International Workshop on Multi-Rate Processes and Hysteresis IOP Publishing
Journal of Physics: Conference Series **138** (2008) 012005 doi:10.1088/1742-6596/138/1/012005

On a parabolic equation with hysteresis and convection: a uniqueness result

M Eleuteri[1] **and J Kopfová**[2]

[1] Dipartimento di Matematica di Trento, via Sommarive 14, 38100 Povo (Trento), Italy
[2] Mathematical Institute of the Silesian University, Na Rybníčku 1, 746 01 Opava, Czech Republic

E-mail: `eleuteri@science.unitn.it, jana.kopfova@math.slu.cz`

Abstract. A uniqueness result for a parabolic partial differential equation with hysteresis and convection is established. This equation is a part of a model system which describes the magnetohydrodynamic (MHD) flow of a conducting fluid between two ferromagnetic plates. The result of this paper complements the content of [6], where existence of the solution has been proved under fairly general assumptions on the hysteresis operator and the uniqueness was obtained only for a restricted class of hysteresis operators

1. Introduction

In this paper we deal with the following model equation

$$\frac{\partial}{\partial t}(u + \mathcal{W}[u]) + \mathbf{v} \cdot \nabla (u + \mathcal{W}[u]) - \triangle u = 0 \qquad \text{in } \Omega \times (0,T) \tag{1}$$

coupled with homogeneous Dirichlet boundary conditions, where Ω is an open bounded set of $\mathbb{R}^2$, $\triangle$ is the Laplace operator, $\mathcal{W}$ is a Preisach hysteresis operator and $\mathbf{v} : \Omega \times (0,T) \to \mathbb{R}^2$ is known. This equation is a part of the following model system

$$\begin{cases} \dfrac{\partial b}{\partial t} + \mathbf{v} \cdot \nabla b - \triangle u = 0\,, \\ \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - \triangle \mathbf{v} + b\,\nabla u + \nabla p = 0\,, \\ \operatorname{div} \mathbf{v} = 0\,, \\ b = u + \mathcal{W}[u], \end{cases} \tag{2}$$

in which equation (1) is coupled with a momentum equation of Navier-Stokes kind for the velocity field $\mathbf{v}$. Here u is the non-zero component of the magnetic field after some suitable assumptions on the geometry of the model, $\mathbf{v}$ is the velocity of the fluid and p the pressure. System (2), which has been derived in detail in [9], represents a model for MHD flow of a conducting fluid between two ferromagnetic plates.

Equation (1) has been studied in [6] (see also [4], Chapter 3 and [5]), where the existence of the solution has been proved under fairly general assumptions on the hysteresis operator; the uniqueness was obtained only for a restricted class of hysteresis operators. Here we want to extend this result by deriving a more general uniqueness result (and complementary to this the existence) under some suitable restrictions on the initial data.

We follow a general idea used to prove existence and uniqueness for the complete system (2) (contained in the paper [8]). We first deal with a time discrete scheme with a convexified Preisach operator under the time derivative and a cut-off Preisach operator in the remaining hysteresis term. The key point is to get enough regularity for the solution to (1) to be able to apply a discrete version of the Moser iteration lemma, which will bring the desired uniqueness result. We present here only the main points of the proof of our results; further details can be found in [7].

2. Hysteresis operators

2.1. Some remarks concerning hysteresis operators

2.1.1. The play operator We briefly recall the definition and some properties of the *play* operator, which is the simplest example of a continuous hysteresis operator. It is defined as the mapping that with a given input function $u \in W^{1,1}(0,T)$, a parameter $r > 0$, and an initial condition $x_r^0 \in [-r, r]$, associates the solution $\xi_r \in W^{1,1}(0,T)$ of the variational inequality

$$\begin{array}{lll} (i) & |u(t) - \xi_r(t)| \le r & \forall t \in [0,T], \\ (ii) & (\dot{\xi}_r(t))\,(u(t) - \xi_r(t) - y) \ge 0 & \text{a.e.} \ \ \forall\, y \in [-r, r]\,, \\ (iii) & \xi_r(0) = u(0) - x_r^0\,, & \end{array} \tag{3}$$

see [10, 13], and we denote, for $r > 0$, $\mathcal{P}_r[x_r^0, u] : [-r, r] \times W^{1,1}(0,T) \to W^{1,1}(0,T) : (x_r^0, u) \mapsto \xi_r$. It was shown in [3, Theorem 2.7.7] that the whole class of the so-called *Preisach type hysteresis operators* (also called *operators with return point memory* in engineering literature) can be represented by the one-parametric family of play operators $\{\mathcal{P}_r; r > 0\}$. Following [10, Section II.2], we introduce the *configuration space* as well as its subspaces

$$\Lambda := \left\{ \lambda \in W^{1,\infty}(0,\infty);\ \left| \frac{d\lambda(r)}{dr} \right| \le 1 \ \text{ a.e.} \right\},$$

$$\Lambda_K := \{\lambda \in \Lambda; \lambda(r) = 0 \ \text{ for } r \ge K\}, \qquad \Lambda_0 := \bigcup_{K>0} \Lambda_K. \tag{4}$$

Elements $\lambda \in \Lambda$ are called *memory configurations*. For a given $\lambda \in \Lambda$, it is convenient to define the initial condition x_r^0 by the formula $x_r^0 := Q_r(u(0) - \lambda(r))$, where $Q_r : \mathbb{R} \to [-r, r]$ is the projection $Q_r(x) := \operatorname{sign}(x) \min\{r, |x|\} = \min\{r, \max\{-r, x\}\}$. Then λ is called the *initial configuration* of the play system, and we define for $r > 0$ a mapping $\wp_r : \Lambda \times W^{1,1}(0,T) \to W^{1,1}(0,T)$ by the formula $\wp_r[\lambda, u] := \mathcal{P}_r[x_r^0, u]$.

2.1.2. The Preisach operator We briefly recall here the definition and some properties of the *Preisach operator.* In the *Preisach half-plane*

$$\mathbb{R}^2_+ = \{(r, v) \in \mathbb{R}^2 : r > 0\}, \tag{5}$$

we assume that a function $\psi \in L^1_{\text{loc}}(\mathbb{R}^2_+)$ (the *Preisach density*) is given with the following property.

Assumption 2.1. There exists $\beta_1 \in L^1_{\text{loc}}(0,\infty)$, such that

$$0 \le \psi(r, v) \le \beta_1(r) \quad \textit{for a. e.} \quad (r, v) \in \mathbb{R}^2_+\,.$$

We put

$$\tilde{b}_1(K) := \int_0^K \beta_1(r)\, dr \qquad \text{for } K > 0 \qquad g(r,v) := \int_0^v \psi(r,z)\, dz \qquad \text{for } (r,v) \in \mathbb{R}^2_+, \tag{6}$$

and define the Preisach operator as follows.

Definition 2.2. *Consider $\psi \in L^1_{\rm loc}(\mathbb{R}^2_+)$ satisfying Assumption 2.1 and g as in (6). Then the Preisach operator $\mathcal{W} : \Lambda_0 \times G_+(0,T) \to G_+(0,T)$ generated by the function g is defined by*

$$\mathcal{W}[\lambda, u](t) := \int_0^\infty g(r, \wp_r[\lambda, u](t))\, dr = \int_0^\infty \int_0^{\wp_r[\lambda,u](t)} \psi(r,z)\, dz\, dr \tag{7}$$

for any given $\lambda \in \Lambda_0$, $u \in G_+(0,T)$ and $t \in [0,T]$, where Λ_0 is introduced in (4) and $G_+(0,T)$ is the space of right-continuous regulated functions.

As a counterpart of [10], Section II.3, Proposition 3.11, we have the following estimate.

Proposition 2.3. *Let Assumption 2.1 be satisfied and let $K > 0$ be given. Then for every $\lambda_1, \lambda_2 \in \Lambda_K$ and $u, v \in G_+(0,T)$ such that $||u||_{[0,T]}$, $||v||_{[0,T]} \le K$, we have*

$$|\mathcal{W}[\lambda_1, u](t) - \mathcal{W}[\lambda_2, v](t)| \le \int_0^K |\lambda_1(r) - \lambda_2(r)|\, \beta_1(r)\, dr \,+\, \tilde{b}_1(K)\, ||u - v||_{[0,t]} \quad \forall t \in [0,T]\,.$$

We finally quote the following result (see [10, Proposition II.4.13]) which will be used in Subsection 3.8 to establish the uniqueness of the solution to our model problem.

Proposition 2.4. *Let $\mathcal{W}$ be a Preisach operator (7) satisfying Assumption 2.1. For given $u_1, u_2 \in W^{1,1}(0,T)$ and $\lambda_1, \lambda_2 \in \Lambda_0$ put $\xi^i_r := \wp_r[\lambda_i, u_i]$, $w_i := \mathcal{W}[\lambda_i, u_i] = \int_0^\infty g(r, \xi^i_r)\, dr$, $i = 1, 2$. Then for a.e. $t \in (0,T)$ we have*

$$(\dot{w}_1(t) - \dot{w}_2(t))\,(u_1(t) - u_2(t)) \ge \int_0^\infty (\xi^1_r(t) - \xi^2_r(t))\, \frac{\partial}{\partial t}(g(r, \xi^1_r(t)) - g(r, \xi^2_r(t)))\, dr. \tag{8}$$

3. Convexification and cut-off

3.1. The convexity domain of the Preisach operator

Let $R > 0$ be fixed; set

$$\mathscr{D}_R := \{(r,v) \in \mathbb{R}^2_+ :\, |v| + r \le\, R\}.$$

In addition to Assumption 2.1 we prescribe the following conditions.

Assumption 3.1.

$$(i) \quad \frac{\partial \psi}{\partial v} \in L^\infty_{\rm loc}(\mathbb{R}^2_+);$$
$$(ii) \quad A_R := \inf\{\psi(r,v);\, (r,v) \in \mathscr{D}_R\} > 0.$$

Furthermore, denote

$$C_R := \sup\left\{ \left| \frac{\partial}{\partial v} \psi(r,v) \right| ;\, (r,v) \in \mathscr{D}_R \right\}.$$

Taking possibly a smaller $R > 0$, if necessary, we may assume that $K_R := \frac{1}{2}\, A_R - R\, C_R > 0$. We modify the density ψ outside $\mathscr{D}_R$ by setting

$$\psi_R(r,v) = \begin{cases} \psi(r,v) & (r,v) \in \mathscr{D}_R \\ \psi(r, -R + r) & v < -R + r,\ r \le\, R \\ \psi(r, R - r) & v > R - r,\ r \le R \\ \psi(R, 0) & r > R. \end{cases} \tag{9}$$

Journal of Physics: Conference Series **138** (2008) 012005 doi:10.1088/1742-6596/138/1/012005

3.2. Convexification

We define a new Preisach operator $\mathcal{W}_R$ by the formula

$$\mathcal{W}_R[\lambda, u](t) = \int_0^\infty \int_0^{\wp_r[\lambda,u](t)} \psi_R(r,v)\,dv\,dr \tag{10}$$

for $\lambda \in \Lambda_0$ and $u \in W^{1,1}(0,T)$. It has the property that all increasing hysteresis branches are convex and all decreasing branches are concave, see (25). This plays an important role in the higher order energy inequalities.

3.3. Cut-off

We also introduce the cut-off density

$$\widetilde{\psi}_R(r,v) = \begin{cases} \psi(r,v) & (r,v) \in \mathscr{D}_R \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

and the corresponding cut-off operator

$$\widetilde{\mathcal{W}}_R[\lambda, u](t) = \int_0^\infty \int_0^{\wp_r[\lambda,u](t)} \widetilde{\psi}_R(r,v)\,dv\,dr. \tag{12}$$

Remark 3.2. We remark that $\mathcal{W}_R$ is convex but non globally bounded while $\widetilde{\mathcal{W}}_R$ is globally bounded but non convex. In the hysteresis terms that will remain on the left hand side when developing the estimates in Subsections 3.3 - 3.5, we need to use the discrete version of $\mathcal{W}_R$ as we have to exploit the convexity of the loops and the corresponding second order energy inequality; in the hysteresis terms that will be in the right-hand side, we use the discrete version of $\widetilde{\mathcal{W}}_R$, as we need instead to establish a global bound. This motivates the introduction of both the operators (10) and (12).

4. The main result

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitzian boundary, and set $\Omega_T := \Omega \times (0,T)$. We set $V := W_0^{1,2}(\Omega)$, and introduce the spaces of divergence free functions

$$\mathbf{H} := \left\{ \mathbf{u} \in L^\infty(\Omega;\mathbb{R}^2); \int_\Omega \mathbf{u}(x)\cdot\nabla\phi(x)\,dx = 0 \quad \forall \phi \in V \right\} \qquad \mathscr{H} := L^\infty(0,T;\mathbf{H}).$$

We propose to solve the following problem.

Problem 4.1. *Consider a Preisach operator $\mathcal{W}$ of the form (7), and let $u_0 \in L^2(\Omega)$, $\mathbf{v}_0 \in \mathbf{H}$, and $\lambda : \Omega \to \Lambda$ be given initial data. For given $\mathbf{v} \in L^2(0,T;\mathbf{H})$ we search for a function u with appropriate regularity, such that*

$$u(x,0) = u_0(x) \quad \text{a.e. in} \quad \Omega \tag{13}$$

and for any $\phi \in V$ and for a. e. $t \in (0,T)$ we have

$$\int_\Omega \frac{\partial}{\partial t}(u + \mathcal{W}[\lambda,u])\,\phi\,dx - \int_\Omega \mathbf{v}\cdot\nabla\phi\,(u + \mathcal{W}[\lambda,u])\,dx + \int_\Omega \nabla u\cdot\nabla\phi\,dx = 0. \tag{14}$$

The main result can be stated as follows.

Theorem 4.2. *Let $\mathcal{W}$ be the Preisach operator satisfying Assumptions* 2.1 *and* 3.1*, and let $R > 0$ be fixed as in Subsection* 3.2*. Let $K \in [0, R]$ and $\lambda : \Omega \to \Lambda_K$ be given. Let the data have the regularity*

$$u_0 \in V, \qquad \triangle u_0 \in L^2(\Omega), \quad \mathbf{v} \in \mathscr{H}, \quad \mathbf{v}_t \in \mathscr{H} \tag{15}$$

and set

$$\alpha := \max \left\{ ||u_0||_V, ||\triangle u_0||_{L^2(\Omega)}, ||\mathbf{v}||_{\mathscr{H}}, ||\mathbf{v}_t||_{\mathscr{H}} \right\}. \tag{16}$$

Then there exists $\alpha_1 > 0$ such that if $\alpha \le \alpha_1$, then Problem 4.1 *has a unique solution u with the regularity*

$$u \in \mathcal{C}^0(\bar{\Omega}_T) \qquad \nabla u_t \in L^2(\Omega_T; \mathbb{R}^2) \qquad u_t \in L^\infty(\Omega_T). \tag{17}$$

5. Proof of Theorem 4.2

5.1. The discrete problem

Let us fix some $m \in \mathbb{N}$; then define the time step $\tau = \dfrac{T}{m}$. We consider for $k = 1, \ldots, m$ and for any $\phi \in V$ the following recurrent system

$$\int_\Omega \frac{u_k - u_{k-1}}{\tau}\, \phi \, dx + \int_\Omega \frac{w_k - w_{k-1}}{\tau}\, \phi \, dx - \int_\Omega [\mathbf{v}_k \cdot \nabla \phi]\, b_{k-1} \, dx + \int_\Omega \nabla u_k \cdot \nabla \phi \, dx = 0, \tag{18}$$

where:

- $\mathbf{v}_k$ is defined by $\mathbf{v}_k(x) = \dfrac{1}{\tau} \displaystyle\int_{(k-1)\,\tau}^{k\,\tau} \mathbf{v}(x,t)\, dt$

- b_k satisfies the equation

$$b_k(x) = u_k(x) + \widetilde{w}_k(x), \qquad \text{a.e. in } \Omega, \quad k = 0, \ldots, m, \tag{19}$$

with homogeneous Dirichlet boundary conditions, with

$$w_k(x) = \int_0^\infty g_R(r, \xi_k(x,r))\, dr, \qquad \widetilde{w}_k(x) = \int_0^\infty \widetilde{g}_R(r, \xi_k(x,r))\, dr, \tag{20}$$

and

$$g_R(r,v) = \int_0^v \psi_R(r,v')\, dv', \qquad \widetilde{g}_R(r,v) = \int_0^v \widetilde{\psi}_R(r,v')\, dv'. \tag{21}$$

Moreover ψ_R, $\widetilde{\psi}_R$ are the functions introduced in (9), (11) respectively and the sequence ξ_k is defined recursively by

$$\xi_0(x,r) := P[\lambda(x,\cdot), u_0(x)](r), \qquad \xi_k(x,r) := P[\xi_{k-1}(x,\cdot), u_k(x)](r), \tag{22}$$

with $P : \Lambda \times \mathbb{R} \to \Lambda$ defined as

$$P[\lambda, v](r) := \max\{v - r, \min\{v + r, \lambda(r)\}\}. \tag{23}$$

The solution to (18) can be constructed by induction over k, using the Browder-Minty fixed point theorem and the monotonicity of the mappings $g(r,\cdot)$ and $P[\lambda,\cdot]$ (a similar argument has been employed in [8], Section 4.2).

5.2. A discrete first order energy inequality

We recall here a discrete counterpart of the first order energy inequality presented in [10], Section II.4, which is proved in detail in [8].

Setting $\xi_k^r(x) := \xi_k(x,r)$ where $\xi_k(x,r)$ has been introduced in (22), assuming ψ is an arbitrary function satisfying Assumption 2.1, the discrete version of the first order energy inequality can be stated as follows.

$$(w_k - w_{k-1})\, u_k - (E_k - E_{k-1}) \geq \int_0^\infty \int_{\xi_{k-1}^r}^{\xi_k^r} r\,\psi(r,v)\,dv\,dr = |S_k - S_{k-1}|, \tag{24}$$

where

$$E_k(x) = \int_0^\infty G(r, \xi_k^r(x))\,dr, \qquad S_k(x) = \int_0^\infty r\,g(r,\xi_k^r(x))\,dr\,,$$

with G given by $G(r,v) := v\,g(r,v) - \int_0^v g(r,z)\,dz = \int_0^v z\,\psi(r,z)\,dz$, are the discrete versions of the Preisach potential energy $\mathcal{E}$ and dissipation operator $\mathcal{S}$, introduced in [10], Section II.4.

5.3. A discrete second order energy inequality

Let $p \geq 2$ be arbitrary and set $F_k = U_k\,|U_k|^{p-2}$, where $U_k := \frac{u_k - u_{k-1}}{\tau}$ in agreement with the notations we will introduce later in (27). We recall here a discrete version of the second order energy inequality which can be stated as follows: For every $k = 2, \ldots, n$, $n \in \{1, \ldots, m\}$ and a. e. $x \in \Omega$

$$(W_k - W_{k-1})\,F_k \geq \frac{1}{p}\,(W_k F_k - W_{k-1} F_{k-1})\,. \tag{25}$$

A detailed proof can be found in Section 6.2 of [8]; the time continuous case with $p = 2$ is treated in [10, Sections II.3 and II.4].

5.4. First a priori estimate

In the estimates below, we denote for simplicity with C every constant independent of α and τ; the value of C may vary from line to line. We choose $\phi = u_k$ in (18). This yields

$$\int_\Omega \frac{u_k - u_{k-1}}{\tau}\,u_k\,dx + \int_\Omega \frac{w_k - w_{k-1}}{\tau}\,u_k\,dx + \int_\Omega |\nabla u_k|^2\,dx \leq \int_\Omega (\mathbf{v}_k \cdot \nabla u_k)\,b_{k-1}\,dx$$

$$\overset{(15)}{\leq} \alpha^2 \int_\Omega |b_{k-1}|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u_k|^2\,dx \leq \alpha^2 \int_\Omega (|u_{k-1}|^2 + 1)\,dx + \frac{1}{2}\int_\Omega |\nabla u_k|^2\,dx;$$

in the last inequality we used the pointwise estimate $|\tilde{w}_{k-1}(x)| \leq C$, which follows from the definition of the Preisach operator. Therefore, using (24) and summing for $k = 1, \ldots, n$, for every $n \in \{1, \ldots, m\}$ we obtain

$$\int_\Omega |u_n|^2\,dx + \tau \sum_{k=1}^n \int_\Omega |\nabla u_k|^2\,dx \leq C \int_\Omega |u_0|^2 dx + \int_\Omega E_0\,dx + \sum_{k=1}^n \int_\Omega |u_k|^2\,dx + \alpha^2.$$

Using a discrete Gronwall argument it follows that

$$\max_{n=1,\ldots,m} \int_\Omega |u_n|^2\,dx + \tau \sum_{k=1}^m \int_\Omega |\nabla u_k|^2\,dx \leq C\,\alpha^2. \tag{26}$$

5.5. Estimate of the initial condition

In equation (18) corresponding to $k = 1$ we choose $\phi := \frac{u_1 - u_0}{\tau}$. Due to the monotonicity and local Lipschitz continuity of the functions $g(r, \cdot)$ and $P[\lambda, \cdot](r)$, we have the pointwise inequality

$$\left(\frac{u_1 - u_0}{\tau}\right)\left(\frac{w_1 - w_0}{\tau}\right) \geq 0 \quad \forall x \in \Omega.$$

Using this and the assumptions on the function $\mathbf{v}$, we deduce

$$\int_\Omega \left|\frac{u_1 - u_0}{\tau}\right|^2 dx + \tau \int_\Omega \left|\nabla\left(\frac{u_1 - u_0}{\tau}\right)\right|^2 dx \leq \int_\Omega |\triangle u_0|^2\, dx + C \int_\Omega |\nabla b_0|^2\, dx \ \leq\ C\,\alpha^2.$$

5.6. Second a priori estimate

We set for brevity

$$U_k := \frac{u_k - u_{k-1}}{\tau}, \quad B_k := \frac{b_k - b_{k-1}}{\tau} \quad W_k := \frac{w_k - w_{k-1}}{\tau}. \tag{27}$$

We take the time increments in (18) and then test by U_k. We get

$$\begin{aligned} &\int_\Omega (U_k - U_{k-1})\, U_k\, dx + \int_\Omega (W_k - W_{k-1})\, U_k\, dx \\ - \quad &\tau \int_\Omega (\mathbf{V}_k \cdot \nabla U_k)\, b_{k-1} - \tau \int_\Omega (\mathbf{v}_{k-1} \cdot \nabla U_k)\, B_{k-1}\, dx + \tau \int_\Omega |\nabla U_k|^2\ dx = 0. \end{aligned}$$

Using (25) for $p = 2$, we have, for $k \geq 2$

$$\frac{1}{2}\int_\Omega [(U_k + W_k)U_k - (U_{k-1} + W_{k-1})U_{k-1}]\, dx + \frac{\tau}{2}\int_\Omega |\nabla U_k|^2\, dx \leq C\,\tau \int_\Omega (|U_{k-1}|^2 + |u_{k-1}|^2)\, dx,$$

where we used also Assumption (15). At this point we sum up for $k = 1, ..., n$, $n \in \{1, ..., m\}$, we use the estimate on the initial condition and apply a discrete version of the Gronwall lemma, to get

$$\max_{n=1,\ldots,m} \int_\Omega |U_n|^2 dx + \tau \sum_{k=1}^{m} \int_\Omega |\nabla U_k|^2\, dx \leq C\,\alpha^2. \tag{28}$$

5.7. Third a priori estimate

In this subsection will establish a further a priori estimate for ∇u_k; we will use Theorem 6.1 from Section 6.1. Set, for any $\phi \in V$

$$F(\phi) = -\int_\Omega (U_k + W_k)\,\phi\, dx + \int_\Omega [\mathbf{v}_k \cdot \nabla\phi]\, b_{k-1}\, dx.$$

Now take any $q > 2$ and $\phi \in W_0^{1,q'}(\Omega)$. Using the Sobolev embeddings and the fact that $b_{k-1} \in L^p(\Omega)$, for every $p \geq 2$ (this can be seen using the fact that $u_k \in L^p(\Omega)$ from (26) and $\widetilde{w}_k \in L^p(\Omega)$ directly by the definition of $\widetilde{w}_k$), we have the following estimate

$$|F(\phi)| \leq \int_\Omega \left(|U_k|(1 + \max_{j=1,\ldots,k} |u_j|)\,|\phi| + |\mathbf{v}_k|\,|b_{k-1}|\,|\nabla\phi|\right) dx \leq\ C_F\,\alpha^2\,||\phi||_{W^{1,q'}(\Omega)},$$

with a constant C_F dependent on q. This is equivalent to

$$||F||_{W^{-1,q}(\Omega)} \leq\ C_F\,\alpha^2.$$

Thus, as we proved that $F \in W^{-1,q}(\Omega)$, we can use Theorem 6.1 from Section 6.1 to obtain the following estimate

$$||\nabla u_k||_{L^q(\Omega)} \leq\ C\,\alpha^2. \tag{29}$$

5.8. An L^∞−bound for the solution: applying the Moser iteration technique
We apply the Gagliardo-Nirenberg inequality (see e.g. [1]) to deduce

$$||U_{k-1}||_{L^4(\Omega)} \leq C. \tag{30}$$

Set $\mathbf{M}_k := \mathbf{V}_k\, b_{k-1} + \mathbf{v}_{k-1}\, B_{k-1}$. Due to the assumptions on the function $\mathbf{v}$, we have that

$$\tau \sum_{k=1}^{n} \int_\Omega |\mathbf{M}_k|^q\, dx \leq C, \tag{31}$$

with $q > 2$. We take the time increments in (18) and test by $U_k\, |U_k|^{p-2}$ for any $p \geq 2$. Using the monotonicity of the functions g and P (see (6) and (23)), we get

$$\begin{aligned} & \frac{1}{p} \max_{k=1,\ldots,n} \int_\Omega |U_k|^p\, dx + \tau\,(p-1) \sum_{k=1}^{n} \int_\Omega |\nabla U_k|^2\, |U_k|^{p-2}\, dx \\ \leq\ & \frac{1}{p} \int_\Omega |U_0|^p\, dx + C\,\tau\,(p-1) \sum_{k=1}^{n} \int_\Omega |\mathbf{M}_k|\, |\nabla (U_k\, |U_k|^{p-2})|\, dx \\ \leq\ & \frac{1}{p} C^p + C\,\tau\, \frac{(p-1)}{2} \int_\Omega |\nabla U_k|^2\, |U_k|^{p-2}\, dx + C\,\tau\,(p-1) \int_\Omega |\mathbf{M}_k|^2\, |U_k|^{p-2}\, dx. \end{aligned} \tag{32}$$

Using (31) with $q = 4$, we deduce

$$\max_{k=1,\ldots,n} \int_\Omega |U_k|^p\, dx + \tau \sum_{k=1}^{n} \int_\Omega |\nabla U_k|^2\, |U_k|^{p-2}\, dx \leq C \left(1 + \tau \sum_{k=1}^{n} \int_\Omega |U_k|^{2p-4} \right).$$

At this point it is easy to prove the following implication

$$\tau \sum_{k=1}^{n} \int_\Omega |U_k|^{2p-2}\, dx \leq C \quad \Rightarrow \quad \tau \sum_{k=1}^{n} \int_\Omega |U_k|^{2p}\, dx \leq C,$$

which yields, together with (30), that (31) holds with $q > 4$. Now we come back to (32) and set

$$Z_p^{(k)} := U_k\, |U_k|^{\frac{p}{2}-1} \text{ so that } |Z_p^{(k)}|^2 = |U_k|^p \text{ and } |\nabla Z_p^{(k)}|^2 = |\nabla U_k|^2\, |U_k|^{p-2}\, \frac{p^2}{4}. \tag{33}$$

This in the notation (33) implies

$$\max_{k=1,\ldots,n} \int_\Omega |Z_p^{(k)}|^2\, dx + \tau \sum_{k=1}^{n} \int_\Omega |\nabla Z_p^{(k)}|^2\, dx \leq C^p + C\,\tau\, p^2 \sum_{k=1}^{n} \int_\Omega \left[|\mathbf{M}_k|^2\, |Z_p^{(k)}|^{2\frac{p-2}{p}} \right] dx.$$

A combined use of the Gagliardo-Nirenberg, Hölder and generalized Young inequalities yield (here we need (31) with $q > 4$)

$$\left(\tau \sum_{k=1}^{n} \int_\Omega |Z_p^{(k)}|^4\, dx \right)^{1/2} \leq C \left(C^p + C\,\tau\, p^3 \sum_{k=1}^{n} ||Z_p^{(k)}||^2_{L^{2q'}(\Omega)} \right). \tag{34}$$

We choose

$$p = (1+\kappa)^j\, q \qquad \kappa = \frac{2}{q'} - 1 \qquad X_j := \tau \sum_{k=1}^{j} \int_\Omega |U_k|^{2q\,(1+\kappa)^j}\, dx. \tag{35}$$

We have

$$\left(\tau\sum_{k=1}^{n}\int_\Omega \left|Z_p^{(k)}\right|^4\,dx\right)^{1/2} =: X_j^{1/2} \qquad \tau\sum_{k=1}^{n}\left|\left|Z_p^{(k)}\right|\right|^2_{L^{2q'}(\Omega)} =: X_{j-1}^{1/q'}.$$

From (34) we therefore deduce

$$X_j^{1/2} \le C\,\max\left\{C^{(1+\kappa)^j},(1+\kappa)^j\,X_{j-1}^{1/q'}\right\}.$$

This inequality is the first step from which we can start the application of the Moser iteration technique (see Lemma 5.6 of Chapter II of [12]). We have

$$X_j^{\frac{1}{2q(1+\kappa)^j}} \quad\le\quad C^{\frac{1}{(1+\kappa)^j}}\,\max\{C,(1+\kappa)^{\frac{j}{q(1+\kappa)^j}}\,X_{j-1}^{\frac{1}{2q\,(1+\kappa)^{j-1}}}\}.$$

We set

$$Y_j := X_j^{\frac{1}{2q\,(1+\kappa)^j}} \qquad \delta_j := \frac{j}{q\,(1+\kappa)^j},$$

from which we deduce (after have taken the logarithm of both members of the inequality)

$$\max\{\log C,\log Y_j\} \le \frac{1}{(1+\kappa)^j}\log C + \delta_j\,\log(1+\kappa) + \max\{\log C,\log Y_{j-1}\}.$$

We set for brevity $a_j := \max\{\log C,\log Y_j\}$, getting

$$a_j \le \frac{1}{(1+\kappa)^j}\log C + \delta_j\,\log(1+\kappa) + a_{j-1} \le\; a_0 + \log C\sum_{\ell=1}^{j}\frac{1}{(1+\kappa)^\ell} + \log(1+\kappa)\sum_{\ell=1}^{j}\delta_\ell.$$

As long as $\sum_{\ell=1}^\infty \delta_\ell < \infty$ and $\sum_{\ell=1}^\infty \frac{1}{(1+\kappa)^\ell} < \infty$ for $\kappa > 0$ (the fact that $q > 2$ assures that $q' < 2$ and therefore that $\kappa > 0$) we finally obtain $a_j \le C$, for all j, with constant C independent of j. This implies that

$$\left(\tau\sum_{k=1}^{n}\int_\Omega |U_k|^{2p}\,dx\right)^{\frac{1}{2p}} \le C, \tag{36}$$

and after letting $p \to \infty$ we obtain the bound

$$\sup_{k=1,\dots,n} ||U_k||_{L^\infty(\Omega)} \le C, \qquad n\in\{1,\dots,m\}. \tag{37}$$

5.9. Passage to the limit

For each fixed time step τ, we associate with the sequence $\{u_k\}$ constructed above their piecewise linear and piecewise constant time interpolates according to the following scheme:

$$\left.\begin{aligned} \bar u_+^{(\tau)}(x,t) &= u_k(x), & \bar w_+^{(\tau)}(x,t) &= w_k(x),\\ \bar u_-^{(\tau)}(x,t) &= u_{k-1}(x), & \bar w_-^{(\tau)}(x,t) &= \widetilde w_{k-1}(x), \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} \hat u^{(\tau)}(x,t) &= u_{k-1}(x) + \tfrac{t-(k-1)\tau}{\tau}(u_k(x)-u_{k-1}(x))\\ \hat w^{(\tau)}(x,t) &= w_{k-1}(x) + \tfrac{t-(k-1)\tau}{\tau}(w_k(x)-w_{k-1}(x))\\ \bar b^{(\tau)}(x,t) &= \bar u_-^{(\tau)}(x,t) + \bar w_-^{(\tau)}(x,t) \end{aligned}\right\} \tag{39}$$

for $x \in \Omega$ and $t \in [(k-1)\tau, k\tau)$, $k = 1, 2, \ldots, m$, continuously extended to $t = T$. We have

$$\bar{w}_{+}^{(\tau)} = \mathcal{W}_R[\lambda, \bar{u}_{+}^{(\tau)}], \qquad \bar{w}_{-}^{(\tau)} = \widetilde{\mathcal{W}}_R[\lambda, \bar{u}_{-}^{(\tau)}]. \tag{40}$$

As a consequence of the estimates (26) and (28), we see that there exists a function $u \in L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$, such that $u_t \in L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$, and, along a subsequence as $\tau \to 0$,

$$\left.\begin{array}{lllll} \hat{u}^{(\tau)} & \to & u & \text{weakly star in} & L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)), \\ \hat{u}_t^{(\tau)} & \to & u_t & \text{weakly star in} & L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)). \end{array}\right\} \tag{41}$$

On the other hand, from (29) we deduce, for any $p > 2$ (and thus for any $p > 1$)

$$\sup_{t \in (0,T)} \operatorname{ess} ||\nabla \bar{u}_{+}^{(\tau)}||_{L^p(\Omega; \mathbb{R}^2)} \leq \alpha^2 \, C_F^*, \tag{42}$$

with a constant $C_F^* > 0$ independent of τ. The space $\{z \in L^1(\Omega \times (0, T)); \nabla z \in L^{\infty}(0, T; L^p(\Omega; \mathbb{R}^2)),\, z_t \in L^{\infty}(0, T; L^2(\Omega))\}$ is compactly embedded in $\mathcal{C}^0(\bar{\Omega} \times [0, T])$. Hence, there exists a constant C_u such that

$$|\hat{u}^{(\tau)}(x, t)| \leq \alpha^2 \, C_u \qquad \forall \, (x, t) \in \bar{\Omega} \times [0, T]. \tag{43}$$

Hence, we have, passing again to a subsequence, if necessary,

$$\left.\begin{array}{lllll} \nabla \hat{u}^{(\tau)} & \to & \nabla u & \text{strongly in} & L^2(\Omega_T; \mathbb{R}^2), \\ \hat{u}^{(\tau)} & \to & u & \text{uniformly in} & \mathcal{C}^0(\bar{\Omega}_T). \end{array}\right\} \tag{44}$$

The passage to the limit in the hysteresis terms can be obtained as in [8], using Proposition 2.3 from Section 3 and the theory of right-continuous regulated functions $G_+(0, T)$ (which are functions $u : [0, T] \to \mathbb{R}$ which admit the left limit $u(t_-)$ at each point $t \in (0, T]$ and the right limit $u(t_+)$ exists and coincides with $u(t)$ at each point $t \in [0, T)$). At the end we are able to deduce that

$$\hat{w}^{(\tau)} \to w := \mathcal{W}_R[\lambda, u] \text{ strongly in } L^2(\Omega; G_+(0, T)). \tag{45}$$

At this point, the convergences (41), (44), (45) enable us to pass to the limit as $\tau \to 0$ and obtain (we set $\tilde{w} := \widetilde{\mathcal{W}}_R[\lambda, u]$)

$$\int_{\Omega} \left((u_t + w_t) \, \phi - (u + \tilde{w}) \, \mathbf{v} \cdot \nabla \phi + \nabla u \cdot \nabla \phi \right) \, dx = 0. \tag{46}$$

The L^{∞} bound (43) is preserved in the limit. Hence, choosing α sufficiently small, we obtain

$$|u(x, t)| \leq R, \qquad \text{a.e. in } \Omega_T.$$

Since $K \leq R$, it follows e. g. from [10, Lemma II.2.4] that the integration domain in (10) and (12) is contained in $\mathscr{D}_R$, hence the truncations in (9) and (11) never become active, and we have

$$w = \tilde{w} = \mathcal{W}[\lambda, u] \, .$$

Moreover, from (37) we deduce the following regularity for u

$$||u_t||_{L^{\infty}(\Omega_T)} \leq C. \tag{47}$$

This concludes the existence part for Theorem 4.2.

Journal of Physics: Conference Series **138** (2008) 012005 doi:10.1088/1742-6596/138/1/012005

5.10. Uniqueness

Suppose now by contradiction that Problem 4.1 admits two solutions u_1 and u_2. We write (14) first for u_1 then for u_2; then we choose $\phi = u_1 - u_2$. We set, for $i = 1, 2$

$$w_i = \mathcal{W}_R(u_i) := \int_0^\infty g_R(r, \wp_r[\lambda, u_i])\, dr \qquad \xi_r^i := \wp_r[\lambda, u_i],$$

with $g_R(r, v) := \int_0^v \psi_R(r, z)\, dz$ (ψ_R being introduced in (11)), for $(r, v) \in \mathbb{R}_+^2$ and we deduce

$$\int_\Omega \frac{\partial}{\partial t}(u_1 - u_2 + w_1 - w_2)\,(u_1 - u_2)\, dx - \int_\Omega [\mathbf{v} \cdot \nabla(u_1 - u_2)]\,(b_1(u_1) - b_2(u_2))\, dx$$
$$+ \int_\Omega |\nabla(u_1 - u_2)|^2\, dx = 0 \tag{48}$$

In order to deal with the hysteresis term under the ime derivative, we have to use Proposition 2.4 from Section 3 and the estimate (47); we notice that in (8) we can replace the integral $\int_0^\infty$ with $\int_0^R$ as it vanishes for $r > R$. The key point is the following identity

$$(\xi_r^1 - \xi_r^2)\, \frac{\partial}{\partial t}(g_R(r, \xi_r^1) - g_R(r, \xi_r^2)) = \frac{1}{2}\, \frac{\partial}{\partial t}\left[\psi_R(r, \xi_r^1)\, |\xi_r^1 - \xi_r^2|^2\right]$$
$$- \quad \frac{1}{2}\, \frac{\partial \psi_R}{\partial v}\, \frac{\partial \xi_r^1}{\partial t}\, |\xi_r^1 - \xi_r^2|^2 + \frac{\partial}{\partial t}\xi_r^2\, (\xi_r^1 - \xi_r^2)\, (\psi_R(r, \xi_r^1) - \psi_R(r, \xi_r^2)), \tag{49}$$

which gives, using estimate (47)

$$\frac{\partial}{\partial t}(w_1 - w_2)\,(u_1 - u_2) \;\geq\; \frac{1}{2}\,\frac{\partial}{\partial t}\int_0^R \psi_R(r, \xi_r^1)\, |\xi_r^1 - \xi_r^2|^2\, dr - \frac{3}{2}\, C \int_0^R |\xi_r^1 - \xi_r^2|^2\, dr.$$

Integrating in time, for $\tau \in (0, t)$, for any $t \in (0, T)$ we deduce (due to the causality of the hysteresis operator, the terms evaluated at $t = 0$ vanish)

$$\int_0^t \frac{\partial}{\partial t}(w_1 - w_2)\,(u_1 - u_2)\, ds \tag{50}$$
$$\geq \quad \tilde{C}_1\, |w_1(t) - w_2(t)|^2 + \tilde{C}_2 \int_0^R |\xi_r^1(t) - \xi_r^2(t)|^2\, dr - 2\,\tilde{C}_2 \int_0^t \int_0^R |\xi_r^1(s) - \xi_r^2(s)|^2\, dr\, ds,$$

where A_R has been introduced in Assumption 3.1.

We now deal with the remaining terms of (48).

$$\int_\Omega [\mathbf{v} \cdot \nabla(u_1 - u_2)]\,(b_1(u_1) - b_2(u_2))\, dx \quad \leq \quad \frac{1}{4}\int_\Omega |\nabla(u_1 - u_2)|^2\, dx + C\int_\Omega |u_1 - u_2|^2\, dx.$$

Therefore summing up we deduce in particular

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1 - u_2|^2\, dx + \int_\Omega \frac{\partial}{\partial t}(w_1 - w_2)(u_1 - u_2)\, dx + \int_\Omega |\nabla(u_1 - u_2)|^2\, dx \leq C\int_\Omega (|u_1 - u_2|^2 dx.$$

Integrating in time and using (50) we deduce (we once more use the causality of the hysteresis operator and the fact that the two solutions are supposed to have the same initial data) the uniqueness of the solution in a standard way by the application of the Gronwall lemma. This finishes the proof of Theorem 4.2. □

6. Appendix

6.1. Maximal regularity theorem

Theorem 6.1. *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$; for a given $1 < q < \infty$, let $F \in (W_0^{1,q'}(\Omega))^* = W^{-1,q}(\Omega)$ and let z be the solution of the Poisson equation $-\triangle z = F$ associated with homogeneous Dirichlet boundary conditions. Then we have the following estimate for ∇z*

$$||\nabla z||_{L^q(\Omega)} \leq C(q)\,||F||_{W^{-1,q}}$$

with a constant $C(q)$ only dependent on q.

Acknowledgment. The authors appreciate several interesting and stimulating discussions with Pavel Krejčí on this and related topics. This work was supported by the project MSM4781305904 of the Czech Ministry of Education.

References

[1] O. V. Besov, V. P. Il'in, S. M. Nikol'skii: *Integral representations of functions and embedding theorems*, Nauka, Moscow, 1975 (in Russian), English translation edited by Mitchell H. Taibleson. V. H. Winston & Sons, Washington, D.C.; Halsted Press [John Wiley & Sons], New York-Toronto, Ont.-London, 1978 (Vol. 1), 1979 (Vol. 2).

[2] H. Brezis: *Analyse fonctionnelle, Théorie et applications*, Masson, Paris (1983). (In French)

[3] M. Brokate, J. Sprekels: *Hysteresis and phase transitions*, Applied Mathematical Sciences, **121**, Springer-Verlag, New York (1996).

[4] M. Eleuteri: *"On some P.D.E.s with hysteresis"*, Ph.D. Thesis, University of Trento (2006).

[5] M. Eleuteri: *An existence result for a P.D.E. with hysteresis, convection and a nonlinear boundary condition*, Discrete and Continuous Dynamical Systems, supplement (2007), 344-353.

[6] M. Eleuteri: *Well posedness results for a class of parabolic partial differential equations with hysteresis*, NoDEA Nonlinear Differential Equations Appl., to appear.

[7] M. Eleuteri, J. Kopfová: *A uniqueness result for a class of parabolic equation with hysteresis*, in preparation.

[8] M. Eleuteri, J. Kopfová, P. Krejčí: *Magnetohydrodynamic flow with hysteresis*, WIAS Preprint No. 1319, 2008.

[9] M. Eleuteri, J. Kopfová, P. Krejčí: *On a model with hysteresis arising in magnetohydrodynamics*, Physica B: Condensed Matter, "Proceedings of the Sixth International Symposium on Hysteresis and Micromagnetic Modeling", **403**, no. 3, (2008), 448-450.

[10] P. Krejčí: *Hysteresis, convexity and dissipation in hyperbolic equations*, Tokyo: Gakkotosho (1996).

[11] P. Krejčí: *The Kurzweil integral and hysteresis*, Proceedings of MURPHYS'06, Cork, Ireland, Journal of Physics: Conference Series, **55** (2006), 144-154.

[12] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva: *Linear and quasilinear equations of parabolic type*, American Mathematical Society, 1968. (Russian edition: Nauka, Moskow, 1967).

[13] A. Visintin: *Differential models of hysteresis*, Springer (1994).