities. This is the most important feature of quasilinear hyperbolic equations and an essential difference from linear hyperbolic equations. It is this phenomenon that leads to special difficulties.

For the reasons given above, we shall generalize the notion of solution for equations of the form (2.1):

Definition 1. *A bounded, measurable function $u(x,t)$ is called a weak solution of the problem* (2.1) *with the initial condition $u(x,0) = u_0(x)$ with bounded and measurable initial data $u_0(x)$, provided that*

$$\int_0^T \int_{\mathbb{R}^N} (uf_t + \sum_i^N \phi_i(u) f_{x_i}) dx dt + \int_{\mathbb{R}^N} u_0 f dx = 0 \tag{2.2}$$

holds for all $f \in C_0^1((0,\infty) \times \mathbb{R}^N)$.

Note that if (2.2) holds for all $f \in C_0^1((0,\infty) \times \mathbb{R}^N)$, and if u is in $C^1((0,T) \times \mathbb{R}^N)$, then u is a classical solution (this is easy to see, using integration by parts).

In our effort to solve initial-value problems which are not solvable classically, we are led to extend the class of solutions. In doing this, we run the risk of losing uniqueness. That this concern is well-founded follows from the next example.

Example 1. (see [8]): Consider the equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

with the initial condition

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x > 0. \end{cases}$$

For each $\alpha \geq 1$, this problem has a solution u_α defined by

$$u_\alpha(x,t) = \begin{cases} 1 & \text{if } 2x \leq (1-\alpha)t \\ -\alpha & \text{if } (1-\alpha)t < 2x \leq 0 \\ \alpha & \text{if } 0 < 2x \leq (\alpha-1)t \\ -1 & \text{if } (\alpha-1)t < 2x. \end{cases}$$

Thus, our problem has a continuum of solutions (see Figure 2).

Equations of the above form arise in the physical sciences and so we must have some mechanism to pick out the "physically relevant" solution. Thus, we are led to impose an a priori condition on solutions which distinguishes the "correct" one from the others.

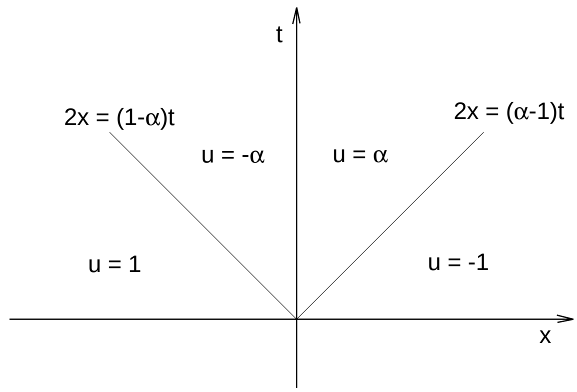

Figure 2. Continuum of solutions in Example 1.

In the case of the equation when $N = 1$

$$u_t + [\phi(u)]_x = 0, \qquad \text{with } \phi'' > 0,$$

there is a unique solution which satisfies the "entropy" condition

$$\frac{u(x+a,t) - u(x,t)}{a} \leq \frac{E}{t} \qquad \forall a > 0, \forall t > 0, \tag{2.3}$$

where E is independent of x, t and a.

This condition implies that if we fix $t > 0$ and let x increase from $-\infty$ to $+\infty$, then we can only jump down, as we cross a discontinuity - hence the reason for the word "entropy".

If we return to the previous example, then we see that (2.3) is satisfied only when $\alpha = 1$.

So far we considered only the case $\phi''(u) > 0$. O.A. Olejnik [5] gives a uniqueness condition for (2.1) in a special case when $N = 1$, namely

$$\frac{\partial u}{\partial t} + \frac{\partial \phi(u,x,t)}{\partial x} = 0, \tag{2.4}$$

now called the E condition, without any restriction on $\phi \in C^1$ as follows.

We introduce the notation

$$u(x+0,t) = u_+(x,t)$$
$$u(x-0,t) = u_-(x,t)$$
$$l(u) = \frac{\phi(u_+,x,t) - \phi(u_-,x,t)}{u_+ - u_-}(u - u_+) + \phi(u_+,x,t)).$$

Consider the straight line $w = l(u)$ in the $u-w$ plane, which joins the points $(u_+, \phi(u_+, x, t))$ and $(u_-, \phi(u_-, x, t))$. We shall say that the generalized solution $u(x,t)$ of (2.4) satisfies condition E if at all points of discontinuity of $u(x,t)$ (except possibly a finite number of them), the following condition is satisfied: when $u_+ > u_-$, $l(u) \leq \phi(u,x,t)$ for all u in $[u_-, u_+]$, while when $u_+ < u_-$, $l(u) \geq \phi(u,x,t)$ for all u in $[u_+, u_-]$.

It is easy to see that if the function $\phi(u,x,t)$ is such that $\phi_{uu} \neq 0$, then condition E is identical with (2.3), namely $u_+ < u_-$ if $\phi_{uu} > 0$, and $u_+ > u_-$ if $\phi_{uu} < 0$.

We have the following

Theorem 1. (Olejnik [5]) *A weak solution of* (2.4) *with* $u(x,0) = u_0(x)$, *which satisfies condition E, is unique.*

A different approach to the question of existence of a unique solution of (2.4), $N \geq 1$, was given by Kružkov [4]. He defines a generalized solution of (2.1) as follows:

Definition 2. *A bounded measurable function* $u(x,t)$ *is called a generalized solution of* (2.1) *with* $u(x,0) = u_0(x)$ *in* $Q_T = [0,T] \times \mathbb{R}^N$ *if*

1) *for any constant* k *and any smooth function* $f(x,t) \geq 0$ *the following inequality holds:*

$$\iint_{Q_T} \Big\{ |u(x,t) - k| f_t + [\mathit{sign}\ (u(x,t) - k)] \sum_{i=1}^{N} [\phi_i(u(x,t),x,t) - \phi_i(k,x,t)] f_{x_i} \Big\} dx\, dt \geq 0; \quad (2.5)$$

2) *there exists a set* $\mathcal{E}$ *of measure zero on* $[0,T]$ *such that for* $t \in [0,T] \setminus \mathcal{E}$ *the function* $u(x,t)$ *is defined almost everywhere in* $\mathbb{R}^N$, *and for any ball*

$$K_r = \{|x| \leq r\} \subset \mathbb{R}^N, \qquad \lim_{\substack{t \to 0 \\ t \in [0,T] \setminus \mathcal{E}}} \int_{K_r} |u(x,t) - u_0(x)| dx = 0.$$

Since the smooth function $f \geq 0$ is arbitrary, it is obvious that inequality (2.5) for $k = \pm \sup |u(x,t)|$ implies (2.1). But Definition 2 also contains a condition which characterizes the permissible discontinuities of solutions. This condition is especially easy to visualize when the generalized solution is a piecewise smooth function in some neighborhood of the point of discontinuity; in this case, using integration by parts and the fact that f was chosen arbitrarily, we obtain from (2.5) that for any constant k, along the surface of discontinuity we have

$$\begin{aligned} &|u_+ - k|\cos(\nu,t) + \text{sign } (u_+ - k)[\phi(u_+,x,t) - \phi(k,x,t)]\cos(\nu,x) \\ \le &|u_- - k|\cos(\nu,t) + \text{sign } (u_- - k)[\phi(u_-,x,t) - \phi(k,x,t)]\cos(\nu,x), \end{aligned} \tag{2.6}$$

where ν is the normal vector to the surface of discontinuity at the point (x,t) and u_+, u_- are the one-sided limits of the generalized solution at the point (x,t) from the positive and negative side of the surface of discontinuity, respectively. It can be shown that in the case $N = 1$ (2.6) is equivalent to condition E introduced above (we just need to express $\cos(\nu,t)$ and $\cos(\nu,x)$ by using (2.4) and choose $k = u \in [u_-, u_+]$).

Kružkov shows that there exists a unique generalized solution of (2.4) in the sense of Definition 2.

Inspired by the results of Kružkov, Crandall in his paper [1] treats the Cauchy problem for the equation

$$u_t + \sum_{i=1}^{N} (\phi_i(u))_{x_i} = 0, \qquad t > 0, \quad x \in \mathbb{R}^N$$

from the point of view of semigroups of nonlinear transformations.

The following notation will be used whenever it is meaningful:

$$\phi = (\phi_1, ..., \phi_N) : \mathbb{R} \to \mathbb{R}^N \tag{2.7}$$

$$[\phi(v)]_x = \sum_{i=1}^{N} (\phi_i(v(x)))_{x_i} \qquad \text{if } v : \mathbb{R}^N \to \mathbb{R} \tag{2.8}$$

$$f_x = (f_{x_1}, ... f_{x_N}) \qquad \text{if } f : \mathbb{R}^N \to \mathbb{R} \tag{2.9}$$

$$ab = \sum_{i=1}^{N} a_i b_i \qquad \text{if } a, b \in \mathbb{R}^N. \tag{2.10}$$

Given $\phi_i(v)$ with $\phi_i(0) = 0$, $i = 1, ..., N$, he defines

$$Av = \sum_{i=1}^{N} (\phi_i(v))_{x_i}, \qquad v \in D(A)$$

as the closure of A_0 given in the next definition.

Definition 3. *A_0 is the operator in $L^1(\mathbb{R}^N)$ defined by $v \in D(A_0)$ and $w \in A_0(v)$ if v, $w \in L^1(\mathbb{R}^N)$, $\phi(v) \in L^1(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} [sign\ (v(x) - k)]\{(\phi(v(x)) - \phi(k)) f_x(x) + w(x) f(x)\} dx \ge 0 \tag{2.11}$$

for every $k \in \mathbb{R}$*, and every* $f \in C_0^\infty(\mathbb{R}^N)$ *such that* $f \geq 0$.

Lemma 2. (Crandall, [1]) *Let* $\phi \in C^1$ *and* A_0 *be as given by Definition* 3. *If* $v \in C_0^1(\mathbb{R}^N)$*, then* $v \in D(A_0)$ *and* $A_0 v = \{[\phi(v)]_x\}$.

The lemma shows that A extends A_0 from $C_0^1(\mathbb{R}^N)$. Crandall then shows that A, the closure of A_0, is an m-accretive operator, thus generates a semigroup of contractions $S(t)$, and $S(t)u_0$ is the (unique) integral solution of (2.4). Then he shows that this solution constructed by the method of semigroups satisfies indeed the entropy condition introduced by Kružkov:

Theorem 3. (Crandall) *Let* S *be the semigroup of contractions generated by* A*. Let* $v \in \overline{D(A)}$ *and* $t \geq 0$*. If also* $v \in L^\infty(\mathbb{R}^N)$*, then*

$$\int_0^T \int_{\mathbb{R}^N} \{|S(t)v(x) - k| f_t + [sign(S(t)v(x) - k)][\phi(S(t)v(x)) - \phi(k)] f_x\} dx\, dt \geq 0,$$

for every $f(x,t) \in C_0^\infty((0,T) \times \mathbb{R}^N)$ *such that* $f \geq 0$ *and every* $k \in \mathbb{R}$ *and* $T > 0$.

Proof. Let $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u_\epsilon(t)$ satisfy

$$\begin{aligned} \epsilon^{-1}(u_\epsilon(t) - u_\epsilon(t-\epsilon)) + A_0 u_\epsilon(t) &= 0 & t &\geq 0 \\ u_\epsilon(t) &= v & t &< 0. \end{aligned}$$

Let $u_\epsilon(x,t) = u_\epsilon(t)(x)$. By the definition of A_0:

$$\begin{aligned} \int_{\mathbb{R}^N} &\{\text{sign } (u_\epsilon(x,t) - k)(\phi(u_\epsilon(x,t)) - \phi(k)) f_x(x,t) \\ &+ \epsilon^{-1}[\text{sign } (u_\epsilon(x,t) - k)](u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t)) f(x,t)\} dx \geq 0 \end{aligned} \tag{2.12}$$

for every $k \in \mathbb{R}$ and nonnegative $f \in C_0^\infty((0,T) \times \mathbb{R}^N)$.

Let $h_\epsilon(x,t) = [\text{sign } (u_\epsilon(x,t) - k)](u_\epsilon(x,t) - k) = |u_\epsilon(x,t) - k|$. Notice that

$$\begin{aligned} &(u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t))[\text{sign } (u_\epsilon(x,t) - k)] \\ &= (u_\epsilon(x,t-\epsilon) - k)[\text{sign}(u_\epsilon(x,t) - k)] - (u_\epsilon(x,t) - k)[\text{sign}(u_\epsilon(x,t) - k)] \\ &\leq h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t). \end{aligned} \tag{2.13}$$

Using (2.13) and integrating (2.12) over $0 \leq t \leq T$ yields

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} &\{[\text{sign } (u_\epsilon(x,t) - k)](\phi(u_\epsilon(x,t)) - \phi(k)) f_x(x,t) \\ &+ \epsilon^{-1}(h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t)) f(x,t)\} dxdt \geq 0. \end{aligned} \tag{2.14}$$

Now

$$\begin{aligned}
&\epsilon^{-1}\int_0^T\int_{\mathbb{R}^N}\{(h_\epsilon(x,t-\epsilon)-h_\epsilon(x,t))f(x,t)\}dxdt\\
&\quad=\epsilon^{-1}\left(\int_0^\epsilon\int_{\mathbb{R}^N}h_\epsilon(x,t-\epsilon)f(x,t)dxdt-\int_{T-\epsilon}^T\int_{\mathbb{R}^N}h_\epsilon(x,t)f(x,t)dxdt\right)\\
&\qquad\qquad+\int_0^{T-\epsilon}\int_{\mathbb{R}^N}h_\epsilon(x,t)(\epsilon^{-1})(f(x,t+\epsilon)-f(x,t))dxdt.
\end{aligned}$$

The first and the second integrals vanish for ϵ small enough since f is in $C_0^\infty((0,T)\times\mathbb{R}^N)$. The convergence $u_\epsilon(x,t)\to S(t)v(x)$ in $L^1(\mathbb{R}^N)$, uniformly in t as $\epsilon\to 0$, implies that the third term tends to

$$\int_0^T\int_{\mathbb{R}^N}|S(t)v(x)-k|f_t(x,t)dx\,dt$$

as $\epsilon\downarrow 0$. So the theorem follows by letting $\epsilon\downarrow 0$ in (2.14). $\square$

3. Quasilinear hyperbolic equation with hysteresis

Let b_j and c be given smooth functions, Ω an open subset of $\mathbb{R}^n$ of Lipschitz class, $T>0, Q=\Omega\times[0,T]$. In this section we consider the equation

$$\frac{\partial}{\partial t}(u+w)+\sum_{j=1}^N\frac{\partial}{\partial x_j}(b_ju)+cu=f \qquad \text{in } Q \tag{3.1}$$

and couple it with the hysteresis relation

$$w(x,t)=\left[\mathcal{E}(u(x,.),w_0(x))\right](t) \qquad \text{in } [0,T], \text{ a.e. in } \Omega.$$

Here $\mathcal{E}$ is a multivalued functional. Its values depend not only on the current value of $u(.,t)$ at $t>0$, but on the past history $u(.,s), 0<s<t$. We consider at first $\mathcal{E}$ to be a generalized play operator. Let

$$\begin{gathered}
\gamma_r,\gamma_l \text{ be maximal monotone (possibly multivalued) functions,}\\
\text{and} \qquad \inf\gamma_r(u)\le\sup\gamma_l(u), \qquad \forall u\in\mathbb{R}.
\end{gathered}\tag{3.2}$$

Now, given $w_0\in\mathbb{R}$, we construct the hysteresis operator $\mathcal{E}(\cdot,w_0)$ as follows. Let u be any continuous, piecewise linear function on $\mathbb{R}^+$ such that u is linear on $[t_{i-1},t_i]$ for $i=1,2,....$ We then define $w:=\mathcal{E}(u,w_0):\mathbb{R}^+\to\mathbb{R}$ by

$$w(t):=\begin{cases}\min\{\gamma_l(u(0)),\max\{\gamma_r(u(0)),w_0\}\} & \text{if } t=0,\\ \min\{\gamma_l(u(t)),\max\{\gamma_r(u(t)),w(t_{i-1})\}\} & \text{if } t\in(t_{i-1},t_i], i=1,2,...\end{cases}$$

Note that $w(0) = w_0$ only if $\gamma_r(u(0)) \leq w_0 \leq \gamma_l(u(0))$. The hysteresis relation is assumed to hold pointwise in space :

$$w(x,t) = [\mathcal{E}(u(x,.), w_0(x))](t) \qquad \text{in } [0,T], \text{ a.e. in } \Omega. \tag{3.3}$$

As proved in Visintin [9], Section III.2, $\mathcal{E}(\cdot, w_0)$ has a unique continuous extension, denoted by $\mathcal{E}(\cdot, w_0)$ again, to an operator

$$\mathcal{E} : C(\mathbb{R}^+) \times \mathbb{R} \to C(\mathbb{R}^+). \tag{3.4}$$

This operator is called a generalized play; see Figure 3.

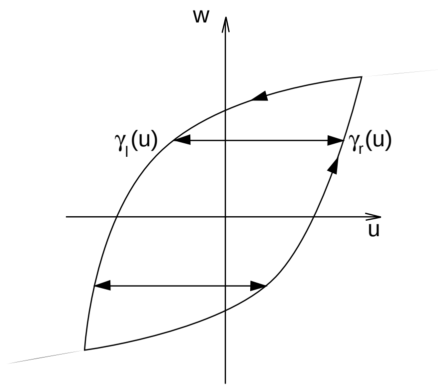

FIGURE 3. The generalized play.

The system (3.1) is formally equivalent to

$$\begin{cases} \frac{\partial u}{\partial t} + \xi + \sum_{j=1}^N \frac{\partial}{\partial x_j}(b_j u) + cu = f & \text{in } Q \\ \frac{\partial w}{\partial t} - \xi = 0 & \text{in } Q \\ \xi \in \phi(u,w) & \text{in } Q, \end{cases} \tag{3.5}$$

where

$$\phi(u,w) = \begin{cases} +\infty & \text{if } w < \inf \gamma_r(u) \\ \tilde{\mathbb{R}}^+ & \text{if } w \in \gamma_r(u) \backslash \gamma_l(u) \\ \{0\} & \text{if } \sup \gamma_r(u) < w < \inf \gamma_l(u) \\ \tilde{\mathbb{R}}^- & \text{if } w \in \gamma_l(u) \backslash \gamma_r(u) \\ -\infty & \text{if } w > \sup \gamma_l(u) \\ \tilde{\mathbb{R}} & \text{if } w \in \gamma_l(u) \cap \gamma_r(u) \end{cases} \tag{3.6}$$

and $\tilde{\mathbb{R}} := [-\infty, +\infty]$, $\tilde{\mathbb{R}}^+ := [0, \infty]$, $\tilde{\mathbb{R}}^- := [-\infty, 0]$.

To simplify the discussion, we assume that

$$\Big\{b_j \in C^1(\overline{\Omega})\Big\}_{j=1,\dots,N}\ , \quad \sum_{j=1}^{N} b_j \nu_j = 0 \ \text{ a.e. on } \partial\Omega$$

and $c \in L^\infty(\Omega)$, where $\overrightarrow{\nu}$ denotes a field normal to $\partial\Omega$.

By introducing the following operators

$$\begin{aligned}
D(A) &:= \{U := (u,w) \in \mathbb{R}^2 : \inf \gamma_r(u) \le w \le \sup \gamma_l(u)\},\\
A(U) &:= \{(\xi,-\xi) : \xi \in \phi(U) \cap \mathbb{R}\} \qquad \forall U \in D(A),\\
B(u) &:= \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j u) + cu, \quad R(U) := (B(u),0)
\end{aligned}$$

and by setting $U := (u,w)$, $U_0 := (u_0,w_0)$, $F := (f,0)$ the Cauchy problem for the system (3.5) can be written in the form

$$\begin{gathered}
\frac{\partial U}{\partial t} + A(U) + R(U) \ni F \qquad \text{in } Q \\
U(0) = U_0.
\end{gathered} \tag{3.7}$$

This approach can be easily extended to the case in which $\mathcal{E}$ is replaced by a generalized Prandtl-Ishlinskii operator of play type, which is defined as a weighted superposition of generalized plays with boundary curves $\gamma_{\rho l}$, $\gamma_{\rho r}$, $\rho \in \mathcal{P}$; here $\mathcal{P}$ denotes an index set. For the precise definition see, e.g., Visintin [9].

Then we have the following theorem [9]:

Theorem 4. *Let Ω be an open subset of $\mathbb{R}^N$ $(N \ge 1)$ of Lipschitz class. Let $L^1(\Omega;\mathbb{R}^2)$ be endowed with the norm*

$$\|U\|_{L^1(\Omega;\mathbb{R}^2)} := \int_\Omega (|u(x)| + |w(x)|)\, dx \quad \forall U := (u,w) \in L^1(\Omega;\mathbb{R}^2).$$

Define the operator R as

$$R(U) := (Bu, 0) \quad \forall U \in D(R) := \{U \in L^1(\Omega;\mathbb{R}^2) : Bu \in L^1(\Omega)\},$$

A is defined for

$$\begin{cases} \gamma_l, \gamma_r \ \textit{maximal monotone (possibly multivalued) functions:} \\ \mathbb{R} \to P(\widetilde{\mathbb{R}}),\ \textit{such that} \ \inf \gamma_r(u) \le \sup \gamma_l(u) \qquad \forall u \in \mathbb{R}\,. \end{cases}$$

Also assume that γ_l, γ_r are affinely bounded; that is, there exist constants $C_1, C_2 > 0$, such that $\forall v \in \mathbb{R}, \forall z \in \gamma_h(v)$,

$$|z| \leq C_1|v| + C_2 \quad (h = l, r).$$

Take any $U_0 := (u_0, w_0) \in L^1(\Omega;\mathbb{R}^2)$, such that $U_0 \in D(A)$ almost everywhere in Ω, and any $f \in L^1(\Omega \times (0,T))$. Then the Cauchy problem (3.7) *has one and only one integral solution $U : [0,T] \to L^1(\Omega,\mathbb{R}^2)$, which depends continuously on the data u_0, w_0, f. Moreover, if $f \in BV(0,T;L^1(\Omega))$ and $Ru_0 \in L^1(\Omega)$, then U is Lipschitz continuous.*

A similar statement is true for a generalized Prandtl-Ishlinskii operator of play type. It was conjectured (see [9]), that the integral solution from Theorem 4 fulfils a condition of the type introduced by Kružkov. The next Theorem establishes this in a precise form.

Theorem 5. *Let the assumptions of Theorem* 4 *hold and let $F = (f, 0) = (0, 0)$. Assume also that the hysteresis operator is symmetric around $w = u$. Let $A_0U = A(U) + R(U)$ on $D(A_0)$, and let $S(t) = (S_1(t), S_2(t))$ be the corresponding semigroup of contractions.*

Let $v \in \overline{D(A)}$ and $t \geq 0$. Then if $v = (v_1, v_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$,

$$\begin{aligned} &\int_0^T \int_\Omega |S_1(t)v_1(x) - k|\psi_t(x,t)dxdt + \int_0^T \int_\Omega |S_2(t)v_2(x) - k|\psi_t(x,t)dx\,dt \\ &+ \int_0^T \int_\Omega \Big\{ \sum_{j=1}^N b_j|S_1(t)v_1(x) - k|\frac{\partial}{\partial x_j}\psi(x,t) - c|S_1(t)v_1(x) - k|\psi(x,t) \\ &\qquad - [sign(S_1(t)v_1(x) - k)]k\Big(\sum_{j=1}^N \frac{\partial}{\partial x_j}b_j + c\Big)\psi(x,t)\Big\}dx\,dt \geq 0 \end{aligned}$$

for every $\psi(x,t) \in C_0^\infty((0,T) \times \Omega)$ such that $\psi \geq 0$ and every $k \in \mathbb{R}$.

Remark 1. The hysteresis operator can be also discontinuous, e.g., the relay operator, as long as the boundary curves γ_l, γ_r satisfy the symmetry relation given above.

Proof. Let $v \in \overline{D(A_0)} \cap L^\infty(\Omega,\mathbb{R}^2)$ and $u_\epsilon(t)$, $w_\epsilon(t)$ satisfy

$$\left.\begin{array}{c} \frac{u_\epsilon(t)-u_\epsilon(t-\epsilon)}{\epsilon} + \xi + \sum_{j=1}^N \frac{\partial}{\partial x_j}(b_j u_\epsilon(t)) + cu_\epsilon(t) = 0 \\ \frac{w_\epsilon(t)-w_\epsilon(t-\epsilon)}{\epsilon} - \xi = 0 \end{array}\right\} \text{ for } t \geq 0, \tag{3.8}$$

$$\begin{pmatrix} u_\epsilon(t) \\ w_\epsilon(t) \end{pmatrix} = v \qquad \text{for } t < 0. \tag{3.9}$$

If $k \in \mathbb{R}$ is any constant, then we have

$$-\Big(\sum_{j=1}^{N}\frac{\partial}{\partial x_j}b_j k + ck\Big) = -k\Big(\sum_{j=1}^{N}\frac{\partial}{\partial x_j}b_j + c\Big). \tag{3.10}$$

We get from the second equation in (3.8) that

$$\xi = \frac{w_\epsilon(t) - w_\epsilon(t-\epsilon)}{\epsilon},$$

which we can put into the first equation in (3.8). Adding the resulting equation to (3.10) gives us

$$\frac{u_\epsilon(t) - u_\epsilon(t-\epsilon)}{\epsilon} + \frac{w_\epsilon(t) - w_\epsilon(t-\epsilon)}{\epsilon} + \sum_{j=1}^{N}\frac{\partial}{\partial x_j}[b_j(u_\epsilon(t) - k)] \\ + [c(u_\epsilon(t) - k)] + k\Big(\sum_{j=1}^{N}\frac{\partial}{\partial x_j}b_j + c\Big) = 0.$$

Let $u_\epsilon(x,t) = u_\epsilon(t)(x)$ and $w_\epsilon(x,t) = w_\epsilon(t)(x)$. Multiply the last equation by $[\text{sign}(u_\epsilon(x,t) - k)]$ to get

$$[\text{sign}(u_\epsilon(x,t) - k)]\Big[\frac{u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t)}{\epsilon} + \frac{w_\epsilon(x,t-\epsilon) - w_\epsilon(x,t)}{\epsilon}\Big] \\ - [\text{sign}(u_\epsilon(x,t) - k)]\Big[\sum_{j=1}^{N}\frac{\partial}{\partial x_j}[b_j(u_\epsilon(x,t) - k)] + [c(u_\epsilon(x,t) - k)] \\ + k\Big(\sum_{j=1}^{N}\frac{\partial}{\partial x_j}b_j + c\Big)\Big] = 0.$$

As before, let $h_\epsilon(x,t) = (u_\epsilon(x,t) - k)[\text{sign }(u_\epsilon(x,t) - k)] = |u_\epsilon(x,t) - k|$. Recall that

$$\begin{aligned} &(u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t))[\text{sign }(u_\epsilon(x,t) - k)] \\ &= (u_\epsilon(x,t-\epsilon) - k)[\text{sign }(u_\epsilon(x,t) - k)] - (u_\epsilon(x,t) - k)[\text{sign }(u_\epsilon(x,t) - k)] \\ &\le h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t). \end{aligned} \tag{3.11}$$

Also

$$\begin{aligned} &(w_\epsilon(x,t-\epsilon) - w_\epsilon(x,t))[\text{sign }(u_\epsilon(x,t) - k)] \\ &\le (w_\epsilon(x,t-\epsilon) - w_\epsilon(x,t))[\text{sign }(w_\epsilon(x,t) - k)]. \end{aligned} \tag{3.12}$$

This last inequality is true because of the following: The only way it could fail would be if either: $\text{sign}(u_\epsilon(x,t)-k)=1$, $\text{sign}(w_\epsilon(x,t)-k)=-1$, and $w_\epsilon(x,t-\epsilon)-w_\epsilon(x,t)>0$, so $u_\epsilon(x,t)>k$, $w_\epsilon(x,t)<k$ and $w_\epsilon(x,t-\epsilon)>w_\epsilon(x,t)$ or $\text{sign}(u_\epsilon(x,t)-k)=-1$, $\text{sign}(w_\epsilon(x,t)-k)=1$, and $w_\epsilon(x,t-\epsilon)-w_\epsilon(x,t)<0$, so $u_\epsilon(x,t)<k$, $w_\epsilon(x,t)>k$, and $w_\epsilon(x,t-\epsilon)<w_\epsilon(x,t)$.

It can be easily seen from Figures 4a and 4b that these situations are not possible because of the properties of the hysteresis operator; thus (3.12) must be true.

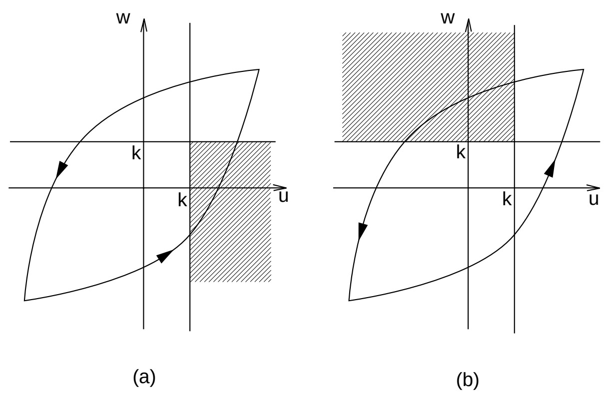

FIGURE 4. Possible situations in the proof of (3.12).

If we introduce the notation $g_\epsilon(x,t)=(w_\epsilon(x,t)-k)[\text{sign }(w_\epsilon(x,t)-k)]=|w_\epsilon(x,t)-k|$, then, similarly, we get

$$\begin{aligned}
&(w_\epsilon(x,t-\epsilon)-w_\epsilon(x,t))[\text{sign }(u_\epsilon(x,t)-k)] \\
&\le (w_\epsilon(x,t-\epsilon)-w_\epsilon(x,t))[\text{sign }(w_\epsilon(x,t)-k)] \\
&= (w_\epsilon(x,t-\epsilon)-k)[\text{sign }(w_\epsilon(x,t)-k)]-(w_\epsilon(x,t)-k)[\text{sign }(w_\epsilon(x,t)-k)] \\
&\le g_\epsilon(x,t-\epsilon)-g_\epsilon(x,t).
\end{aligned} \tag{3.13}$$

Therefore,

$$\Big[\frac{(h_\epsilon(x,t-\epsilon)-h_\epsilon(x,t))}{\epsilon}+\frac{(g_\epsilon(x,t-\epsilon)-g_\epsilon(x,t))}{\epsilon}\Big] \\
-[\text{sign}(u_\epsilon(x,t)-k)]\Big[\sum_{j=1}^{N}\frac{\partial}{\partial x_j}[b_j(u_\epsilon(x,t)-k)]+[c(u_\epsilon(x,t)-k)]$$

$$+ k\Big(\sum_{j=1}^{N} \frac{\partial}{\partial x_j} b_j + c\Big)\Big] \geq 0.$$

Now we can multiply by any $\psi \geq 0$, $\psi(x,t) \in C_0^\infty((0,T)\times\Omega)$, and integrate over $[0,T]\times\Omega$ to get the following inequality:

$$\begin{aligned}
0 \leq \epsilon^{-1} \int_0^T \int_\Omega &\{[h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t)]\, \psi(x,t)\}\, dxdt \\
&+ \epsilon^{-1} \int_0^T \int_\Omega \{[g_\epsilon(x,t-\epsilon) - g_\epsilon(x,t)]\, \psi(x,t)\}\, dxdt \\
- \int_0^T \int_\Omega \Big\{ &\sum_{j=1}^{N} \frac{\partial}{\partial x_j} b_j(|u_\epsilon(x,t) - k|)\psi(x,t) + c|u_\epsilon(x,t) - k|\psi(x,t) \\
&+ [\text{ sign } (u_\epsilon(x,t) - k)]k\Big(\sum_{j=1}^{N} \frac{\partial}{\partial x_j} b_j + c\Big)\psi(x,t)\Big\} dx\, dt. \qquad (3.14)
\end{aligned}$$

Now

$$\begin{aligned}
&\epsilon^{-1} \int_0^T \int_\Omega \{[h_\epsilon(x,t-\epsilon) - h_\epsilon(x,t)]\, \psi(x,t)\}\, dxdt \\
&= \epsilon^{-1} \left(\int_0^\epsilon \int_\Omega \{h_\epsilon(x,t-\epsilon)\psi(x,t)\}\, dxdt - \int_{T-\epsilon}^T \int_\Omega \{h_\epsilon(x,t) f(x,t)\} dxdt \right) \\
&\qquad + \int_0^{T-\epsilon} \int_\Omega \big\{ h_\epsilon(x,t)\epsilon^{-1}(\psi(x,t+\epsilon) - \psi(x,t)) \big\}\, dxdt.
\end{aligned}$$

The first and the second integrals vanish for ϵ small enough, since ψ is in $C_0^\infty((0,T)\times\Omega)$. The convergence $u_\epsilon(x,t) \to S_1(t)v_1(x)$ in $L^1(\Omega)$, uniformly in t as $\epsilon \to 0$, implies that the third term tends to

$$\int_0^T \int_\Omega |S_1(t)v_1(x) - k| f_t(x,t) dxdt \qquad \text{as } \epsilon \downarrow 0.$$

By a similar argument, using the convergence $w_\epsilon(x,t) \to S_2(t)v_2(x)$ in $L^1(\Omega)$, we have that

$$\int_0^{T-\epsilon} \int_\Omega \big\{ g_\epsilon(x,t)\epsilon^{-1}(\psi(x,t+\epsilon) - \psi(x,t)) \big\}\, dxdt \qquad \text{tends to}$$

$$\int_0^T \int_\Omega |S_2(t)v_2(x) - k| \psi_t(x,t) dxdt \qquad \text{as } \epsilon \downarrow 0.$$

If we now let $\epsilon \downarrow 0$ in (3.14), we get

$$\int_0^T \int_\Omega |S_1(t)v_1(x) - k|\psi_t(x,t)dxdt + \int_0^T \int_\Omega |S_2(t)v_2(x) - k|\psi_t(x,t)dxdt$$
$$+ \int_0^T \int_\Omega \Big\{ \sum_{j=1}^N b_j |S_1(t)v_1(x) - k| \frac{\partial}{\partial x_j}\psi(x,t) - c|S_1(t)v_1(x) - k|\psi(x,t)$$
$$- [\operatorname{sign}(S_1(t)v_1(x) - k)]k\Big(\sum_{j=1}^N \frac{\partial}{\partial x_j} b_j + c\Big)\psi(x,t)\Big\} dxdt \geq 0,$$

which is the claim of our theorem. □

4. Conclusions

Peszyńska and Showalter in [6] showed the existence of a unique differentiable solution of (1.3). In that special case the presence of hysteresis in the equation prevents the formation of shocks if the initial conditions are chosen inside the hysteresis loop. Consider (1.3) with a generalized play operator, pictured in Figure 1, with increasing hysteresis boundary curves γ_l and γ_r, $\gamma_r \leq \gamma_l$. Then from (1.3) and the definition of the generalized play operator we have

$$\frac{\partial u}{\partial t} + \gamma_r'(u)\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \text{ if } w = \gamma_r(u) \text{ and } u \text{ is increasing in } t \tag{4.1}$$
$$\frac{\partial u}{\partial t} + \gamma_l'(u)\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \text{ if } w = \gamma_l(u) \text{ and } u \text{ is decreasing in } t \tag{4.2}$$
$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \text{ if } \gamma_r(u) \leq w \leq \gamma_l(u). \tag{4.3}$$

Consider just the first equation (4.1). It is equivalent to the equation

$$\frac{\partial u}{\partial t} + \frac{1}{1+\gamma_r'(u)}\frac{\partial u}{\partial x} = 0, \text{ if } w = \gamma_r(u) \text{ and } u \text{ is increasing in } t. \tag{4.4}$$

A simple computation gives us

$$\frac{\partial u}{\partial t} = -\frac{f'(u)u_0'}{1+u_0'f''(u)t}, \tag{4.5}$$

where we denoted $f'(u) = \frac{1}{1+\gamma_r'(u)}$, which is greater than zero, because γ_r is increasing by assumption. If we also assume that γ_r is a convex function,

$$f''(u) = -\frac{\gamma_r''(u)}{[1+\gamma_r'(u)]^2} < 0. \tag{4.6}$$

We see that discontinuities for (4.1) can occur for the choice of an initial condition $u_0'(x) \geq 0$. But in this case $\frac{\partial u}{\partial t} < 0$ and following (4.1) the solution moves inside the hysteresis region and is governed by the third equation (4.3). Analogously for γ_l, shocks can happen for ${u_0}' \leq 0$, but in this case the solution is increasing and moves again inside the hysteresis region. The precise statement is in [6].

In the case of $N \geq 2$, discontinuous hysteresis, and other choices of initial data, we established a criteria, which selects a unique integral solution of (1.2) and satisfies a kind of condition introduced by Kružkov. In this way it is an answer to an open problem stated in Visintin's book. However it would be interesting to establish if shocks can happen in any of those cases. On the other hand, as it is easy to see, shocks appear even when $N = 1$ in a continuous case, when the initial conditions are chosen to lie on the hysteresis curves. The argument of Peszyńska and Showalter does not work for, e.g., discontinuous hysteresis, so other techniques have to be applied.

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