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MAGNETOHYDRODYNAMIC FLOW WITH HYSTERESIS*

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Abstract. We consider a model system describing the two-dimensional flow of a conducting fluid surrounded by a ferromagnetic solid under the influence of the hysteretic response of the surrounding medium. We assume that this influence can be represented by the Preisach hysteresis operator. Existence and uniqueness of solutions for the resulting system of PDEs with hysteresis nonlinearities is established in the convexity domain of the Preisach operator.

Key words. Preisach hysteresis operator, magnetohydrodynamics

AMS subject classifications. 76W05, 47J40

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1. Introduction.

The flow of a conducting fluid surrounded by a ferromagnetic solid is strongly influenced by the hysteretic response of the surrounding medium ([16, part G9]). We assume that this influence can be represented by the Preisach model, and we show below in section 3 that this assumption is in agreement with general thermodynamics. A similar problem was recently considered in [8], where, however, the typical hysteresis magnetization curve is approximated by two linear parts.

Principles of the magnetohydrodynamic (MHD) flow theory with linear relation between the magnetic field and magnetic induction are explained, e. g., in [9]. In order to take hysteretic effects in MHD into account, we consider the following problem, which has been derived in detail in [13], as a model for MHD flow of a conducting fluid between two ferromagnetic plates:

$$
\left.
\begin{aligned}
&\frac{\partial b}{\partial t} + \mathbf{v}\cdot\nabla b - \Delta u = 0,\\
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \Delta\mathbf{v} + b\,\nabla u + \nabla p = 0,\\
&\operatorname{div}\mathbf{v} = 0,\\
&b = u + \mathcal{W}[u]
\end{aligned}
\right\}
\tag{1.1}
$$

in $\Omega\times(0,T)$, coupled with initial conditions and homogeneous Dirichlet boundary conditions, with unknowns u (represents the magnetic field), b (magnetic induction), $\mathbf{v}$ (fluid velocity), and p (pressure), where Ω is an open bounded set in $\mathbb{R}^2$ with $\mathcal{C}^{1,1}$ boundary, and $\mathcal{W}$ is a Preisach hysteresis operator. All positive material constants are normalized to 1.

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The first equation in (1.1) for $\mathbf{v}$ fixed is studied in [12] (see also [11]), where existence of the solution is proved under fairly general assumptions on the hysteresis operator. Uniqueness and stable dependence on the data are established in the special case of the so-called *Prandtl–Ishlinskii operator* and suitable regularity assumptions on $\mathbf{v}$. The problem of obtaining this regularity in the coupled system (1.1) is difficult due to the occurrence of the hysteresis terms. On the one hand, hysteresis operators are not continuous with respect to weak or strong L^p topologies for $p < \infty$; on the other hand, they are not differentiable as mappings in function spaces and the chain rule does not hold. Therefore, a refined estimation technique using a new hysteresis energy inequality is necessary to obtain the desired bounds for the solution. This energy inequality holds only in the domain where all hysteresis loops are convex. This is not a restriction for Prandtl–Ishlinskii operators, which are globally convex. For a general Preisach operator, however, only small amplitude loops have this property. This is why we are able to construct the solution only for small initial data, which ensure that the solution does not leave the Preisach convexity domain.

The existence proof is based on a time discrete scheme with a convexified Preisach operator under the time derivative and a cut-off Preisach operator in the other two hysteresis terms. Uniform bounds enable us to pass to the limit using compact embeddings and check that the limit is a solution of the original problem. Under more regular initial data, we prove via a Moser iteration technique that the solution has sufficient regularity for uniqueness.

The text is organized as follows. In section 2, we recall some basic facts about the Preisach hysteresis model. The main results are stated in section 3; sections 4 and 5 are devoted to the proof of the existence and uniqueness theorem. The appendix contains some general results we use throughout the paper: the Gagliardo–Nirenberg inequality, a detailed derivation of the discrete first and second order energy inequalities for the Preisach operator, and a discrete Moser iteration lemma.

2. Hysteresis operators. Hysteresis is characterized (cf. [26]) by the *memory effect* and *rate independence*. To illustrate the meaning of these concepts, consider a system described by the input-output pair (u, w). The memory effect means that at any instant t the value of the output $w(t)$ is not simply determined by the value $u(t)$ of the input at the same instant but it depends also on the previous evolution of the input u. The rate independence means that the path $(u(t), w(t))$ is invariant with respect to any increasing time homeomorphism. On scalar monotone inputs, rate independent memory operators behave like usual superposition (Nemytskii) operators. Their generating functions are called *trajectories* of the hysteresis operators, and depend on the history of the process. Here, we substantially use the fact that trajectories corresponding to small amplitude oscillations form convex hysteresis loops.

A basic contribution to the theory of hysteresis has been brought by Krasnosel'skiĭ and his collaborators, summarized in the monograph [17]. In this fundamental work, they introduced the concept of hysteresis operator and started a systematic investigation of its properties. Since then, other monographs devoted to more special questions have been published; see, e. g., [1, 6, 10, 19, 22, 26].

2.1. The play operator. Now we briefly recall the definition and some properties of the *play* operator, which is the simplest example of a continuous hysteresis operator; see Figure 1. It is defined as the mapping that with a given input function $u \in W^{1,1}(0, T)$, a parameter $r > 0$, and an initial condition $x_r^0 \in [-r, r]$, associates

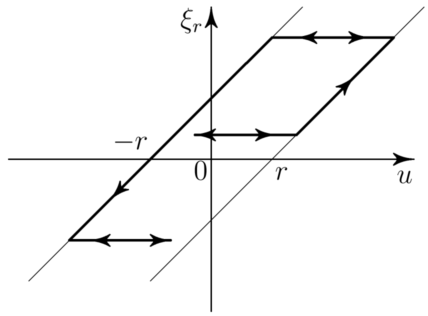

FIG. 1. *A diagram of the play operator.*

the solution $\xi_r \in W^{1,1}(0,T)$ of the variational inequality

$$\text{(2.1)}\qquad \begin{array}{lll} \text{(i)} & |u(t)-\xi_r(t)| \le r & \forall t \in [0,T], \\ \text{(ii)} & (\dot{\xi}_r(t))\,(u(t)-\xi_r(t)-y) \ge 0 & \text{a.e.}\ \ \forall\, y \in [-r,r]\,, \\ \text{(iii)} & \xi_r(0) = u(0) - x_r^0\,; & \end{array}$$

see [19, 26], and we denote for $r > 0$

$$\text{(2.2)}\qquad \mathcal{P}_r[x_r^0, u] : [-r,r] \times W^{1,1}(0,T) \to W^{1,1}(0,T) : (x_r^0, u) \mapsto \xi_r\,.$$

It was shown in [6, Theorem 2.7.7] that the whole class of the so-called *Preisach type hysteresis operators* (also called *operators with return point memory* in engineering literature) can be represented by the one-parametric family of play operators $\{\mathcal{P}_r; r > 0\}$. For given $u \in W^{1,1}(0,T)$, $t \in [0,T]$, and $x_r^0 \in [-r,r]$, the distribution of plays $r \mapsto \mathcal{P}_r[x_r^0, u](t)$ represents the state of the system at time t. Following [19, section II.2], we introduce the *configuration space*

$$\Lambda := \left\{ \lambda \in W^{1,\infty}(0,\infty);\ \left| \frac{\mathrm{d}\lambda(r)}{\mathrm{d}r} \right| \le 1 \ \text{ a.e.} \right\},$$

as well as its subspaces

$$\text{(2.3)}\qquad \Lambda_K := \{\lambda \in \Lambda; \lambda(r) = 0 \ \text{ for } r \ge K\}, \qquad \Lambda_0 := \bigcup_{K>0} \Lambda_K.$$

Elements $\lambda \in \Lambda$ are called *memory configurations.* For a given $\lambda \in \Lambda$, it is convenient to define the initial condition x_r^0 by the formula

$$x_r^0 := Q_r(u(0) - \lambda(r)),$$

where $Q_r : \mathbb{R} \to [-r,r]$ is the projection

$$\text{(2.4)}\qquad Q_r(x) := \operatorname{sign}(x) \min\{r, |x|\} = \min\{r, \max\{-r, x\}\}.$$

Then λ is called the *initial configuration* of the play system, and we define for $r > 0$ a mapping $\wp_r : \Lambda \times W^{1,1}(0,T) \to W^{1,1}(0,T)$ by the formula

$$\wp_r[\lambda, u] := \mathcal{P}_r[x_r^0, u]\,.$$

The reason for introducing the space Λ is that for every fixed $t \in [0,T]$ and $\lambda \in \Lambda$, the *state mapping* $r \mapsto \wp_r[\lambda, u](t)$ belongs to Λ.

In [20], the play operator has been extended to the space $G_+(0,T)$ of right-continuous regulated functions. This is the space of functions $u : [0,T] \to \mathbb{R}$ which admit the left limit $u(t_-)$ at each point $t \in (0,T]$, and the right limit $u(t_+)$ exists and coincides with $u(t)$ at each point $t \in [0,T)$. We define the seminorms

$$||u||_{[0,t]} \; = \; \sup\{|u(\tau)|\,;\, \tau \in [0,t]\} \quad \text{for } u \in G_+(0,T) \text{ and } t \in [0,T]. \tag{2.5}$$

Indeed, $||\cdot||_{[0,T]}$ is a norm and $G_+(0,T)$ endowed with this norm is a Banach space.

By Theorem 2.1 and Proposition 2.4 of [20], this extension is Lipschitz continuous in the sense that

$$|\wp_r[\lambda, u](t) - \wp_r[\mu, v](t)| \leq \; \max\left\{|\lambda(r) - \mu(r)|,\, \|u - v\|_{[0,t]}\right\}, \tag{2.6}$$

for any $\lambda, \mu \in \Lambda$, $u, v \in G_+(0,T)$, and $t \in [0,T]$. For an initial configuration $\lambda \in \Lambda$ and a step function $u \in G_+(0,T)$ of the form

$$u(t) = \sum_{k=1}^{m} u_{k-1}\, \chi_{[t_{k-1}, t_k)}(t) + u_m\, \chi_{\{T\}}(t), \tag{2.7}$$

where $0 = t_0 < t_1 < \cdots < t_m = T$ is a division of $[0,T]$, we have in particular

$$\wp_r[\lambda, u](t) = \sum_{k=1}^{m} \xi_{k-1}(r)\, \chi_{[t_{k-1}, t_k)}(t) + \xi_m(r)\, \chi_{\{T\}}(t), \tag{2.8}$$

where χ_ω is the characteristic function of a set $\omega \subset [0,T]$, and

$$\xi_0(r) = P[\lambda, u_0](r), \qquad \xi_k(r) = P[\xi_{k-1}, u_k](r), \tag{2.9}$$

with $P : \Lambda \times \mathbb{R} \to \Lambda$ defined as

$$P[\lambda, v](r) := \max\{v - r, \min\{v + r, \lambda(r)\}\} = Q_r(v - \lambda(r)). \tag{2.10}$$

2.2. The Preisach operator. We briefly recall here the definition and some properties of the *Preisach operator.* The construction presented here was introduced in [18] as an equivalent alternative to the classical model proposed in [23]. More information about the Preisach model can be found in [4, 5, 6, 7, 17, 19, 22, 25, 26, 27].

In the *Preisach half-plane*

$$\mathbb{R}^2_+ = \{(r, v) \in \mathbb{R}^2 : r > 0\}, \tag{2.11}$$

we assume that a function $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ (the *Preisach density*) is given with the following property.

Assumption 2.1. *There exists $\beta_1 \in L^1_{\mathrm{loc}}(0,\infty)$, such that*

$$0 \leq \psi(r, v) \leq \beta_1(r) \quad \textit{for a. e.} \quad (r, v) \in \mathbb{R}^2_+\,.$$

We put

$$\tilde{b}_1(K) := \int_0^K \beta_1(r)\, \mathrm{d}r \qquad \text{for } K > 0 \tag{2.12}$$

and

$$g(r,v) := \int_0^v \psi(r,z)\,\mathrm{d}z \qquad \text{for } (r,v) \in \mathbb{R}_+^2, \tag{2.13}$$

and define the Preisach operator as follows.

Definition 2.2. *Let $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ satisfying Assumption* 2.1 *be given and let g be as in* (2.13). *Then the Preisach operator $\mathcal{W} : \Lambda_0 \times G_+(0,T) \to G_+(0,T)$ generated by the function g is defined by the formula*

$$\mathcal{W}[\lambda,u](t) := \int_0^\infty g(r,\wp_r[\lambda,u](t))\,\mathrm{d}r = \int_0^\infty \int_0^{\wp_r[\lambda,u](t)} \psi(r,z)\,\mathrm{d}z\,\mathrm{d}r \tag{2.14}$$

for any given $\lambda \in \Lambda_0$, $u \in G_+(0,T)$, and $t \in [0,T]$, where Λ_0 is introduced in (2.3).

As a counterpart of [19, section II.3, Proposition 3.11], we obtain from (2.6) the following estimate.

Proposition 2.3. *Let Assumption* 2.1 *be satisfied and let $K > 0$ be given. Then for every $\lambda_1, \lambda_2 \in \Lambda_K$ and $u, v \in G_+(0,T)$ such that $||u||_{[0,T]}, ||v||_{[0,T]} \le K$, we have*

$$|\mathcal{W}[\lambda_1,u](t) - \mathcal{W}[\lambda_2,v](t)| \le \int_0^K |\lambda_1(r)-\lambda_2(r)|\,\beta_1(r)\,\mathrm{d}r + \tilde{b}_1(K)\,||u-v||_{[0,t]} \quad \forall t \in [0,T]\,.$$

We introduce the *Preisach potential energy* $\mathcal{E}$ as

$$\mathcal{E}[\lambda,u](t) := \int_0^\infty G(r,\wp_r[\lambda,u](t))\,\mathrm{d}r, \tag{2.15}$$

where

$$G(r,v) := v\,g(r,v) - \int_0^v g(r,z)\,\mathrm{d}z = \int_0^v z\,\psi(r,z)\,\mathrm{d}z, \tag{2.16}$$

and the *Preisach dissipation operator* as

$$\mathcal{S}[\lambda,u](t) := \int_0^\infty r\,g(r,\wp_r[\lambda,u](t))\,\mathrm{d}r. \tag{2.17}$$

For $u \in W^{1,1}(0,T)$ and $\xi_r = \wp_r[\lambda,u]$, it is easy to derive the pointwise inequality

$$0 \le \dot{u}(t)\,\dot{\xi}_r(t) \le \dot{u}^2(t) \quad \text{a. e.}, \tag{2.18}$$

which entails in turn the following result (see also [19, Proposition II.4.8]).

Proposition 2.4. *Let Assumption* 2.1 *be satisfied and let $K > 0$ be given. Suppose moreover that $b \ge 0$, $\lambda \in \Lambda_K$, and $u \in W^{1,1}(0,T)$ be given such that $||u||_{\mathcal{C}^0([0,T])} \le K$. Put $w := b\,u + \mathcal{W}[\lambda,u]$. Then for a. e. $t \in (0,T)$ we have*

$$b\,\dot{u}^2(t) \le \dot{w}(t)\,\dot{u}(t) \le (b + \tilde{b}_1(K))\,\dot{u}^2(t). \tag{2.19}$$

Later we will need a discrete counterpart of (2.19); see (A.13).

The following result can be found in [19, Theorem II.4.3].

Proposition 2.5. *Let Assumption* 2.1 *be satisfied and let $K > 0$ be given. For arbitrary $\lambda \in \Lambda_K$ and $u \in W^{1,1}(0,T)$ such that $||u||_{\mathcal{C}^0([0,T])} \le K$, we put*

$$w := \mathcal{W}[\lambda,u], \qquad E := \mathcal{E}[\lambda,u], \qquad S := \mathcal{S}[\lambda,u],$$

where $\mathcal{E}$ and $\mathcal{S}$ are, respectively, the Preisach potential energy and the Preisach dissipation operator introduced in (2.15) *and* (2.17). *Then we have*

$$\text{(i)} \quad E(t) \geq \frac{1}{2\tilde{b}_1(K)} w^2(t) \qquad \forall\, t \in [0,T],$$
$$\text{(ii)} \quad \dot{w}(t)\, u(t) - \dot{E}(t) = |\dot{S}(t)| \qquad \text{a. e.}$$

We will later need a discrete counterpart of equation (ii) in Proposition 2.5, which will be derived in subsection A.1.

We finally quote the following result (see [19, Proposition II.4.13]), which will be used in subsection 5.1 to establish the uniqueness of the solution to our model problem.

PROPOSITION 2.6. *Let $\mathcal{W}$ be a Preisach operator* (2.14) *satisfying Assumption* 2.1. *For given $u_1, u_2 \in W^{1,1}(0,T)$, $\lambda_1, \lambda_2 \in \Lambda_0$, and $i = 1,2$, put $\xi_r^i := \wp_r[\lambda_i, u_i]$, $w_i := \mathcal{W}[\lambda_i, u_i] = \int_0^\infty g(r, \xi_r^i)\,\mathrm{d}r$. Then for a. e. $t \in (0,T)$ we have*

$$(\dot{w}_1(t) - \dot{w}_2(t))\,(u_1(t) - u_2(t)) \geq \int_0^\infty (\xi_r^1(t) - \xi_r^2(t))\,\frac{\partial}{\partial t}(g(r, \xi_r^1(t)) - g(r, \xi_r^2(t)))\,\mathrm{d}r. \tag{2.20}$$

In Problem (1.1), both the input and the initial memory configuration λ additionally depend on the space variable $x \in \Omega$. If $\lambda(x,\cdot)$ belongs to Λ_0 and $u(x,\cdot)$ belongs to $\mathcal{C}^0([0,T])$ for (almost) every $x \in \Omega$, then we define

$$\mathcal{W}[\lambda, u](x,t) \;:=\; \int_0^\infty g(r, \wp_r[\lambda(x,\cdot), u(x,\cdot)](t))\,\mathrm{d}r. \tag{2.21}$$

For $x_1, x_2 \in \Omega$, we have by (2.6) and Assumption 2.1 that

$$\begin{aligned} &|\mathcal{W}[\lambda,u](x_1,t) - \mathcal{W}[\lambda,u](x_2,t)| \\ &\quad \leq \int_0^\infty |g(r, \wp_r[\lambda(x_1,\cdot), u(x_1,\cdot)]) - g(r, \wp_r[\lambda(x_2,\cdot), u(x_2,\cdot)])|(t)\,\mathrm{d}x \\ &\quad \leq \int_0^\infty \beta_1(r)\left(|\lambda(x_1,r) - \lambda(x_2,r)| + \sup_{\tau\in[0,t]} |u(x_1,\tau) - u(x_2,\tau)|\right)\mathrm{d}r. \end{aligned} \tag{2.22}$$

Assume that $\nabla u \in L^2(\Omega; G_+(0,T))$, $\beta_1 \in L^1(0,\infty)$, and that $\lambda : \Omega \to \Lambda_K$ is such that

$$\int_0^K \int_\Omega \beta_1(r)|\nabla\lambda(x,r)|\,\mathrm{d}x\,\mathrm{d}r < \infty.$$

Here and in what follows, the symbol ∇ denotes the gradient with respect to the spatial variable $x \in \Omega$. Set $\tilde{b}_1 = \int_0^\infty \beta_1(r)\,\mathrm{d}r$ and $w(x,t) = \mathcal{W}[\lambda,u](x,t)$. Then we obtain from (2.22) that

$$|\nabla w(x,t)| \leq \int_0^\infty \beta_1(r)\,|\nabla\lambda(x,r)|\,\mathrm{d}r + \tilde{b}_1 \sup_{\tau\in[0,t]} |\nabla u(x,\tau)| \tag{2.23}$$

for all $t \in [0,T]$ a. e. in Ω.

2.3. Convexification and cut-off. Let $R > 0$ be fixed; set

$$\mathscr{D}_R := \{(r,v) \in \mathbb{R}_+^2 : \, |v| + r \leq \, R\}.$$

In addition to Assumption 2.1 we prescribe the following conditions.

Assumption 2.7.

(i) $\dfrac{\partial \psi}{\partial v} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2_+)$,

(ii) $A_R := \inf\{\psi(r,v);\, (r,v) \in \mathscr{D}_R\} > 0$.

Furthermore, denote

$$C_R := \sup\left\{\left|\frac{\partial}{\partial v}\psi(r,v)\right|;\, (r,v) \in \mathscr{D}_R\right\}.$$

Taking possibly a smaller $R > 0$, if necessary, we may assume that

$$K_R := \frac{1}{2}\, A_R - R\, C_R > 0. \tag{2.24}$$

We modify the density ψ outside $\mathscr{D}_R$ by setting

$$\psi_R(r,v) = \begin{cases} \psi(r,v) & (r,v) \in \mathscr{D}_R, \\ \psi(r,-R+r) & v < -R+r,\ r \le R, \\ \psi(r,R-r) & v > R-r,\ r \le R, \\ \psi(R,0) & r > R. \end{cases} \tag{2.25}$$

We define a new Preisach operator $\mathcal{W}_R$ by the formula

$$\mathcal{W}_R[\lambda,u](t) = \int_0^\infty \int_0^{\wp_r[\lambda,u](t)} \psi_R(r,v)\,\mathrm{d}v\,\mathrm{d}r \tag{2.26}$$

for $\lambda \in \Lambda_0$ and $u \in W^{1,1}(0,T)$. In subsection A.2 we prove that all increasing trajectories of $\mathcal{W}_R$ are convex and all decreasing trajectories are concave. This will play an important role in higher order energy estimates in subsections 4.5 and 5.2.

We also introduce the cut-off density

$$\widetilde{\psi}_R(r,v) = \begin{cases} \psi(r,v) & (r,v) \in \mathscr{D}_R, \\ 0 & \text{otherwise}, \end{cases} \tag{2.27}$$

and the corresponding cut-off operator

$$\widetilde{\mathcal{W}}_R[\lambda,u](t) = \int_0^\infty \int_0^{\wp_r[\lambda,u](t)} \widetilde{\psi}_R(r,v)\,\mathrm{d}v\,\mathrm{d}r. \tag{2.28}$$

Remark 2.8. We remark that $\mathcal{W}_R$ is convex (in the sense of trajectories) but not globally bounded, while $\widetilde{\mathcal{W}}_R$ is globally bounded but nonconvex; see Figure 2. The former will appear under the time derivative to ensure the validity of the second order energy inequality; the latter is used in the quadratic terms to keep the growth under control. We eventually show that the whole memory evolution takes place in $\mathscr{D}_R$, so that the truncations never become active.

In what follows, we will often write $\mathcal{W}[u]$ instead of $\mathcal{W}[\lambda,u]$ for brevity when λ is clear from the context.

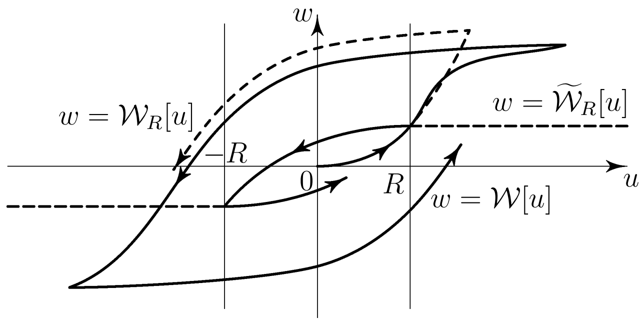

FIG. 2. *Preisach hysteresis diagrams for $\mathcal{W}$, $\mathcal{W}_R$, and $\widetilde{\mathcal{W}}_R$ with the choice $\lambda \equiv 0$.*

3. Main result. Let us consider an open bounded domain $\Omega \subset \mathbb{R}^2$ with $\mathcal{C}^{1,1}$ boundary, and set $\Omega_T := \Omega \times (0,T)$. We set $V := W_0^{1,2}(\Omega)$, and introduce the spaces of divergence free functions

$$\mathbf{H} := \left\{ \mathbf{u} \in L^2(\Omega;\mathbb{R}^2);\ \int_\Omega \mathbf{u}(x) \cdot \nabla\phi(x)\,\mathrm{d}x = 0 \quad \forall \phi \in V \right\}, \quad \mathbf{V} := \mathbf{H} \cap W_0^{1,2}(\Omega;\mathbb{R}^2).$$

For $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathbf{V}$, we denote by $\nabla\boldsymbol{\phi} = (\nabla\phi_1, \nabla\phi_2)$ the Jacobi matrix of $\boldsymbol{\phi}$, with each row being the gradient of a component of $\boldsymbol{\phi}$, and for all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathbf{V}$ we denote with $(\nabla\boldsymbol{\phi}, \nabla\boldsymbol{\psi})$ the canonical scalar product of matrices.

We propose solving the following problem.

PROBLEM 3.1. *Consider a Preisach operator $\mathcal{W}$ of the form* (2.21)*, and let $u_0 \in L^2(\Omega)$, $\mathbf{v}_0 \in \mathbf{H}$, $\lambda : \Omega \to \Lambda$ be given initial data; we search for functions $(u, \mathbf{v})$ with appropriate regularity, such that*

$$u(x,0) = u_0(x), \quad \mathbf{v}(x,0) = \mathbf{v}_0(x) \quad \text{a. e. in } \Omega, \tag{3.1}$$

and for any $\phi \in V$, any $\boldsymbol{\phi} \in \mathbf{V}$, and for a. e. $t \in (0,T)$ we have

$$\int_\Omega \frac{\partial}{\partial t}(u + \mathcal{W}[\lambda,u])\,\phi\,\mathrm{d}x - \int_\Omega \mathbf{v} \cdot \nabla\phi\,(u + \mathcal{W}[\lambda,u])\,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla\phi\,\mathrm{d}x = 0, \tag{3.2}$$

$$\int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \boldsymbol{\phi}\,\mathrm{d}x + \int_\Omega (\mathbf{v}\cdot\nabla)\,\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{d}x + \int_\Omega (\nabla\mathbf{v}, \nabla\boldsymbol{\phi})\,\mathrm{d}x + \int_\Omega (u + \mathcal{W}[\lambda,u])\,\nabla u \cdot \boldsymbol{\phi}\,\mathrm{d}x = 0. \tag{3.3}$$

Interpretation. If the functions u, $\mathcal{W}[\lambda,u]$, $\mathbf{v}$ are smooth enough, we may integrate by parts in (3.2) and (3.3). We see that the function

$$\mathbf{q} := \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \Delta\mathbf{v} + (u + \mathcal{W}[\lambda,u])\,\nabla u$$

is orthogonal to every function $\boldsymbol{\phi} \in \mathbf{V}$; hence (see [15]), there exists p such that $\mathbf{q} = -\nabla p$. System (3.2)–(3.3) thus formally reduces to (1.1) with homogeneous Dirichlet boundary conditions for both u and $\mathbf{v}$.

It is straightforward to check the thermodynamic consistency of system (3.2)–(3.3). Putting $\phi = u$ and $\boldsymbol{\phi} = \mathbf{v}$, we formally obtain from Proposition 2.5 the energy equality

(3.4)
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}u^2 + \mathcal{E}[\lambda,u] + \frac{1}{2}|\mathbf{v}|^2\right)\mathrm{d}x + \int_\Omega \left(\left|\frac{\partial}{\partial t}\mathcal{S}[\lambda,u]\right| + |\nabla u|^2 + |\nabla \mathbf{v}|^2\right)\mathrm{d}x \;=\; 0,$$

where $\frac{1}{2}u^2 + \mathcal{E}[\lambda,u] + \frac{1}{2}|\mathbf{v}|^2 \geq 0$ is the total specific energy, and $|\frac{\partial}{\partial t}\mathcal{S}[\lambda,u]| + |\nabla u|^2 + |\nabla \mathbf{v}|^2 \geq 0$ is the specific dissipation (or entropy production) rate.

The main result of the paper can be stated as follows.

THEOREM 3.2. *Let the Preisach operator $\mathcal{W}$ satisfy Assumptions* 2.1 *and* 2.7*, and let $R > 0$ be fixed as in subsection* 2.3*. Let $K \in [0,R]$ and $\lambda : \Omega \to \Lambda_K$ be given.*

(i) (Existence) *Let the initial data have the regularity*

(3.5)
$$u_0 \in V, \quad \mathbf{v}_0 \in \mathbf{V}, \quad \Delta u_0 \in L^2(\Omega), \quad \Delta \mathbf{v}_0 \in \mathbf{H}, \quad \nabla\lambda \in L^2(\Omega \times (0,K)),$$

and set

(3.6)
$$\alpha := \max\left\{||u_0||_V, ||\mathbf{v}_0||_{\mathbf{V}}, ||\Delta u_0||_{L^2(\Omega)}, ||\Delta \mathbf{v}_0||_{L^2(\Omega;\mathbb{R}^2)}, \lambda||_{L^2(\Omega\times(0,K))}\right\}.$$

Then there exists $\alpha_1 > 0$ such that if $\alpha \leq \alpha_1$, then Problem 3.1 *has a solution $(u, \mathbf{v})$ with the regularity*

$$\begin{aligned} &u \in \mathcal{C}^0(\bar\Omega_T) \cap \mathcal{C}^0(0,T;V), && \mathbf{v} \in \mathcal{C}^0(\bar\Omega_T;\mathbb{R}^2) \cap \mathcal{C}^0(0,T;\mathbf{V}),\\ &u_t, \Delta u \in L^\infty(0,T;L^2(\Omega)), && \mathbf{v}_t, \Delta\mathbf{v} \in L^\infty(0,T;\mathbf{H}),\\ &\nabla u_t \in L^2(\Omega_T;\mathbb{R}^2), && \nabla \mathbf{v}_t \in L^2(\Omega_T;\mathbb{R}^{2\times 2}).\end{aligned}$$

(ii) (Uniqueness) *In addition to* (3.5)*, let the initial data satisfy*

$$\Delta u_0 \in L^\infty(\Omega), \quad \nabla\lambda \in L^{\hat q+1}(\Omega\times(0,K)) \quad \textit{for some} \quad \hat q > 3\,. \tag{3.7}$$

Then there exists a unique solution $(u, \mathbf{v})$ to Problem 3.1 *with additional regularity $u_t \in L^\infty(\Omega_T)$.*

Remark 3.3. The initial data are taken sufficiently small in order to keep the solution inside the convexity domain of the hysteresis operator $\mathcal{W}$; see Remark 2.8. We restrict ourselves to an a priori bounded interval $(0, \alpha_0)$ of admissible values of α.

4. Proof of existence.

4.1. Strategy of the proof. We first replace the Preisach operator $\mathcal{W}$ by $\mathcal{W}_R$ and $\widetilde{\mathcal{W}}_R$ at suitable places, and discretize the PDEs in time. The solution to the discrete problem is found using the Browder–Minty theorem (subsection 4.2). In subsections 4.3–4.6, we derive a priori estimates independent of the discretization parameter based on a discrete version of the second order energy inequality (subsection A.2). If α is sufficiently small, the sup-norm of u is uniformly bounded by the cut-off parameter R, hence $\mathcal{W}[u] = \mathcal{W}_R[u] = \widetilde{\mathcal{W}}_R[u]$. By compactness, we choose a convergent subsequence as $\tau \to 0$, and check that the limit is a solution to Problem 3.1. We will carefully write down explicitly how the estimates depend upon α introduced in (3.6), and upon the discretization parameter τ.

4.2. The discrete problem. Let us fix some $m \in \mathbb{N}$ and define the time step $\tau = \dfrac{T}{m}$. For $k = 1, \ldots, m$, consider a recurrent system with unknowns u_k and $\mathbf{v}_k$,

(4.1)
$$\left.\begin{aligned} \int_\Omega \overset{\bullet}{u}_k\ \phi \,\mathrm{d}x + \int_\Omega \overset{\bullet}{w}_k\ \phi \,\mathrm{d}x - \int_\Omega b_{k-1}\, \mathbf{v}_k \cdot \nabla\phi \,\mathrm{d}x + \int_\Omega \nabla u_k \cdot \nabla \phi \,\mathrm{d}x = 0, \\ \int_\Omega \overset{\bullet}{\mathbf{v}}_k \cdot \boldsymbol{\phi} \,\mathrm{d}x + \int_\Omega (\mathbf{v}_{k-1} \cdot \nabla)\, \mathbf{v}_k \cdot \boldsymbol{\phi} \,\mathrm{d}x + \int_\Omega (\nabla \mathbf{v}_k, \nabla \boldsymbol{\phi}) \,\mathrm{d}x + \int_\Omega b_{k-1}\, \nabla u_k \cdot \boldsymbol{\phi} \,\mathrm{d}x = 0, \end{aligned}\right\}$$

for any $\phi \in V$ and $\boldsymbol{\phi} \in \mathbf{V}$, with u_0 and $\mathbf{v}_0$ as in (3.5), where we set

$$\overset{\bullet}{u}_k := \frac{u_k - u_{k-1}}{\tau}, \qquad \overset{\bullet}{\mathbf{v}}_k := \frac{\mathbf{v}_k - \mathbf{v}_{k-1}}{\tau}, \qquad \overset{\bullet}{w}_k := \frac{w_k - w_{k-1}}{\tau}, \quad k = 1, \ldots, m; \tag{4.2}$$

$$b_k(x) = u_k(x) + \widetilde{w}_k(x), \qquad \text{a.e. in } \Omega, \quad k = 0, \ldots, m; \tag{4.3}$$

$$w_k(x) = \int_0^\infty g_R(r, \xi_k(x,r)) \,\mathrm{d}r, \qquad \widetilde{w}_k(x) = \int_0^\infty \widetilde{g}_R(r, \xi_k(x,r)) \,\mathrm{d}r, \tag{4.4}$$

with

$$g_R(r,v) = \int_0^v \psi_R(r,v') \,\mathrm{d}v', \qquad \widetilde{g}_R(r,v) = \int_0^v \widetilde{\psi}_R(r,v') \,\mathrm{d}v', \tag{4.5}$$

where ψ_R, $\widetilde{\psi}_R$ are the functions introduced in (2.25), (2.27), respectively. As in (2.9)–(2.10), the sequence ξ_k is defined recursively by

$$\xi_0(x,r) := P[\lambda(x,\cdot), u_0(x)](r), \qquad \xi_k(x,r) := P[\xi_{k-1}(x,\cdot), u_k(x)](r). \tag{4.6}$$

Setting

$$\bar{u}^{(\tau)}(x,t) = \sum_{k=1}^m u_{k-1}(x)\, \chi_{[(k-1)\tau,\, k\tau)}(t) + u_m(x)\, \chi_{\{T\}}(t), \tag{4.7}$$

and

$$\bar{\xi}_r^{(\tau)}(x,t) = \sum_{k=1}^m \xi_{k-1}(x,r)\, \chi_{[(k-1)\tau,\, k\tau)}(t) + \xi_m(x,r)\, \chi_{\{T\}}(t), \tag{4.8}$$

we thus have, in view of (2.7)–(2.10),

$$\bar{\xi}_r^{(\tau)}(x,t) = \wp_r[\lambda, \bar{u}^{(\tau)}](x,t). \tag{4.9}$$

We construct the solution to (4.1) by induction over k. Assuming that $u_{k-1} \in V$, $\mathbf{v}_{k-1} \in \mathbf{V}$ are already known, we define the mapping

$$\mathscr{F}_k : \mathscr{W} \to \mathscr{W}',$$

where $\mathscr{W} := V \times \mathbf{V}$, by the formula

$$\begin{aligned} \left\langle \mathscr{F}_k \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix}, \begin{pmatrix} \phi \\ \boldsymbol{\phi} \end{pmatrix} \right\rangle_{\mathscr{W},\mathscr{W}'} &= \frac{1}{\tau} \int_\Omega (u - u_{k-1})\, \phi \,\mathrm{d}x + \frac{1}{\tau} \int_\Omega (w - w_{k-1})\, \phi \,\mathrm{d}x \\ &\quad - \int_\Omega b_{k-1}\, \mathbf{v} \cdot \nabla \phi \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla \phi \,\mathrm{d}x + \frac{1}{\tau} \int_\Omega (\mathbf{v} - \mathbf{v}_{k-1}) \cdot \boldsymbol{\phi} \,\mathrm{d}x \\ &\quad + \int_\Omega (\mathbf{v}_{k-1} \cdot \nabla)\, \mathbf{v} \cdot \boldsymbol{\phi} \,\mathrm{d}x + \int_\Omega (\nabla \mathbf{v}, \nabla \boldsymbol{\phi}) \,\mathrm{d}x + \int_\Omega b_{k-1}\, \nabla u \cdot \boldsymbol{\phi} \,\mathrm{d}x, \end{aligned}$$

where

$$w(x) = \int_0^\infty g_R(r, P[\xi_{k-1}(x,\cdot), u(x)](r))\,\mathrm{d}r. \tag{4.10}$$

For $\begin{pmatrix} u_i \\ \mathbf{v}_i \end{pmatrix} \in \mathscr{W}$, $i = 1, 2$, we have, for some constant $c > 0$,

$$\left\langle \mathscr{F}_k \begin{pmatrix} u_1 \\ \mathbf{v}_1 \end{pmatrix} - \mathscr{F}_k \begin{pmatrix} u_2 \\ \mathbf{v}_2 \end{pmatrix}, \begin{pmatrix} u_1 - u_2 \\ \mathbf{v}_1 - \mathbf{v}_2 \end{pmatrix} \right\rangle_{\mathscr{W},\mathscr{W}'} = \frac{1}{\tau} \int_\Omega |u_1 - u_2|^2\,\mathrm{d}x$$

$$+ \frac{1}{\tau} \int_\Omega (w_1 - w_2)(u_1 - u_2)\,\mathrm{d}x + \frac{1}{\tau} \int_\Omega |\mathbf{v}_1 - \mathbf{v}_2|^2\,\mathrm{d}x + \int_\Omega |\nabla(u_1 - u_2)|^2\,\mathrm{d}x$$

$$+ \int_\Omega |\nabla(\mathbf{v}_1 - \mathbf{v}_2)|^2\,\mathrm{d}x \geq c\,(||u_1 - u_2||_V^2 + ||\mathbf{v}_1 - \mathbf{v}_2||_{\mathbf{V}}^2),$$

where we used the monotonicity of the mapping $u \mapsto w$ defined by (4.10) (which in turn is given by the superposition of the two nondecreasing mappings g and $P[\lambda,\cdot]$). We see that $\mathscr{F}_k$ is bounded, continuous, monotone and coercive, and by the Browder–Minty theorem (see [24, Theorem 9.45]), there exists $\binom{u_k}{\mathbf{v}_k} \in \mathscr{W}$ such that

$$\mathscr{F}_k \begin{pmatrix} u_k \\ \mathbf{v}_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

i.e., (4.1) holds.

4.3. First a priori estimate. In the estimates below, many different constants will appear. For simplicity, we denote every constant independent of α and τ by C. Indeed, the value of C may vary from one formula to another.

We choose $\phi = u_k$ and $\boldsymbol{\phi} = \mathbf{v}_k$ in (4.1). This yields

$$\int_\Omega \overset{\bullet}{u}_k\, u_k\,\mathrm{d}x + \int_\Omega \overset{\bullet}{w}_k\, u_k\,\mathrm{d}x + \int_\Omega |\nabla u_k|^2\,\mathrm{d}x$$

$$+ \int_\Omega \overset{\bullet}{\mathbf{v}}_k \cdot \mathbf{v}_k\,\mathrm{d}x + \int_\Omega (\mathbf{v}_{k-1} \cdot \nabla)\,\mathbf{v}_k \cdot \mathbf{v}_k\,\mathrm{d}x + \int_\Omega |\nabla \mathbf{v}_k|^2\,\mathrm{d}x = 0.$$

We notice that, as $\mathbf{v}_{k-1} \in \mathbf{V}$,

$$\int_\Omega (\mathbf{v}_{k-1} \cdot \nabla)\,\mathbf{v}_k \cdot \mathbf{v}_k\,\mathrm{d}x = \frac{1}{2} \int_\Omega \mathbf{v}_{k-1} \cdot \nabla |\mathbf{v}_k|^2\,\mathrm{d}x = 0;$$

hence, using (A.3), we have for every $k = 1, \ldots, m$, as a discrete counterpart of the energy equality (3.4), that

$$\frac{1}{2} \int_\Omega (|u_k|^2 - |u_{k-1}|^2)\,\mathrm{d}x + \int_\Omega (E_k - E_{k-1})\,\mathrm{d}x + \frac{1}{2} \int_\Omega (|\mathbf{v}_k|^2 - |\mathbf{v}_{k-1}|^2)\,\mathrm{d}x$$

$$+ \tau \int_\Omega |\nabla u_k|^2\,\mathrm{d}x + \tau \int_\Omega |\nabla \mathbf{v}_k|^2\,\mathrm{d}x \leq 0.$$

After summing for $k = 1, \ldots, n$, for every $n \in \{1, \ldots, m\}$, using the regularity of the initial data (3.5), we obtain

$$
\begin{aligned}
&\max_{n=1,\ldots,m} \int_\Omega |u_n|^2 \mathrm{d}x + \max_{n=1,\ldots,m} \int_\Omega |\mathbf{v}_n|^2 \mathrm{d}x + \tau \sum_{k=1}^m \int_\Omega |\nabla u_k|^2 \mathrm{d}x + \tau \sum_{k=1}^m \int_\Omega |\nabla \mathbf{v}_k|^2 \mathrm{d}x \\
&\le \frac{1}{2} \int_\Omega |u_0|^2 \mathrm{d}x + \frac{1}{2} \int_\Omega |\mathbf{v}_0|^2 \mathrm{d}x + \int_\Omega E_0 \, \mathrm{d}x \le C\, \alpha^2. \qquad (4.11)
\end{aligned}
$$

4.4. Estimate of the initial condition. For $k = 1, \ldots, m$ set

$$
H_k := \int_\Omega \left(\dot{u}_k + \dot{w}_k \right) \dot{u}_k \ \mathrm{d}x. \tag{4.12}
$$

Due to the monotonicity and local Lipschitz continuity of the functions $g(r, \cdot)$ and $P[\lambda, \cdot](r)$, we have the pointwise inequality

$$
\dot{u}_k(x)\, \dot{w}_k(x) \ge 0 \quad \text{a.e. in } \ \Omega\,, \quad k = 1, \ldots, m. \tag{4.13}
$$

In (4.1) corresponding to $k = 1$ choose $\phi := \dot{u}_1$, $\boldsymbol{\phi} := \dot{\mathbf{v}}_1$, and sum the two equations. We deduce

$$
\begin{aligned}
H_1 &+ \int_\Omega \left| \dot{\mathbf{v}}_1 \right|^2 \mathrm{d}x + \tau \int_\Omega \left| \nabla \dot{u}_1 \right|^2 \mathrm{d}x + \tau \int_\Omega \left| \nabla \dot{\mathbf{v}}_1 \right|^2 \mathrm{d}x \\
&= - \int_\Omega (\mathbf{v}_0 \cdot \nabla)\, \mathbf{v}_1 \cdot \dot{\mathbf{v}}_1 \ \mathrm{d}x + \int_\Omega b_0\, \mathbf{v}_1 \cdot \nabla \dot{u}_1 \ \mathrm{d}x \\
&\quad - \int_\Omega b_0\, \nabla u_1 \cdot \dot{\mathbf{v}}_1 \ \mathrm{d}x + \int_\Omega \Delta u_0 \ \dot{u}_1 \ \mathrm{d}x + \int_\Omega \Delta \mathbf{v}_0 \cdot \dot{\mathbf{v}}_1 \ \mathrm{d}x.
\end{aligned}
$$

On the right-hand side, we have

$$
\begin{aligned}
&\int_\Omega b_0\, \mathbf{v}_1 \cdot \nabla \dot{u}_1 \ \mathrm{d}x - \int_\Omega b_0\, \nabla u_1 \cdot \dot{\mathbf{v}}_1 \ \mathrm{d}x \\
&\qquad = \int_\Omega b_0\, \mathbf{v}_0 \cdot \nabla \dot{u}_1 \ \mathrm{d}x - \int_\Omega b_0\, \nabla u_0 \cdot \dot{\mathbf{v}}_1 \ \mathrm{d}x =: I_a + II_a.
\end{aligned}
$$

We estimate these two terms as

$$
I_a = - \int_\Omega \dot{u}_1 \ \mathbf{v}_0 \cdot \nabla b_0 \, \mathrm{d}x \le \frac{1}{4} \int_\Omega \left| \dot{u}_1 \right|^2 \mathrm{d}x + \int_\Omega |\mathbf{v}_0|^2\, |\nabla b_0|^2 \, \mathrm{d}x
$$

and

$$
II_a = - \int_\Omega b_0\, \nabla u_0 \cdot \dot{\mathbf{v}}_1 \ \mathrm{d}x \le \frac{1}{4} \int_\Omega \left| \dot{\mathbf{v}}_1 \right|^2 \mathrm{d}x + \int_\Omega |\nabla u_0|^2\, |b_0|^2 \, \mathrm{d}x.
$$

The remaining integrals are estimated similarly as

$$
\begin{aligned}
&\int_\Omega \Delta u_0 \ \dot{u}_1 \ \mathrm{d}x + \int_\Omega \Delta \mathbf{v}_0 \cdot \dot{\mathbf{v}}_1 \ \mathrm{d}x \le \frac{1}{4} \int_\Omega \left| \dot{u}_1 \right|^2 \mathrm{d}x \\
&\qquad + \int_\Omega |\Delta u_0|^2 \, \mathrm{d}x + \frac{1}{4} \int_\Omega \left| \dot{\mathbf{v}}_1 \right|^2 \mathrm{d}x + \int_\Omega |\Delta \mathbf{v}_0|^2 \, \mathrm{d}x
\end{aligned}
$$

and

$$\int_\Omega (\mathbf{v}_0 \cdot \nabla)\,\mathbf{v}_1 \cdot \overset{\bullet}{\mathbf{v}}_1\ \mathrm{d}x = \int_\Omega (\mathbf{v}_0 \cdot \nabla)\,\mathbf{v}_0 \cdot \overset{\bullet}{\mathbf{v}}_1\ \mathrm{d}x$$

$$\leq \frac{1}{4}\int_\Omega \left|\overset{\bullet}{\mathbf{v}}_1\right|^2\ \mathrm{d}x + \int_\Omega |(\mathbf{v}_0 \cdot \nabla)\,\mathbf{v}_0|^2\,\mathrm{d}x.$$

Summing up the above inequalities, we obtain

$$\frac{1}{2}\left(H_1 + \int_\Omega \left|\overset{\bullet}{\mathbf{v}}_1\right|^2\ \mathrm{d}x\right) + \tau\int_\Omega \left|\nabla \overset{\bullet}{u}_1\right|^2\ \mathrm{d}x + \tau\int_\Omega \left|\nabla \overset{\bullet}{\mathbf{v}}_1\right|^2\ \mathrm{d}x$$

$$\leq \int_\Omega |\mathbf{v}_0|^2\,|\nabla b_0|^2\,\mathrm{d}x + \int_\Omega |\nabla u_0|^2\,|b_0|^2\,\mathrm{d}x$$

$$+\int_\Omega |\Delta u_0|^2\,\mathrm{d}x + \int_\Omega |\Delta \mathbf{v}_0|^2\,\mathrm{d}x + \int_\Omega |(\mathbf{v}_0\cdot\nabla)\mathbf{v}_0|^2\,\mathrm{d}x\ \leq\ C\alpha^2. \tag{4.14}$$

The constant C in (4.14) depends on the upper bound α_0 for α, but α_0 is kept fixed as mentioned in Remark 3.3.

4.5. Second a priori estimate. We take the time increments in (4.1) and set for $k = 1, \ldots, m$, in addition to the notations in (4.2),

$$\overset{\bullet}{b}_k := \frac{b_k - b_{k-1}}{\tau}. \tag{4.15}$$

We obtain

$$\int_\Omega \left(\overset{\bullet}{u}_k - \overset{\bullet}{u}_{k-1}\right)\,\phi\,\mathrm{d}x + \int_\Omega \left(\overset{\bullet}{w}_k - \overset{\bullet}{w}_{k-1}\right)\,\phi\,\mathrm{d}x - \tau\int_\Omega b_{k-1}\ \overset{\bullet}{\mathbf{v}}_k \cdot\nabla\phi\,\mathrm{d}x$$

$$-\,\tau\int_\Omega \overset{\bullet}{b}_{k-1}\ \mathbf{v}_{k-1}\cdot\nabla\phi\,\mathrm{d}x + \tau\int_\Omega \nabla\overset{\bullet}{u}_k \cdot\nabla\phi\,\mathrm{d}x = 0 \tag{4.16}$$

for any $\phi \in V$ and

(4.17)

$$\int_\Omega \left(\overset{\bullet}{\mathbf{v}}_k - \overset{\bullet}{\mathbf{v}}_{k-1}\right)\cdot\boldsymbol{\phi}\ \mathrm{d}x + \tau\int_\Omega (\mathbf{v}_{k-1}\cdot\nabla)\ \overset{\bullet}{\mathbf{v}}_k \cdot\boldsymbol{\phi}\ \mathrm{d}x + \tau\int_\Omega \left(\overset{\bullet}{\mathbf{v}}_{k-1}\cdot\nabla\right)\ \mathbf{v}_{k-1}\cdot\boldsymbol{\phi}\,\mathrm{d}x$$

$$+\,\tau\int_\Omega \left(\nabla\overset{\bullet}{\mathbf{v}}_k, \nabla\boldsymbol{\phi}\right)\mathrm{d}x + \tau\int_\Omega b_{k-1}\,\nabla\overset{\bullet}{u}_k \cdot\boldsymbol{\phi}\,\mathrm{d}x + \tau\int_\Omega \overset{\bullet}{b}_{k-1}\ \nabla u_{k-1}\cdot\boldsymbol{\phi}\,\mathrm{d}x = 0$$

for any $\boldsymbol{\phi} \in \mathbf{V}$. Now we choose $\phi = \overset{\bullet}{u}_k$ in (4.16) and $\boldsymbol{\phi} = \overset{\bullet}{\mathbf{v}}_k$ in (4.17). We get

(4.18)

$$\int_\Omega \left(\overset{\bullet}{u}_k - \overset{\bullet}{u}_{k-1}\right)\ \overset{\bullet}{u}_k\ \mathrm{d}x + \int_\Omega \left(\overset{\bullet}{w}_k - \overset{\bullet}{w}_{k-1}\right)\ \overset{\bullet}{u}_k\ \mathrm{d}x -\ \tau\int_\Omega \overset{\bullet}{b}_{k-1}\ \mathbf{v}_{k-1}\cdot\nabla\overset{\bullet}{u}_k\ \mathrm{d}x$$

$$+\ \tau\int_\Omega \overset{\bullet}{b}_{k-1}\ \nabla u_{k-1}\cdot\overset{\bullet}{\mathbf{v}}_k\ \mathrm{d}x + \tau\int_\Omega \left|\nabla\overset{\bullet}{u}_k\right|^2\ \mathrm{d}x + \int_\Omega (\overset{\bullet}{\mathbf{v}}_k - \overset{\bullet}{\mathbf{v}}_{k-1})\cdot\overset{\bullet}{\mathbf{v}}_k\ \mathrm{d}x$$

$$+\,\tau\int_\Omega \left(\overset{\bullet}{\mathbf{v}}_{k-1}\cdot\nabla\right)\mathbf{v}_{k-1}\cdot\overset{\bullet}{\mathbf{v}}_k\ \mathrm{d}x + \tau\int_\Omega \left|\nabla\overset{\bullet}{\mathbf{v}}_k\right|^2\ \mathrm{d}x = 0,$$

where we used the fact that

$$\tau\int_\Omega (\mathbf{v}_{k-1}\cdot\nabla)\ \overset{\bullet}{\mathbf{v}}_k\cdot\overset{\bullet}{\mathbf{v}}_k\ \mathrm{d}x = 0.$$

For simplicity, we will use the notation $|\cdot|_p$ for the $L^p(\Omega)$-norm of both scalar- or vector-valued functions, $1 \le p \le \infty$. Using (A.4) for $p = 2$, we have, for $k \ge 2$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left(\dot{u}_k + \dot{w}_k\right) \dot{u}_k \, dx - \frac{1}{2}\int_\Omega \left(\dot{u}_{k-1} + \dot{w}_{k-1}\right) \dot{u}_{k-1} \, dx + \tau \left|\nabla \dot{u}_k\right|_2^2 \\
&+ \frac{1}{2}\left(\left|\dot{\mathbf{v}}_k\right|_2^2 - \left|\dot{\mathbf{v}}_{k-1}\right|_2^2\right) + \tau \left|\nabla \dot{\mathbf{v}}_k\right|_2^2 \le -\tau \int_\Omega \left(\dot{\mathbf{v}}_{k-1} \cdot \nabla\right) \mathbf{v}_{k-1} \cdot \dot{\mathbf{v}}_k \, dx \\
&+ \tau \int_\Omega \left|\dot{b}_{k-1}\right| |\mathbf{v}_{k-1}| \left|\nabla \dot{u}_k\right| dx + \tau \int_\Omega \left|\dot{b}_{k-1}\right| |\nabla u_{k-1}| \left|\dot{\mathbf{v}}_k\right| dx.
\end{aligned} \tag{4.19}$$

By (A.13), we have

$$\left|\dot{b}_{k-1}(x)\right| \le C \left|\dot{u}_{k-1}(x)\right| \quad \forall x \in \Omega\,. \tag{4.20}$$

With H_k defined in (4.12), it follows from (4.19) for $k \ge 2$ that

$$\begin{aligned}
&\frac{1}{2}\left(H_k - H_{k-1}\right) + \frac{1}{2}\left(\left|\dot{\mathbf{v}}_k\right|_2^2 - \left|\dot{\mathbf{v}}_{k-1}\right|_2^2\right) + \tau \left|\nabla \dot{u}_k\right|_2^2 + \tau \left|\nabla \dot{\mathbf{v}}_k\right|_2^2 \\
&\le \tau \left|\dot{\mathbf{v}}_{k-1}\right|_4 |\nabla \mathbf{v}_{k-1}|_2 \left|\dot{\mathbf{v}}_k\right|_4 + C\tau \left|\dot{u}_{k-1}\right|_4 |\mathbf{v}_{k-1}|_4 \left|\nabla \dot{u}_k\right|_2 \\
&+ C\tau \left|\dot{u}_{k-1}\right|_4 |\nabla u_{k-1}|_2 \left|\dot{\mathbf{v}}_k\right|_4 =: I_b + II_b + III_b.
\end{aligned} \tag{4.21}$$

We now apply (A.15) with the choices $q_1 = 4$, $q_2 = q_3 = 2$, $\rho = 1/2$, and obtain

$$\begin{aligned}
I_b &\le C\tau \left|\dot{\mathbf{v}}_{k-1}\right|_2^{1/2} \left|\nabla \dot{\mathbf{v}}_{k-1}\right|_2^{1/2} |\nabla \mathbf{v}_{k-1}|_2 \left|\dot{\mathbf{v}}_k\right|_2^{1/2} \left|\nabla \dot{\mathbf{v}}_k\right|_2^{1/2} \\
&\le \frac{\tau}{4}\left|\nabla \dot{\mathbf{v}}_{k-1}\right|_2^2 + \frac{\tau}{4}\left|\nabla \dot{\mathbf{v}}_k\right|_2^2 + C\tau |\nabla \mathbf{v}_{k-1}|_2^2 \left|\dot{\mathbf{v}}_{k-1}\right|_2 \left|\dot{\mathbf{v}}_k\right|_2, \\
II_b &\le C\tau \left|\dot{u}_{k-1}\right|_2^{1/2} \left|\nabla \dot{u}_{k-1}\right|_2^{1/2} |\mathbf{v}_{k-1}|_2^{1/2} |\nabla \mathbf{v}_{k-1}|_2^{1/2} \left|\nabla \dot{u}_k\right|_2 \\
&\le \frac{\tau}{4}\left|\nabla \dot{u}_k\right|_2^2 + \frac{\tau}{4}\left|\nabla \dot{u}_{k-1}\right|_2^2 + C\tau |\nabla \mathbf{v}_{k-1}|_2^2 \left|\dot{u}_{k-1}\right|_2^2 |\mathbf{v}_{k-1}|_2^2, \\
III_b &\le C\tau \left|\dot{u}_{k-1}\right|_2^{1/2} \left|\nabla \dot{u}_{k-1}\right|_2^{1/2} |\nabla u_{k-1}|_2 \left|\dot{\mathbf{v}}_k\right|_2^{1/2} \left|\nabla \dot{\mathbf{v}}_k\right|_2^{1/2} \\
&\le \frac{\tau}{4}\left|\nabla \dot{u}_{k-1}\right|_2^2 + \frac{\tau}{4}\left|\nabla \dot{\mathbf{v}}_k\right|_2^2 + C\tau |\nabla u_{k-1}|_2^2 \left|\dot{u}_{k-1}\right|_2 \left|\dot{\mathbf{v}}_k\right|_2.
\end{aligned}$$

Using the fact that $|\mathbf{v}_{k-1}|_2 \le C$ by (4.11), we thus have for $k \ge 2$

$$\begin{aligned}
&\frac{1}{2}\left(H_k - H_{k-1}\right) + \frac{1}{2}\left(\left|\dot{\mathbf{v}}_k\right|_2^2 - \left|\dot{\mathbf{v}}_{k-1}\right|_2^2\right) + \frac{3}{4}\tau \left|\nabla \dot{u}_k\right|_2^2 + \frac{\tau}{2}\left|\nabla \dot{\mathbf{v}}_k\right|_2^2 \\
&\le \frac{\tau}{2}\left|\nabla \dot{u}_{k-1}\right|_2^2 dx + \frac{\tau}{4}\left|\nabla \dot{\mathbf{v}}_{k-1}\right|_2^2 + C\tau |\nabla \mathbf{v}_{k-1}|_2^2 \left|\dot{\mathbf{v}}_{k-1}\right|_2 \left|\dot{\mathbf{v}}_k\right|_2 \\
&+ C\tau |\nabla \mathbf{v}_{k-1}|_2^2 \left|\dot{u}_{k-1}\right|_2^2 + C\tau |\nabla u_{k-1}|_2^2 \left|\dot{u}_{k-1}\right|_2 \left|\dot{\mathbf{v}}_k\right|_2.
\end{aligned} \tag{4.22}$$

We define auxiliary quantities

$$X_k := \frac{1}{2}H_k + \frac{1}{2}\int_\Omega \left|\overset{\bullet}{\mathbf{v}}_k\right|^2 \,\mathrm{d}x + \frac{\tau}{2}\int_\Omega \left|\nabla \overset{\bullet}{u}_k\right|^2 \,\mathrm{d}x + \frac{\tau}{4}\int_\Omega \left|\nabla \overset{\bullet}{\mathbf{v}}_k\right|^2 \,\mathrm{d}x, \tag{4.23}$$

$$Y_k := \frac{\tau}{4}\left|\nabla \overset{\bullet}{u}_k\right|_2^2 + \frac{\tau}{4}\left|\nabla \overset{\bullet}{\mathbf{v}}_k\right|_2^2, \tag{4.24}$$

and

$$a_k := |\nabla u_{k-1}|_2^2 + |\nabla \mathbf{v}_{k-1}|_2^2. \tag{4.25}$$

By virtue of (4.11), we have for $k = 1, \dots, m+1$ the estimate

$$\tau\, a_k \le \tau \sum_{j=1}^{m+1} a_j \le C\,\alpha^2. \tag{4.26}$$

Now (4.22) implies, using (4.13) and (4.23)–(4.25), that

$$X_k - X_{k-1} + Y_k \le C\,\tau\, a_k(X_{k-1} + \sqrt{X_k X_{k-1}}) \le \tau\, a_k(C X_{k-1} + c_* X_k), \tag{4.27}$$

where c_* is a fixed constant such that

$$1 - c_*\,\tau\, a_k > \frac{1}{2} \quad \forall\, k = 1, \dots, m+1\,. \tag{4.28}$$

Such a constant exists as a consequence of (4.26). This enables us to rewrite (4.27) as

$$X_k + Y_k \le \frac{1 + C\,\tau\, a_k}{1 - c_*\,\tau\, a_k}\, X_{k-1} \le (1 + \tau\, d_k)\, X_{k-1} \tag{4.29}$$

for $k = 2, \dots, m$, where we set $d_k = 2(C + c_*)a_k$, with C from (4.29). We now apply the discrete Gronwall argument. Putting

$$R_k = \prod_{j=1}^{k}(1 + \tau\, d_j),$$

we have

$$\frac{X_k}{R_k} + \frac{Y_k}{R_k} \le \frac{X_{k-1}}{R_{k-1}}\,;$$

hence,

$$X_k + \sum_{i=1}^{k} Y_i \prod_{j=i+1}^{k}(1 + \tau\, d_j) \le X_1 \prod_{j=2}^{k}(1 + \tau\, d_j) \le X_1 \mathrm{e}^{\tau \sum_{j=2}^{k} d_j} \le C X_1 \tag{4.30}$$

for $k = 2, \dots, m$. By (4.14), we have $X_1 \le C\alpha^2$; hence, $X_k + \sum_{i=1}^{k} Y_i \le C\alpha^2$ for all $k = 1, \dots, m$. This implies in particular that

$$\left|\overset{\bullet}{u}_n\right|_2^2 + \left|\overset{\bullet}{\mathbf{v}}_n\right|_2^2 + \tau \sum_{k=1}^{n}\left|\nabla \overset{\bullet}{u}_k\right|_2^2 + \tau \sum_{k=1}^{n}\left|\nabla \overset{\bullet}{\mathbf{v}}_k\right|_2^2 \le C\alpha^2 \tag{4.31}$$

for every $n = 1, \dots, m$.

4.6. Third a priori estimate. We prove by induction over $k = 1, \dots, m$ that there exists $B > 0$ independent of k and m such that

$$|\mathbf{v}_k|_\infty \le B\,\alpha, \qquad |u_k|_\infty \le B\,\alpha \qquad \text{for all } k = 0, \dots, m. \tag{4.32}$$

For $k = 0, \dots, m$ set

$$B_k^{(m)} = \frac{1}{\alpha} \max\{|u_j|_\infty, |\mathbf{v}_j|_\infty; j = 0, \dots, k\}. \tag{4.33}$$

We have $B_0^{(m)} \le C$ independently of m by hypotheses on initial data. Now let $1 \le k_0 \le m$ be fixed, and assume that $B_{k_0-1}^{(m)} < \infty$. Let $\{\boldsymbol{\phi}_i; i \in \mathbb{N}\}$ be the complete system of eigenfunctions, orthonormal in $\mathbf{H}$, of the problem

$$-\Delta \boldsymbol{\phi}_i = \lambda_i\, \boldsymbol{\phi}_i, \qquad \operatorname{div} \boldsymbol{\phi}_i = 0, \qquad \boldsymbol{\phi}_i = 0 \text{ on } \partial\Omega.$$

Put $\mathbf{v}_{ki} = \int_\Omega \mathbf{v}_k \cdot \boldsymbol{\phi}_i \,\mathrm{d}x$, and in the second equation of (4.1) set $\boldsymbol{\phi} = -\sum_{i=1}^J \mathbf{v}_{ki}\, \lambda_i\, \boldsymbol{\phi}_i$. We may let J tend to ∞ and obtain for $k = 1, \dots, k_0$ that

$$\begin{aligned} |\Delta \mathbf{v}_k|_2^2 &\le \left(\left|\overset{\bullet}{\mathbf{v}}_k\right|_2 + |\mathbf{v}_{k-1}|_\infty\, |\nabla \mathbf{v}_k|_2 + |b_{k-1}|_\infty\, |\nabla u_k|_2 \right) |\Delta \mathbf{v}_k|_2 \\ &\le \left(\left|\overset{\bullet}{\mathbf{v}}_k\right|_2 + B_{k_0-1}^{(m)}\, \alpha\, |\nabla \mathbf{v}_k|_2 + \left(\tilde{b}_1(R) + B_{k_0-1}^{(m)}\, \alpha\right) |\nabla u_k|_2 \right) |\Delta \mathbf{v}_k|_2; \end{aligned}$$

hence,

$$|\Delta \mathbf{v}_{k_0}|_2 \le \left|\overset{\bullet}{\mathbf{v}}_{k_0}\right|_2 + C\,(1 + B_{k_0-1}^{(m)}\, \alpha)\,(|\nabla \mathbf{v}_{k_0}|_2 + |\nabla u_{k_0}|_2). \tag{4.34}$$

Using (4.31), we estimate

$$|\nabla \mathbf{v}_{k_0}|_2 \le |\nabla \mathbf{v}_0|_2 + \tau \sum_{k=1}^{k_0} \left|\nabla \overset{\bullet}{\mathbf{v}}_k\right|_2 \le |\nabla \mathbf{v}_0|_2 + \left(\tau \sum_{k=1}^{k_0} \left|\nabla \overset{\bullet}{\mathbf{v}}_k\right|_2^2 \right)^{1/2} \le C\alpha,$$

and similarly,

$$|\nabla u_{k_0}|_2 \le C\,\alpha\,, \quad \left|\overset{\bullet}{\mathbf{v}}_{k_0}\right|_2 \le C\,\alpha\,;$$

hence,

$$\|\mathbf{v}_{k_0}\|_{W^{2,2}(\Omega;\mathbb{R}^2)} \le C\,|\Delta \mathbf{v}_{k_0}|_2 \le C\,\alpha\,(1 + B_{k_0-1}^{(m)}\, \alpha)\,. \tag{4.35}$$

The embedding of $W^{2,2}(\Omega;\mathbb{R}^2)$ into $W^{1,4}(\Omega;\mathbb{R}^2)$ yields

$$|\nabla \mathbf{v}_{k_0}|_4 \le C\,\alpha\,(1 + B_{k_0-1}^{(m)}\, \alpha).$$

Using the Gagliardo–Nirenberg inequality (A.15) in the form

$$|\mathbf{v}_{k_0}|_\infty \le C\,|\mathbf{v}_{k_0}|_2^{1/3}\, |\nabla \mathbf{v}_{k_0}|_4^{2/3}$$

and (4.11), we obtain that

$$|\mathbf{v}_{k_0}|_\infty \le C\,\alpha\,(1 + B_{k_0-1}^{(m)}\, \alpha)^{2/3}. \tag{4.36}$$

By direct comparison in the first equation in (4.1), we derive for $k = 1, \dots, k_0$ the estimate

$$|\Delta u_k|_2 \le \left|\overset{\bullet}{u}_k\right|_2 + \left|\overset{\bullet}{w}_k\right|_2 + |\mathbf{v}_k|_\infty \, |\nabla b_{k-1}|_2. \tag{4.37}$$

This yields in particular that $B_{k_0}^{(m)} < \infty$. Using (4.4)–(4.6) and (2.25), we get for every k and a. e. $x \in \Omega$ the pointwise estimate

$$\left|\overset{\bullet}{w}_k(x)\right| \le C \left(1 + \max_{j=0,\dots,k} |u_k(x)|\right) \left|\overset{\bullet}{u}_k(x)\right|.$$

From (2.23) and the hypotheses on $\mathcal{W}$ it follows for a. e. $x \in \Omega$ that

$$\begin{aligned} |\nabla b_{k-1}(x)| &\le C \left(\int_0^R |\nabla \lambda(x,r)| \,\mathrm{d}r + \max_{j=0,\dots,k-1} |\nabla u_j(x)|\right) \\ &\le C \left(\int_0^R |\nabla \lambda(x,r)| \,\mathrm{d}r + |\nabla u_0(x)| + \tau \sum_{j=1}^{k-1} \left|\nabla \overset{\bullet}{u}_j(x)\right|\right). \end{aligned} \tag{4.38}$$

Hence, by (3.6) and (4.31), we obtain from (4.37) that

$$\begin{aligned} |\Delta u_{k_0}|_2 &\le C \left(1 + \max\{B_{k_0-1}^{(m)}\,\alpha, |u_{k_0}|_\infty\}\right) \left|\overset{\bullet}{u}_{k_0}\right|_2 + C\,\alpha\,|\mathbf{v}_{k_0}|_\infty \\ &\le C\,\alpha\,(1 + \max\{B_{k_0-1}^{(m)}\,\alpha, |u_{k_0}|_\infty, |\mathbf{v}_{k_0}|_\infty\}). \end{aligned} \tag{4.39}$$

We proceed as in (4.35)–(4.36) to obtain

$$|u_{k_0}|_\infty \le C\,\alpha\,(1 + \max\{B_{k_0-1}^{(m)}\alpha, |u_{k_0}|_\infty, |\mathbf{v}_{k_0}|_\infty\})^{2/3}.$$

From (4.36) we conclude that

$$\max\{|u_{k_0}|_\infty, |\mathbf{v}_{k_0}|_\infty\} \le C\,\alpha\,(1 + \max\{B_{k_0-1}^{(m)}\alpha, |u_{k_0}|_\infty, |\mathbf{v}_{k_0}|_\infty\})^{2/3}. \tag{4.40}$$

Assume that $B_{k_0}^{(m)} > B_{k_0-1}^{(m)}$. Then

$$B_{k_0}^{(m)} = \frac{1}{\alpha} \max\{|u_{k_0}|_\infty, |\mathbf{v}_{k_0}|_\infty\} \le C\,(1 + \alpha_0\, B_{k_0}^{(m)})^{2/3};$$

hence, $B_{k_0}^{(m)} \le \max\{C, B_{k_0-1}^{(m)}\}$ with a constant C independent of k and m, and the desired estimate (4.32) follows. Inequalities (4.35), (4.39) imply in particular that

$$|\Delta u_k|_2 + |\Delta \mathbf{v}_k|_2 \le C\alpha \quad \text{for all } k = 1, \dots, m. \tag{4.41}$$

4.7. Passage to the limit. For each fixed time step τ, we associate with the sequences $\{u_k\}, \{\mathbf{v}_k\}$ constructed above their piecewise linear and piecewise constant time interpolates according to the following scheme, similar to (4.7)–(4.9):

(4.42)

$$\left.\begin{array}{llllll} \bar{u}_+^{(\tau)}(x,t) = u_k(x), & \bar{w}_+^{(\tau)}(x,t) = w_k(x), & \bar{\mathbf{v}}_+^{(\tau)}(x,t) = \mathbf{v}_k(x), \\ \bar{u}_-^{(\tau)}(x,t) = u_{k-1}(x), & \bar{w}_-^{(\tau)}(x,t) = \widetilde{w}_{k-1}(x), & \bar{\mathbf{v}}_-^{(\tau)}(x,t) = \mathbf{v}_{k-1}(x), \end{array}\right\}$$

and

$$\left.\begin{array}{rcl} \hat{u}^{(\tau)}(x,t) &=& u_{k-1}(x) + \frac{t-(k-1)\tau}{\tau}(u_k(x) - u_{k-1}(x)), \\ \hat{w}^{(\tau)}(x,t) &=& w_{k-1}(x) + \frac{t-(k-1)\tau}{\tau}(w_k(x) - w_{k-1}(x)), \\ \hat{\mathbf{v}}^{(\tau)}(x,t) &=& \mathbf{v}_{k-1}(x) + \frac{t-(k-1)\tau}{\tau}(\mathbf{v}_k(x) - \mathbf{v}_{k-1}(x)), \\ \bar{b}^{(\tau)}(x,t) &=& \bar{u}_-^{(\tau)}(x,t) + \bar{w}_-^{(\tau)}(x,t) \end{array}\right\} \tag{4.43}$$

for $x \in \Omega$ and $t \in [(k-1)\tau, k\tau)$, $k = 1, 2, \ldots, m$, continuously extended to $t = T$. We have

$$\bar{w}_+^{(\tau)} = \mathcal{W}_R[\lambda, \bar{u}_+^{(\tau)}], \qquad \bar{w}_-^{(\tau)} = \widetilde{\mathcal{W}}_R[\lambda, \bar{u}_-^{(\tau)}]. \tag{4.44}$$

As a consequence of the estimates (4.31) and (4.41), we see that there exist functions $u \in L^\infty(0,T; V \cap W^{2,2}(\Omega))$, $\mathbf{v} \in L^\infty(0,T; \mathbf{V} \cap W^{2,2}(\Omega, \mathbb{R}^2))$, $w \in L^\infty(0,T; W^{1,2}(\Omega))$, with $u_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; V)$, $\mathbf{v}_t \in L^\infty(0,T; \mathbf{H}) \cap L^2(0,T; \mathbf{V})$, $w_t \in L^\infty(0,T; L^2(\Omega))$, such that, along a subsequence as $\tau \to 0$, we have

$$\left.\begin{array}{rcll} \hat{u}^{(\tau)} &\to& u & \text{weakly star in} \quad L^\infty(0,T; W^{2,2}(\Omega)), \\ \hat{w}^{(\tau)} &\to& w & \text{weakly star in} \quad L^\infty(0,T; L^2(\Omega)), \\ \hat{\mathbf{v}}^{(\tau)} &\to& \mathbf{v} & \text{weakly star in} \quad L^\infty(0,T; W^{2,2}(\Omega; \mathbb{R}^2)), \\ \hat{u}_t^{(\tau)} &\to& u_t & \text{weakly star in} \quad L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; V), \\ \hat{w}_t^{(\tau)} &\to& w_t & \text{weakly star in} \quad L^\infty(0,T; L^2(\Omega)), \\ \hat{\mathbf{v}}_t^{(\tau)} &\to& \mathbf{v}_t & \text{weakly star in} \quad L^\infty(0,T; \mathbf{H}) \cap L^2(0,T; \mathbf{V}). \end{array}\right\} \tag{4.45}$$

By compact embedding, we have, passing again to a subsequence, if necessary,

$$\left.\begin{array}{rcll} \nabla\hat{u}^{(\tau)} &\to& \nabla u & \text{strongly in} \quad L^2(\Omega_T; \mathbb{R}^2), \\ \nabla\hat{\mathbf{v}}^{(\tau)} &\to& \nabla \mathbf{v} & \text{strongly in} \quad L^2(\Omega_T; \mathbb{R}^{2\times 2}), \\ \hat{u}^{(\tau)} &\to& u & \text{uniformly in} \quad \mathcal{C}^0(\bar{\Omega}_T), \\ \hat{\mathbf{v}}^{(\tau)} &\to& \mathbf{v} & \text{uniformly in} \quad \mathcal{C}^0(\bar{\Omega}_T; \mathbb{R}^2). \end{array}\right\} \tag{4.46}$$

We further have for every τ and every $(x,t) \in \Omega_T$ that

$$\begin{array}{rcccl} |\hat{u}^{(\tau)}(x,t) - \bar{u}_\pm^{(\tau)}(x,t)|^2 &\le& \max\limits_k |u_k(x) - u_{k-1}(x)|^2 &\le& \sum\limits_{k=1}^m |u_k(x) - u_{k-1}(x)|^2, \\ |\hat{\mathbf{v}}^{(\tau)}(x,t) - \bar{\mathbf{v}}_\pm^{(\tau)}(x,t)|^2 &\le& \max\limits_k |\mathbf{v}_k(x) - \mathbf{v}_{k-1}(x)|^2 &\le& \sum\limits_{k=1}^m |\mathbf{v}_k(x) - \mathbf{v}_{k-1}(x)|^2, \\ |\hat{w}^{(\tau)}(x,t) - \bar{w}_+^{(\tau)}(x,t)|^2 &\le& \max\limits_k |w_k(x) - w_{k-1}(x)|^2 &\le& C\sum\limits_{k=1}^m |u_k(x) - u_{k-1}(x)|^2, \end{array}$$

and similarly,

$$\begin{array}{rcl} |\nabla\hat{u}^{(\tau)}(x,t) - \nabla\bar{u}_\pm^{(\tau)}(x,t)|^2 &\le& \sum\limits_{k=1}^m |\nabla u_k(x) - \nabla u_{k-1}(x)|^2, \\ |\nabla\hat{\mathbf{v}}^{(\tau)}(x,t) - \nabla\bar{\mathbf{v}}_\pm^{(\tau)}(x,t)|^2 &\le& \sum\limits_{k=1}^m |\nabla \mathbf{v}_k(x) - \nabla \mathbf{v}_{k-1}(x)|^2. \end{array}$$

From (4.31) it follows that

$$\begin{aligned} &\|\hat{u}^{(\tau)} - \bar{u}_{\pm}^{(\tau)}\|_{L^2(\Omega;G_+(0,T))} \\ &+\|\hat{w}^{(\tau)} - \bar{w}_{+}^{(\tau)}\|_{L^2(\Omega;G_+(0,T))} + \|\hat{\mathbf{v}}^{(\tau)} - \bar{\mathbf{v}}_{\pm}^{(\tau)}\|_{L^2(0,T;\mathbf{H})} \le C\sqrt{\tau}, \end{aligned} \tag{4.47}$$

$$\|\nabla\hat{u}^{(\tau)} - \nabla\bar{u}_{\pm}^{(\tau)}\|_{L^2(\Omega_T;\mathbb{R}^2)} + \|\nabla\hat{\mathbf{v}}^{(\tau)} - \nabla\bar{\mathbf{v}}_{\pm}^{(\tau)}\|_{L^2(\Omega_T;\mathbb{R}^{2\times 2})} \le C\sqrt{\tau}. \tag{4.48}$$

Hence, $\bar{u}_{\pm}^{(\tau)}$ converge to u strongly in $L^2(\Omega; G_+(0,T))$ as $\tau \to 0$. By Proposition 2.3, we may pass to the limit in (4.44) and obtain

$$\left.\begin{array}{llll} \bar{w}_{+}^{(\tau)} & \to & w = \mathcal{W}_R[\lambda, u] \quad \text{strongly in} & L^2(\Omega; G_+(0,T)), \\ \bar{w}_{-}^{(\tau)} & \to & \tilde{w} = \widetilde{\mathcal{W}}_R[\lambda, u] \quad \text{strongly in} & L^2(\Omega; G_+(0,T)). \end{array}\right\} \tag{4.49}$$

This, (4.47), and (4.48) yield

$$\left.\begin{array}{llll} \hat{w}^{(\tau)} & \to & w \quad \text{strongly in} & L^2(\Omega; G_+(0,T)), \\ \bar{\mathbf{v}}_{\pm}^{(\tau)} & \to & \mathbf{v} \quad \text{strongly in} & L^2(0,T; \mathbf{V}). \end{array}\right\} \tag{4.50}$$

System (4.1) is of the form
(4.51)

$$\left.\begin{array}{l} \displaystyle\int_\Omega \Big(\hat{u}_t^{(\tau)}\,\phi + \hat{w}_t^{(\tau)}\,\phi - \bar{b}^{(\tau)}\,\bar{\mathbf{v}}_{+}^{(\tau)}\cdot\nabla\phi + \nabla\bar{u}_{+}^{(\tau)}\cdot\nabla\phi\Big)\,\mathrm{d}x = 0, \\ \displaystyle\int_\Omega \Big(\hat{\mathbf{v}}_t^{(\tau)}\cdot\boldsymbol{\phi} + (\bar{\mathbf{v}}_{-}^{(\tau)}\cdot\nabla)\,\bar{\mathbf{v}}_{+}^{(\tau)}\cdot\boldsymbol{\phi} + (\nabla\bar{\mathbf{v}}_{+}^{(\tau)}, \nabla\boldsymbol{\phi}) + \bar{b}^{(\tau)}\,\nabla\bar{u}_{+}^{(\tau)}\cdot\boldsymbol{\phi}\Big)\,\mathrm{d}x = 0 \end{array}\right\}$$

for every $\phi \in V$, $\boldsymbol{\phi} \in \mathbf{V}$. The convergences (4.45)–(4.46), (4.49)–(4.50), and inequality (4.48) enable us to pass to the limit as $\tau \to 0$ and obtain

$$\left.\begin{array}{l} \displaystyle\int_\Omega ((u_t + w_t)\,\phi - (u + \tilde{w})\,\mathbf{v}\cdot\nabla\phi + \nabla u\cdot\nabla\phi)\,\mathrm{d}x = 0, \\ \displaystyle\int_\Omega \big(\mathbf{v}_t\cdot\boldsymbol{\phi} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\cdot\boldsymbol{\phi} + (\nabla\mathbf{v}, \nabla\boldsymbol{\phi}) + (u + \tilde{w})\,\nabla u\cdot\boldsymbol{\phi}\big)\,\mathrm{d}x = 0. \end{array}\right\} \tag{4.52}$$

The L^∞ bound (4.32) is preserved in the limit; hence, by choosing $\alpha \le R/B$, we obtain

$$|u(x,t)| \le R, \qquad \text{a.e. in } \Omega_T.$$

Since $K \le R$, it follows, e. g., from [19, Lemma II.2.4] that the integration domain in (2.26) and (2.28) is contained in $\mathscr{D}_R$; hence the truncations in (2.25) and (2.27) never become active, and we have

$$w = \tilde{w} = \mathcal{W}[\lambda, u]\,.$$

This completes the existence part of the proof of Theorem 3.2. $\square$

5. Uniqueness for Problem 3.1.

5.1. A uniqueness theorem. We first prove the following theorem.

THEOREM 5.1. *If the solution to Problem* 3.1 *established in Theorem* 3.2(i) *has the additional regularity*

$$u_t \in L^\infty(\Omega_T), \tag{5.1}$$

then it is unique.

In subsection 5.2, we show by means of a discrete Moser iteration scheme that the regularity (5.1) is available under the hypotheses of Theorem 3.2(ii).

Proof of Theorem 5.1. Let $(u_1, \mathbf{v}_1)$ and $(u_2, \mathbf{v}_2)$ be two solutions to Problem 3.1 with the prescribed regularity. We write (3.2) and (3.3) first for $(u_1, \mathbf{v}_1)$ and $(u_2, \mathbf{v}_2)$, choose $\phi = u_1 - u_2$, $\boldsymbol{\phi} = \mathbf{v}_1 - \mathbf{v}_2$, and subtract the two equations. Setting for $i = 1, 2$

$$w_i = \mathcal{W}[\lambda, u_i]\,, \quad b_i = u_i + w_i,$$

and

$$u^\ominus = u_1 - u_2, \quad \mathbf{v}^\ominus = \mathbf{v}_1 - \mathbf{v}_2, \quad b^\ominus = b_1 - b_2,$$

we obtain

$$\begin{aligned}
&\int_\Omega \left(b_t^\ominus\, u^\ominus + \mathbf{v}_t^\ominus \mathbf{v}^\ominus + \left|\nabla u^\ominus\right|^2 + \left|\nabla \mathbf{v}^\ominus\right|^2\right) \mathrm{d}x \\
&\qquad = \int_\Omega \left(b^\ominus \left(\mathbf{v}_2 \cdot \nabla u^\ominus - \mathbf{v}^\ominus \cdot \nabla u_2\right) - \left(\mathbf{v}^\ominus \cdot \nabla\right) \mathbf{v}_2 \cdot \mathbf{v}^\ominus\right) \mathrm{d}x.
\end{aligned} \tag{5.2}$$

We first estimate the right-hand side of (5.2). We use the symbol C to denote any constant independent of $t \in [0, T]$. Note that $\Delta u_i, \Delta \mathbf{v}_i$ are bounded in $L^\infty(0, T; L^2(\Omega))$. By Sobolev embedding, this yields a uniform bound in time for $|\nabla u_i|_p, |\nabla \mathbf{v}_i|_p$ for every $p < \infty$. Using the Gagliardo–Nirenberg inequality (A.15), we thus obtain

$$\left|\int_\Omega (\mathbf{v}^\ominus \cdot \nabla)\mathbf{v}_2 \cdot \mathbf{v}^\ominus \mathrm{d}x\right| \le \left|\mathbf{v}^\ominus\right|_4^2 |\nabla \mathbf{v}_2|_2 \le C \left|\mathbf{v}^\ominus\right|_2 \left|\nabla \mathbf{v}^\ominus\right|_2 \le C \left|\mathbf{v}^\ominus\right|_2^2 + \frac{1}{4} \left|\nabla \mathbf{v}^\ominus\right|_2^2, \tag{5.3}$$

$$\left|\int_\Omega b^\ominus\, \mathbf{v}_2 \cdot \nabla u^\ominus \,\mathrm{d}x\right| \le \left|b^\ominus\right|_2 |\mathbf{v}_2|_\infty \left|\nabla u^\ominus\right|_2 \le C \left|b^\ominus\right|_2^2 + \frac{1}{2} \left|\nabla u^\ominus\right|_2^2, \tag{5.4}$$

$$\begin{aligned}
\left|\int_\Omega b^\ominus\, \mathbf{v}^\ominus \cdot \nabla u_2 \,\mathrm{d}x\right| &\le \left|b^\ominus\right|_2 \left|\mathbf{v}^\ominus\right|_4 |\nabla u_2|_4 \le C \left|b^\ominus\right|_2 \left|\mathbf{v}^\ominus\right|_2^{1/2} \left|\nabla \mathbf{v}^\ominus\right|_2^{1/2} \\
&\le C \left|b^\ominus\right|_2^2 + \left|\mathbf{v}^\ominus\right|_2^2 + \frac{1}{4} \left|\nabla \mathbf{v}^\ominus\right|_2^2.
\end{aligned} \tag{5.5}$$

The term $|b^\ominus|_2^2$ has to be estimated carefully. The generating function g of the Preisach operator $\mathcal{W}$ in Assumption 2.7 has for every $(r, v_1), (r, v_2) \in \mathscr{D}_R$ the property

$$A_R\,(v_1 - v_2)^2 \;\le\; (g(r, v_1) - g(r, v_2))(v_1 - v_2) \;\le\; (A_R + R\,C_R)\,(v_1 - v_2)^2. \tag{5.6}$$

As in Proposition 2.6, set

$$\xi_r^i(x, t) = \wp_r[\lambda, u_i](x, t)\,, \quad \xi_r^\ominus = \xi_r^1 - \xi_r^2.$$

The memory evolution takes place only in $\mathscr{D}_R$, and we obtain directly from (2.21) that

$$\left|b^{\ominus}(x,t)\right| \le \left|u^{\ominus}(x,t)\right| + C\int_0^R \left|\xi_r^{\ominus}(x,t)\right| \,\mathrm{d}r \quad \text{a.e.},$$

and

$$\left|b^{\ominus}(t)\right|_2^2 \le C\left(\left|u^{\ominus}(t)\right|_2^2 + \int_0^R \left|\xi_r^{\ominus}(t)\right|_2^2 \,\mathrm{d}r\right). \tag{5.7}$$

We now need a lower bound for the term $b_t^{\ominus}\, u^{\ominus}$ of (5.2). By hypothesis (5.1) and inequality (2.18), we have

$$\left|\frac{\partial \xi_r^i}{\partial t}(x,t)\right| \le C;$$

from the elementary identity

$$\begin{aligned}(\xi_r^1 - \xi_r^2)\frac{\partial}{\partial t}(g(r,\xi_r^1) - g(r,\xi_r^2)) = {} & \frac{\partial \xi_r^2}{\partial t}(\xi_r^1 - \xi_r^2)(\psi(r,\xi_r^1) - \psi(r,\xi_r^2)) \\ & + \frac{1}{2}\frac{\partial}{\partial t}\left(\psi(r,\xi_r^1)\,|\xi_r^{\ominus}|^2\right) - \frac{1}{2}\frac{\partial \xi_r^1}{\partial t}\frac{\partial \psi}{\partial v}(r,\xi_r^1)\,\left|\xi_r^{\ominus}\right|^2\end{aligned}$$

and Assumption 2.7 we thus deduce the inequality

$$(\xi_r^1 - \xi_r^2)\frac{\partial}{\partial t}(g(r,\xi_r^1) - g(r,\xi_r^2)) \ge \frac{1}{2}\frac{\partial}{\partial t}\left(\psi(r,\xi_r^1)\,\left|\xi_r^{\ominus}\right|^2\right) - C\left|\xi_r^{\ominus}\right|^2, \tag{5.8}$$

and Proposition 2.6 yields the pointwise inequality

$$b_t^{\ominus}\, u^{\ominus} \ge \frac{1}{2}\frac{\partial}{\partial t}\left(\left|u^{\ominus}\right|^2 + \int_0^R \psi(r,\xi_r^1)\,\left|\xi_r^{\ominus}\right|^2 \,\mathrm{d}r\right) - C\int_0^R \left|\xi_r^{\ominus}\right|^2 \,\mathrm{d}r. \tag{5.9}$$

We now integrate (5.2) from 0 to t and use the fact that the solutions satisfy the same initial condition. Using (5.3)–(5.5), (5.7), (5.9), and Assumption 2.7, we obtain

$$\begin{aligned}&\left|u^{\ominus}(t)\right|_2^2 + A_R\int_0^R \left|\xi_r^{\ominus}(t)\right|_2^2 \,\mathrm{d}r + \left|\mathbf{v}^{\ominus}(t)\right|_2^2 + \int_0^t \left(\left|\nabla u^{\ominus}(t')\right|_2^2 + \left|\nabla \mathbf{v}^{\ominus}(t')\right|_2^2\right)\mathrm{d}t' \\ &\le C\int_0^t \left(\left|u^{\ominus}(t')\right|_2^2 + \int_0^R \left|\xi_r^{\ominus}(t')\right|_2^2 \,\mathrm{d}r + \left|\mathbf{v}^{\ominus}(t')\right|_2^2\right)\mathrm{d}t'.\end{aligned} \tag{5.10}$$

From the Gronwall argument it follows that $u_1 = u_2, \mathbf{v}_1 = \mathbf{v}_2$, which we wanted to prove. ∎

5.2. Further regularity. We go back to the time discrete system (4.1), for which we already have the bounds (4.31) and (4.41); more specifically,

$$\left.\begin{aligned} |u_n|_\infty + |\mathbf{v}_n|_\infty &\le R \\ \left|\dot{u}_n\right|_2 + \left|\dot{\mathbf{v}}_n\right|_2 + |\Delta u_n|_2 + |\Delta \mathbf{v}_n|_2 &\le C \\ \tau\sum_{k=1}^n \left(\left|\nabla \dot{u}_k\right|_2^2 + \left|\nabla \dot{\mathbf{v}}_k\right|_2^2\right) &\le C \end{aligned}\right\} \quad \text{for all} \quad n = 1,\dots,m. \tag{5.11}$$

The dependence of C on α is not relevant anymore. As in section 4, C denotes any constant independent of τ. We now come back to the time increment equation (4.16), and choose $\phi = F_k := \overset{\bullet}{u}_k \mid \overset{\bullet}{u}_k \mid^{p-2}$ for $p \geq 2$. This is admissible, as by (5.11), $\overset{\bullet}{u}_k$ belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ (with a bound that for the moment still depends on τ, indeed). Using the Young inequality, (A.4), (5.11), and the pointwise inequality (4.20), we get

$$\begin{aligned}
&\frac{1}{p}\int_\Omega \left(\left|\overset{\bullet}{u}_k\right|^p + \overset{\bullet}{w}_k\, F_k - \left|\overset{\bullet}{u}_{k-1}\right|^p - \overset{\bullet}{w}_{k-1}\, F_{k-1}\right) \mathrm{d}x + \tau\,(p-1)\int_\Omega \left|\nabla \overset{\bullet}{u}_k\right|^2 \left|\overset{\bullet}{u}_k\right|^{p-2} \mathrm{d}x \\
&\leq -\tau \int_\Omega \overset{\bullet}{\mathbf{v}}_k\, \nabla b_{k-1}\, \overset{\bullet}{u}_k \left|\overset{\bullet}{u}_k\right|^{p-2} \mathrm{d}x \,+\, \tau\,(p-1)\int_\Omega \mathbf{v}_{k-1}\, \overset{\bullet}{b}_{k-1}\, \nabla \overset{\bullet}{u}_k \left|\overset{\bullet}{u}_k\right|^{p-2} \mathrm{d}x \\
&\leq \tau \int_\Omega \left|\overset{\bullet}{\mathbf{v}}_k\right| \left|\nabla b_{k-1}\right| \left|\overset{\bullet}{u}_k\right|^{p-1} \mathrm{d}x \,+\, C\,\tau\,(p-1)\int_\Omega \left|\overset{\bullet}{u}_{k-1}\right| \left|\nabla \overset{\bullet}{u}_k\right| \left|\overset{\bullet}{u}_k\right|^{p-2} \mathrm{d}x.
\end{aligned} \tag{5.12}$$

We first estimate the initial condition as in subsection 4.4. In the first equation of (4.1) corresponding to $k = 1$, we set $\phi = F_1$ and obtain

$$\int_\Omega \left(\left|\overset{\bullet}{u}_1\right|^p + \overset{\bullet}{w}_1\, F_1\right) \mathrm{d}x + (p-1)\int_\Omega \nabla u_1 \cdot \nabla \overset{\bullet}{u}_1 \left|\overset{\bullet}{u}_1\right|^{p-2} \mathrm{d}x \leq \int_\Omega |\mathbf{v}_1|\, |\nabla b_0| \left|\overset{\bullet}{u}_1\right|^{p-1} \mathrm{d}x.$$

Using the estimates (5.11) and hypothesis (3.7), we obtain

$$\begin{aligned}
&\int_\Omega \left(\left|\overset{\bullet}{u}_1\right|^p + \overset{\bullet}{w}_1\, F_1\right) \mathrm{d}x + \tau\,(p-1)\int_\Omega \left|\nabla \overset{\bullet}{u}_1\right|^2 \left|\overset{\bullet}{u}_1\right|^{p-2} \mathrm{d}x \\
&\quad \leq \int_\Omega |\Delta u_0| \left|\overset{\bullet}{u}_1\right|^{p-1} \mathrm{d}x + C\int_\Omega \left|\overset{\bullet}{u}_1\right|^{p-1} \mathrm{d}x \;\leq\; C\int_\Omega \left|\overset{\bullet}{u}_1\right|^{p-1} \mathrm{d}x.
\end{aligned} \tag{5.13}$$

Using Hölder's inequality it follows that

$$\int_\Omega \left(\left|\overset{\bullet}{u}_1\right|^p + \overset{\bullet}{w}_1\, F_1\right) \mathrm{d}x + \tau(p-1)\int_\Omega \left|\nabla \overset{\bullet}{u}_1\right|^2 \left|\overset{\bullet}{u}_1\right|^{p-2} \mathrm{d}x \leq C^p. \tag{5.14}$$

Summing (5.14) with (5.12) over $k = 2, \ldots, n$ for $n = 2, \ldots, m$, we deduce

$$\begin{aligned}
&\frac{1}{p}\int_\Omega \left(\left|\overset{\bullet}{u}_n\right|^p + \overset{\bullet}{w}_n\, F_n\right) \mathrm{d}x + \tau\,(p-1)\sum_{k=1}^n \int_\Omega \left|\nabla \overset{\bullet}{u}_k\right|^2 \left|\overset{\bullet}{u}_k\right|^{p-2} \mathrm{d}x \\
&\qquad \leq \frac{1}{p}\, C^p + C\,\tau \sum_{k=2}^n \int_\Omega \left|\overset{\bullet}{\mathbf{v}}_k\right| \left|\nabla b_{k-1}\right| \left|\overset{\bullet}{u}_k\right|^{p-1} \mathrm{d}x \\
&\qquad\quad + C\,\tau\,(p-1)\sum_{k=2}^n \int_\Omega \left|\overset{\bullet}{u}_{k-1}\right| \left|\nabla \overset{\bullet}{u}_k\right| \left|\overset{\bullet}{u}_k\right|^{p-2} \mathrm{d}x.
\end{aligned} \tag{5.15}$$

This, (4.13), and Hölder's inequality imply for $n = 1, \ldots, m$ that

$$\begin{aligned}
&\frac{1}{p}\int_\Omega \left|\overset{\bullet}{u}_n\right|^p \mathrm{d}x + \tau\,\frac{(p-1)}{2}\sum_{k=1}^n \int_\Omega \left|\nabla \overset{\bullet}{u}_k\right|^2 \left|\overset{\bullet}{u}_k\right|^{p-2} \mathrm{d}x \\
&\leq \frac{1}{p}\, C^p + C\tau \sum_{k=2}^n \int_\Omega \left|\overset{\bullet}{\mathbf{v}}_k\right| \left|\nabla b_{k-1}\right| \left|\overset{\bullet}{u}_k\right|^{p-1} \mathrm{d}x + C\tau(p-1)\sum_{k=2}^n \int_\Omega \left|\overset{\bullet}{u}_{k-1}\right|^2 \left|\overset{\bullet}{u}_k\right|^{p-2} \mathrm{d}x.
\end{aligned}$$

We have

$$\left|\overset{\bullet}{u}_{k-1}\right| \left|\overset{\bullet}{u}_k\right|^{p-2} \leq \frac{1}{p-1} \left|\overset{\bullet}{u}_{k-1}\right|^{p-1} + \frac{p-2}{p-1} \left|\overset{\bullet}{u}_k\right|^{p-1} ;$$

hence, setting $\overset{\bullet}{u}_0 := 0$,

$$\begin{aligned}\frac{1}{p} \int_\Omega \left|\overset{\bullet}{u}_n\right|^p \mathrm{d}x + \tau \frac{(p-1)}{2} \sum_{k=1}^n \int_\Omega \left|\nabla \overset{\bullet}{u}_k\right|^2 \left|\overset{\bullet}{u}_k\right|^{p-2} \mathrm{d}x \\ \leq \frac{1}{p} C^p + C\,\tau\,(p-1) \sum_{k=1}^n \int_\Omega \left(\left|\overset{\bullet}{\mathbf{v}}_k\right| |\nabla b_{k-1}| + \left|\overset{\bullet}{u}_{k-1}\right| + \left|\overset{\bullet}{u}_k\right|\right) \left|\overset{\bullet}{u}_k\right|^{p-1} \mathrm{d}x.\end{aligned}$$

With the intention to apply Lemma A.3, we check that the sequence

$$f_k := \left|\overset{\bullet}{\mathbf{v}}_k\right| |\nabla b_{k-1}| + \left|\overset{\bullet}{u}_{k-1}\right| + \left|\overset{\bullet}{u}_k\right|$$

has the property

$$\tau \sum_{k=1}^n \left|\overset{\bullet}{u}_k\right|_4^4 \leq C\,, \quad \tau \sum_{k=1}^n |f_k|_q^q \leq C \quad \text{for some } q > 2. \tag{5.16}$$

The inequality for $|\overset{\bullet}{u}_k|$ holds as a consequence of (5.11) and the Gagliardo–Nirenberg inequality

$$\left|\overset{\bullet}{u}_k\right|_4^4 \leq C \left|\overset{\bullet}{u}_k\right|_2^2 \left|\nabla \overset{\bullet}{u}_k\right|_2^2,$$

and similarly for $|\overset{\bullet}{u}_{k-1}|$ and $|\overset{\bullet}{\mathbf{v}}_k|$. To estimate $|\nabla b_{k-1}|$, we use formula (4.38), choose $\hat{q}$ from hypothesis (3.7), and obtain

$$\begin{aligned}|\nabla b_{k-1}|_{\hat{q}+1}^{\hat{q}+1} &\leq C \left(1 + \int_\Omega \max_{j=0,\dots,k-1} |\nabla u_j(x)|^{\hat{q}+1} \mathrm{d}x\right) \\ &\leq C \left(1 + |\nabla u_0|_{\hat{q}+1}^{\hat{q}+1} + \int_\Omega \sum_{j=1}^{k-1} \left| |\nabla u_j(x)|^{\hat{q}+1} - |\nabla u_{j-1}(x)|^{\hat{q}+1} \right| \mathrm{d}x\right) \\ &\leq C \left(1 + \tau\,(\hat{q}+1) \sum_{j=1}^{k-1} \int_\Omega \left|\nabla \overset{\bullet}{u}_j(x)\right| |\nabla u_j(x)|^{\hat{q}} \mathrm{d}x\right) \\ &\leq C \left(1 + \tau \sum_{j=1}^{k-1} \left|\nabla \overset{\bullet}{u}_j\right|_2 |\nabla u_j|_{2\hat{q}}^{\hat{q}}\right) \\ &\leq C \left(1 + \left(\tau \sum_{j=1}^{k-1} \left|\nabla \overset{\bullet}{u}_j\right|_2^2\right)^{1/2} \left(\tau \sum_{j=1}^{k-1} |\nabla u_j|_{2\hat{q}}^{2\hat{q}}\right)^{1/2}\right);\end{aligned} \tag{5.17}$$

hence, by virtue of (5.11) and the embedding of $W^{2,2}(\Omega)$ in $W^{1,2\hat{q}}(\Omega)$, we have

$$\max_k |\nabla b_{k-1}|_{\hat{q}+1} \leq C\,. \tag{5.18}$$

In particular, the product $|\overset{\bullet}{\mathbf{v}}_k|\,|\nabla b_{k-1}|$ satisfies (5.16) with $q = 4(\hat{q}+1)/(\hat{q}+5) > 2$. Hence, from Lemma A.3, we conclude that the norms $|\overset{\bullet}{u}_n|_\infty$ are bounded independently of n and τ. This property is preserved when passing to the limit as $\tau \to 0$, which means for the solution u of Problem 3.1 that

$$||u_t||_{L^\infty(\Omega_T)} \le C. \tag{5.19}$$

By Theorem 5.1, the solution to Problem 3.1 is unique, which completes the proof of Theorem 3.2. ☐

Appendix. Auxiliary results.

A.1. A discrete first order energy inequality. We establish here a discrete counterpart of the equation (ii) in Proposition 2.5. We set $\xi_k^r(x) := \xi_k(x,r)$, where $\xi_k(x,r)$ has been introduced in (4.6). As a discrete counterpart of (2.1) and (2.2), it follows from (4.6) that

$$(\xi_k^r - \xi_{k-1}^r)\,(x_k - z) \ge 0 \qquad \forall\, |z| \le\, r,$$

where $x_k := u_k - \xi_k^r$. For $z = r\,\mathrm{sign}\,(\xi_k^r - \xi_{k-1}^r)$, this yields

$$(\xi_k^r - \xi_{k-1}^r)\, x_k \ge\, r\,|\xi_k^r - \xi_{k-1}^r|. \tag{A.1}$$

Let ψ be an arbitrary function satisfying Assumption 2.1. We define the discrete versions of the Preisach potential energy $\mathcal{E}$ and dissipation operator $\mathcal{S}$, introduced in (2.15) and (2.17), respectively, as

$$E_k(x) = \int_0^\infty G(r, \xi_k^r(x))\,\mathrm{d}r,$$

and

$$S_k(x) = \int_0^\infty r\, g(r, \xi_k^r(x))\,\mathrm{d}r,$$

with G given by (2.16). This implies that

$$E_k - E_{k-1} = \int_0^\infty \int_{\xi_{k-1}^r}^{\xi_k^r} v\,\psi(r,v)\,\mathrm{d}v\,\mathrm{d}r.$$

We also have, by (4.4), that

$$w_k - w_{k-1} = \int_0^\infty \int_{\xi_{k-1}^r}^{\xi_k^r} \psi(r,v)\,\mathrm{d}v\,\mathrm{d}r. \tag{A.2}$$

Now, suppose that $\xi_k^r > \xi_{k-1}^r$ (the other case is analogous); we have

$$(w_k - w_{k-1})\,u_k - (E_k - E_{k-1}) = \int_0^\infty \int_{\xi_{k-1}^r}^{\xi_k^r} (u_k - v)\,\psi(r,v)\,\mathrm{d}v\,\mathrm{d}r$$

$$= \int_0^\infty \frac{1}{\xi_k^r - \xi_{k-1}^r} \int_{\xi_{k-1}^r}^{\xi_k^r} (\xi_k^r - \xi_{k-1}^r)\,(x_k + \xi_k^r - v)\,\psi(r,v)\,\mathrm{d}v\,\mathrm{d}r.$$

Now we remark that

$$(\xi_k^r - \xi_{k-1}^r)\,(x_k + \xi_k^r - v) = (\xi_k^r - \xi_{k-1}^r)\,x_k + (\xi_k^r - \xi_{k-1}^r)\,(\xi_k^r - v) \overset{(A.1)}{\geq} r\,|\xi_k^r - \xi_{k-1}^r|,$$

as $v \in (\xi_{k-1}^r, \xi_k^r)$; therefore, we deduce

$$(w_k - w_{k-1})\,u_k - (E_k - E_{k-1}) \geq \int_0^\infty \int_{\xi_{k-1}^r}^{\xi_k^r} r\,\psi(r,v)\,\mathrm{d}v\,\mathrm{d}r = |S_k - S_{k-1}|. \tag{A.3}$$

Remark A.1. Inequality (A.3) is valid for every function ψ satisfying Assumption 2.1. We use it in subsection 4.4 in the special case $\psi = \psi_R$.

A.2. A discrete second order energy inequality. We show here the connection between the convexity of the Preisach hysteresis loops and a second order energy inequality in the time discrete case. The time continuous case with $p = 2$ is treated in detail in [19, sections II.3 and II.4]. Let $p \geq 2$ be arbitrary and set $F_k = \overset{\bullet}{u}_k \mid \overset{\bullet}{u}_k \mid^{p-2}$, with the notations in (4.2). Our aim is to prove that for every $k = 2, \dots, n$, $n \in \{1, \dots, m\}$ and a. e. $x \in \Omega$ we have

$$\left(\overset{\bullet}{w}_k - \overset{\bullet}{w}_{k-1}\right) F_k \geq \frac{1}{p}\left(\overset{\bullet}{w}_k\, F_k - \overset{\bullet}{w}_{k-1}\, F_{k-1}\right). \tag{A.4}$$

To prove (A.4), let $\Omega' \subset \Omega$ be the set of full measure (meas $(\Omega \setminus \Omega') = 0$) for which (4.4)–(4.6) hold for all $k = 1, \dots, m$, and fix $x \in \Omega'$. Let us define the function $\hat{p}(r,v) = P[\xi_{k-2}(x,\cdot), v](r)$, where P has been introduced in (2.10). By (4.4)–(4.6) we have (omitting the argument x)

$$w_{k-1} = \int_0^\infty g_R(r, \hat{p}(r, u_{k-1}))\,\mathrm{d}r, \qquad w_{k-2} = \int_0^\infty g_R(r, \hat{p}(r, u_{k-2}))\,\mathrm{d}r. \tag{A.5}$$

In the second identity we used the obvious implication

$$v - r \leq \lambda(r) \leq v + r \;\; \Rightarrow \;\; P[\lambda, v](r) = \lambda(r). \tag{A.6}$$

Hence $w_{k-1} = w_{k-2}$ whenever $u_{k-1} = u_{k-2}$. Inequality (A.4) is automatically fulfilled if $u_k = u_{k-1}$ or $u_{k-1} = u_{k-2}$.

We may assume from now on that $u_k \neq u_{k-1}$, $u_{k-1} \neq u_{k-2}$ and set

$$L_k = \frac{w_k - w_{k-1}}{u_k - u_{k-1}} \geq 0.$$

Then (A.4) reads

$$\left(1 - \frac{1}{p}\right) L_k \left|\overset{\bullet}{u}_k\right|^p + \frac{1}{p} L_{k-1} \left|\overset{\bullet}{u}_{k-1}\right|^p \geq L_{k-1}\, \overset{\bullet}{u}_{k-1}\, F_k. \tag{A.7}$$

If $\overset{\bullet}{u}_k\, \overset{\bullet}{u}_{k-1} < 0$, then (A.7) holds automatically, since its right-hand side is nonpositive. Otherwise, we estimate it as

$$L_{k-1}\, \overset{\bullet}{u}_{k-1}\, F_k \leq L_{k-1} \left|\overset{\bullet}{u}_{k-1}\right| \left|\overset{\bullet}{u}_k\right|^{p-1} \leq L_{k-1} \left(\left(1 - \frac{1}{p}\right) \left|\overset{\bullet}{u}_k\right|^p + \frac{1}{p} \left|\overset{\bullet}{u}_{k-1}\right|^p\right);$$

hence, (A.7) will be proved if we can show that

$$L_k \geq L_{k-1}, \;\; \text{whenever } u_{k-2} < u_{k-1} < u_k \text{ or } u_{k-2} > u_{k-1} > u_k. \tag{A.8}$$

Assume first that $u_{k-2} < u_{k-1} < u_k$. In addition to (A.5), we have in this case

$$w_k = \int_0^\infty g_R(r, \hat{p}(r, u_k))\,\mathrm{d}r, \tag{A.9}$$

using the fact that $P[\xi_{k-1}, u_k](r) = P[\xi_{k-2}, u_k](r) = \max\{u_k - r, \xi_{k-2}(r)\}$. Hence,

$$w_j = \Phi(u_j) \qquad \text{for } j = k-2, k-1, k, \tag{A.10}$$

where

$$\Phi(v) = \int_0^\infty g_R(r, \max\{v - r, \xi_{k-2}(r)\})\,\mathrm{d}r. \tag{A.11}$$

Set

$$m_{k-2}(v) = \min\{r \geq 0;\, v \leq r + \xi_{k-2}(r)\}.$$

Then

$$\Phi'(v) = \int_0^{m_{k-2}(v)} \psi_R(r, v - r)\,\mathrm{d}r.$$

The function m_{k-2} is increasing; for $u_{k-2} \leq v_1 < v_2 \leq u_k$ we have $m_{k-2}(v_2) - m_{k-2}(v_1) \geq \frac{1}{2}(v_2 - v_1)$ and

$$\Phi'(v_2) - \Phi'(v_1) = \int_{m_{k-2}(v_1)}^{m_{k-2}(v_2)} \psi_R(r, v_2 - r)\,\mathrm{d}r + \int_0^{m_{k-2}(v_1)} (\psi_R(r, v_2 - r) - \psi_R(r, v_1 - r))\,\mathrm{d}r.$$

Using Assumption 2.7(ii), we see that

$$\int_{m_{k-2}(v_1)}^{m_{k-2}(v_2)} \psi_R(r, v_2 - r)\,\mathrm{d}r \geq A_R(m_{k-2}(v_2) - m_{k-2}(v_1)) \geq \frac{1}{2} A_R\,(v_2 - v_1)$$

and

$$\left| \int_0^{m_{k-2}(v_1)} (\psi_R(r, v_2 - r) - \psi_R(r, v_1 - r))\,\mathrm{d}r \right| \leq R\,C_R\,(v_2 - v_1);$$

hence,

$$\Phi'(v_2) - \Phi'(v_1) \geq \left(\frac{1}{2} A_R - R\,C_R\right)(v_2 - v_1) \overset{(2.24)}{=} K_R\,(v_2 - v_1).$$

We see that Φ is convex (as $K_R > 0$); hence $L_k \geq L_{k-1}$ and (A.4) follows.

The case $u_{k-2} > u_{k-1} > u_k$ can be treated in an analogous way.

Similarly to (A.5), we have

$$\widetilde{w}_{k-1} = \int_0^\infty \widetilde{g}_R(r, \hat{p}(r, u_{k-1}))\,\mathrm{d}r, \qquad \widetilde{w}_{k-2} = \int_0^\infty \widetilde{g}_R(r, \hat{p}(r, u_{k-2}))\,\mathrm{d}r; \tag{A.12}$$

hence,

$$|\widetilde{w}_{k-1} - \widetilde{w}_{k-2}| \leq \left| \int_0^\infty \int_{\hat{p}(r, u_{k-2})}^{\hat{p}(r, u_{k-1})} \widetilde{\psi}_R(r, v)\,\mathrm{d}v\,\mathrm{d}r \right| \leq \tilde{b}_1(R)\,|u_{k-1} - u_{k-2}|, \tag{A.13}$$

with $\tilde{b}_1$ given by (2.12).

A.3. The Gagliardo–Nirenberg inequality. We recall the *Gagliardo–Nirenberg inequality* (for more details see, for example, [2, 3, 14]).

PROPOSITION A.2. *Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$ be a bounded Lipschitzian domain, and let $1 \leq q_1, q_2, q_3 \leq \infty$ be given. Then there exists a constant $C_{q_1,q_2,q_3} > 0$ such that for every $v \in W^{1,q_3}(\Omega)$ we have*

$$|v|_{q_1} \leq C_{q_1,q_2,q_3} \, (|v|_{q_2} + |v|_{q_2}^{1-\rho} \, |\nabla v|_{q_3}^{\rho}), \tag{A.14}$$

provided

$$\rho = \frac{\frac{1}{q_2} - \frac{1}{q_1}}{\frac{1}{N} + \frac{1}{q_2} - \frac{1}{q_3}}, \qquad \frac{1}{q_2} > \frac{1}{q_1} > \frac{1}{q_3} - \frac{1}{N}\,,$$

with the convention $1/\infty = 0$. If moreover $v \in W_0^{1,q_3}(\Omega)$, then (A.14) *can be written in the form*

$$|v|_{q_1} \leq C_{q_1,q_2,q_3} \, |v|_{q_2}^{1-\rho} \, |\nabla v|_{q_3}^{\rho}. \tag{A.15}$$

A.4. A discrete Moser iteration lemma. We prove here the following lemma, inspired by [21, Lemma 5.6, Chapter II]).

LEMMA A.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitzian boundary, $N \geq 2$, and let $q > q_0 := (N/2) + 1$ and sequences $\{f_{km}; m \in \mathbb{N}, k = 1, \ldots, m\}$ in $L^q(\Omega)$, $\{U_{km}; m \in \mathbb{N}, k = 1, \ldots, m\}$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ be given. Assume that there exist constants $M > 0$, $E > 0$, and a polynomial H, all independent of m, such that*

$$\left(\frac{1}{m} \sum_{k=1}^{m} |U_{km}|_{q'}^{q'}\right)^{1/q'} \leq M, \quad \left(\frac{1}{m} \sum_{k=1}^{m} |f_{km}|_{q}^{q}\right)^{1/q} \leq M, \tag{A.16}$$

where q' is the conjugate exponent to q, and

$$\begin{aligned} &\frac{1}{p} \int_\Omega |U_{nm}|^p \,\mathrm{d}x + \frac{p}{m} \sum_{k=1}^{n} \int_\Omega |\nabla U_{km}|^2 \, |U_{km}|^{p-2} \,\mathrm{d}x \\ &\leq \frac{1}{p} \, E^p + \frac{H(p)}{m} \sum_{k=1}^{n} \int_\Omega |f_{km}| \, |U_{km}|^{p-1} \,\mathrm{d}x \quad \forall \, p \geq 2 \;\; \forall n = 1, \ldots, m. \end{aligned} \tag{A.17}$$

Then we have

$$\sup_{m \in \mathbb{N}} \max_{k=1,\ldots,m} |U_{km}|_\infty < \infty. \tag{A.18}$$

Proof. We denote by p' the conjugate exponent to p for every $p \geq 2$, and by C any constant independent of k, p, and m. For $j \in \mathbb{N} \cup \{0\}$ we define the sequence

$$p_j = 2(1 + \kappa)^j, \qquad \kappa = \frac{q_0'}{q'} - 1 > 0. \tag{A.19}$$

Let $\{Z_{km}^{(j)}\}$ be the sequence

$$Z_{km}^{(j)} := U_{km} \, |U_{km}|^{\frac{p_j}{2} - 1}, \tag{A.20}$$

so that

$$\left|Z_{km}^{(j)}\right|^2 = |U_{km}|^{p_j}, \qquad \left|\nabla Z_{km}^{(j)}\right|^2 = \frac{p_j^2}{4}|\nabla U_{km}|^2\,|U_{km}|^{p_j-2}. \tag{A.21}$$

By (A.17), Hölder's inequality with exponents q, $p_j q'$, and $p_j' q'$, and hypothesis (A.16), we have for all admissible indices

$$\begin{aligned}
&\int_\Omega \left|Z_{nm}^{(j)}\right|^2 \,\mathrm{d}x + \frac{4}{m}\sum_{k=1}^{n}\int_\Omega \left|\nabla Z_{km}^{(j)}\right|^2 \,\mathrm{d}x \\
&\le\ E^{p_j} + p_j H(p_j)\,\frac{1}{m}\sum_{k=1}^{n}\int_\Omega |f_{km}|\ \left|Z_{km}^{(j)}\right|^{2/p_j'} \mathrm{d}x \\
&\le\ E^{p_j} + p_j H(p_j)\left(\frac{1}{m}\sum_{k=1}^{n}|f_{km}|_q^q\right)^{1/q}\left(\frac{1}{m}\sum_{k=1}^{n}|1|_{p_jq'}^{p_jq'}\right)^{1/p_jq'}\left(\frac{1}{m}\sum_{k=1}^{n}\left|Z_{km}^{(j)}\right|_{2q'}^{2q'}\right)^{1/p_j'q'} \\
&\le\ E^{p_j} + |\Omega|^{1/p_jq'} M\, p_j H(p_j)\left(\frac{1}{p_j} + \frac{1}{p_j'}\left(\frac{1}{m}\sum_{k=1}^{n}\left|Z_{km}^{(j)}\right|_{2q'}^{2q'}\right)^{1/q'}\right).
\end{aligned} \tag{A.22}$$

From the Gagliardo–Nirenberg inequality (A.15) with $q_1 := 2q_0'$, $q_2 = q_3 := 2$, $\rho = N/(N+2)$, it follows that

$$\left|Z_{km}^{(j)}\right|_{2q_0'}^{2q_0'} \le C\left|Z_{km}^{(j)}\right|_2^{4/N}\left|\nabla Z_{km}^{(j)}\right|_2^2;$$

hence, by Young's inequality,

$$\left(\frac{1}{m}\sum_{k=1}^{n}\int_\Omega \left|Z_{km}^{(j)}\right|^{2q_0'} \mathrm{d}x\right)^{1/q_0'} \le\ C\left(\max_{k=1,\dots,n}\int_\Omega \left|Z_{km}^{(j)}\right|^2 \mathrm{d}x + \frac{1}{m}\sum_{k=1}^{n}\int_\Omega \left|\nabla Z_{km}^{(j)}\right|^2 \mathrm{d}x\right). \tag{A.23}$$

By virtue of (A.22)–(A.23), there exists another polynomial $\tilde H$ independent of m and j such that

$$\left(\frac{1}{m}\sum_{k=1}^{m}\int_\Omega \left|Z_{km}^{(j)}\right|^{2q_0'} \mathrm{d}x\right)^{1/q_0'} \le \tilde H(p_j)\max\left\{1, E^{p_j}, \left(\frac{1}{m}\sum_{k=1}^{m}\int_\Omega \left|Z_{km}^{(j)}\right|^{2q'} \mathrm{d}x\right)^{1/q'}\right\}. \tag{A.24}$$

By (A.19), we have $q'p_j = q_0' p_{j-1}$; in view of (A.20), inequality (A.24) is thus equivalent to

$$\left(\frac{1}{m}\sum_{k=1}^{m}\int_\Omega |U_{km}|^{q_0'p_j} \mathrm{d}x\right)^{1/q_0'} \le \tilde H(p_j)\max\left\{1, E^{p_j}, \left(\frac{1}{m}\sum_{k=1}^{m}\int_\Omega |U_{km}|^{q_0'p_{j-1}} \mathrm{d}x\right)^{1/q'}\right\}. \tag{A.25}$$

Set

$$d_j := \left(\frac{1}{m}\sum_{k=1}^{m}\int_\Omega |U_{km}|^{q_0'p_j} \mathrm{d}x\right)^{1/q_0'p_j}. \tag{A.26}$$

Then (A.25) can be written as

$$d_j \leq \tilde{H}(p_j)^{1/p_j} \max\{1, E, d_{j-1}\}. \tag{A.27}$$

For $D_j := \max\{1, E, d_j\}$, this yields in particular

$$D_j \leq \tilde{H}(p_j)^{1/p_j} D_{j-1}; \tag{A.28}$$

hence,

$$D_j \leq D_0 \prod_{i=1}^{j} \tilde{H}(p_i)^{1/p_i} \leq C\, D_0. \tag{A.29}$$

A bound for D_0 follows from the inequalities (A.23) with $j = 0$, (A.17) with $p = 2$, and (A.16). Consequently, for all $j \in \mathbb{N}$ we have the estimate

$$\left(\frac{1}{m} \sum_{k=1}^{m} \int_\Omega |U_{km}|^{q_0' p_j} \, \mathrm{d}x \right)^{1/q_0' p_j} \leq C. \tag{A.30}$$

Assume that there exist $\varepsilon > 0$, $1 \leq k \leq m$, and a set $\Omega_{km} \subset \Omega$ such that $|U_{km}(x)| \geq C + \varepsilon$ for $x \in \Omega_{km}$, with C from (A.30). Then $(|\Omega_{km}|/m)(C+\varepsilon)^{q_0' p_j} \leq C^{q_0' p_j}$ for all $j \in \mathbb{N}$ by virtue of (A.30). Letting $j \to \infty$ we obtain $|\Omega_{km}| = 0$; hence, (A.18) holds and the proof is complete. $\square$

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