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1 Introduction

In several papers and books one can find that the next four conditions for a continuous map f of the interval are equivalent:

(P_1) f has zero topological entropy;

(P_2) the set of periodic points of f is a G_δ set (countable intersection of open sets);

(P_3) the set of recurrent points of f is an F_σ set (countable union of closed sets);

(P_4) f is Lyapunov stable on the set of its periodic points.

These results were published by A. N. Sharkovsky and his group, cf., e.g., [F], [FSS], [K], [KS], [SKSF1], [Sh5], [SMR], [SKSF2], [Sh3]. Unfortunately, it is not true. We disprove the original equivalences

$$(P_2) \Leftrightarrow (P_3) \Leftrightarrow (P_1) \Leftrightarrow (P_4)$$

and show that

$$(P_2) \underset{\not\Leftarrow}{\overset{?}{\Rightarrow}} (P_3) \underset{\not\Leftarrow}{\Rightarrow} (P_1) \underset{\not\Rightarrow}{\Leftarrow} (P_4)$$

Note, that other authors supplied counterexamples to different conjectures and theorems of Sharkovsky, e.g., [ChX] or [ACS].

The pricipal part of the Thesis consist of three papers. Paper [Si1] proves that (P_4) implies (P_1) and exhibits a map satisfying (P_1), but not (P_4). The next paper [Si2] proves that (P_2) implies (P_1) and exhibits a map satisfying (P_1), but not (P_2). This map, however, has the property (P_3) (in other words, that $(P_3) \not\Rightarrow (P_2)$). In paper [Si3], we further show that there is a continuous map f satisfying (P_1), but neither (P_2) nor (P_3). We also show that (P_3) implies (P_1). We can still ask about relations between (P_4) and (P_2) and between (P_4) and (P_3). There is also an open problem if (P_2) implies (P_3). Our conjecture here is that yes. Summarizing the papers [Si2] and [Si3] including the conjectured implication, we get the following ordering: (P_2) is stronger than (P_3), and (P_3) is stronger than (P_1).

2 Terminology and preliminaries

In the sequel, $\mathbb{N}$ denotes the set of all positive integers, $I = [0, 1]$ is the the unit compact interval, and I^n is the n-dimensional cube. For a compact metric space X, $C(X, X)$ denotes the space of continuous maps of X into itself. Let $\overline{A}$ be the closure of a set A and $|| \cdot ||$ the uniform norm.

We define the nth iterate f^n of a map f by $f \circ f^{n-1}$. If for some $n \in \mathbb{N}$, $f^n(x) = x$ then the point x is called a *periodic point* with period n. If $f(x) = x$ then x is a *fixed point* of f. By the period of a periodic point we will mean its smallest period. The set of periodic points of f is denoted by $\mathrm{Per}(f)$ and the set of fixed points by $\mathrm{Fix}(f)$. A periodic point p is *repelling* if the one-sided derivatives of f^n at p have absolute values greater than 1 for each $n \in \mathbb{N}$.

The *orbit* (resp. *trajectory*) of a set A, written $\mathrm{Orb}_f(A)$ (resp. $\mathrm{Traj}_f(A)$), is the smallest set containing A that is closed under both images and preimages with respect to f (resp. the smallest set containing A closed under f).

The ω-limit set of the trajectory of $x \in I$ is the set of accumulation points of this trajectory and is denoted by $\omega_f(x)$, and $\omega(f) = \bigcup\{\omega_f(x);\ x \in I\}$ is the set of ω-limit points of f.

If the periods of points in $\mathrm{Per}(f)$ are the numbers $1, 2, \ldots, 2^n$ then f is of *type* 2^n, and if the periods are all powers of 2 then f is called of *type* 2^∞.

A map $f \in C(I, I)$ is *unimodal* if there exists $c \in (0, 1)$ such that f is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$. A map f is *weakly unimodal* if there exists $c \in (0, 1)$ such that f is non-decreasing on $[0, c]$ and non-increasing on $[c, 1]$.

A map $f \in C(I, I)$ is *Lyapunov stable* on a set $A \subseteq I$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ for x and y in A then $|f^n(x) - f^n(y)| < \varepsilon$ for any n.

Let $f \in C(I, I)$. Then $E \subset I$ is an *(n, ε)-separated set* if, for every two different points x, y from E, there is a j, $0 \leq j < n$, with $|f^j(x) - f^j(y)| > \varepsilon > 0$. If M is a compact subset of I denote by $s_n(\varepsilon, M, f)$ the maximum possible number of points in an (n, ε)-separated subsets of M. Put $\overline{s}(\varepsilon, M, f) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, M, f)$. The *topological entropy of the map f with respect to the compact subset M* and the *topological entropy of the map f* are defined by $h(f, M) = \lim_{\varepsilon\to 0} \overline{s}(\varepsilon, M, f)$ and $h(f) = h(f, I)$, respectively. Topological entropy is an important topological invariant and it is a measure of the dynamical complexity of the map.

A map $F \in C(I^3, I^3)$ such that $F(x, y, z) = (f(x), g(x, y), h(x, y, z))$ is a *triangular map*, f is the *base* of F, and the set $I_x := \{x\} \times I^2$ is the *layer over x*.

Sharkovsky Theorem ([BC] and [SKSF2]). *Let $f \in C(I, I)$. In the set of positive integers, define an ordering as follows*

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^2 \cdot 7 \prec 2^2 \cdot 5 \prec 2^2 \cdot 3 \prec \cdots \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \cdots \prec 9 \prec 7 \prec 5 \prec 3.$$

If f has a cycle of order m and $n \prec m$, then f has a cycle of order n as well. Moreover, for any m there exists a map with cycle of period m and no cycles of periods n if $m \prec n$.

Thus, by the Sharkovsky's theorem there are functions f in $C(I, I)$ such that the set of periods of points in $\mathrm{Per}(f)$ is the set $\{2^n\}_{n=0}^{\infty}$, i.e. f is of type 2^{∞}.

The following theorem gives a characterization of continuous maps of the unit interval I with zero topological entropy: A map f in the class $C(I, I)$ has zero topological entropy if and only if it is of type $\preceq 2^{\infty}$.

Misiurewicz Theorem ([M1], cf. also [BC]). *Let $f \in C(I, I)$. Then f has positive topological entropy if and only if f has a periodic point whose period is not a power of* 2.

3 Lyapunov stability

In Section 5 of the thesis, we find a class of weakly unimodal C^{∞} maps of an interval with zero topological entropy such that no such map f is Lyapunov stable on the set $\mathrm{Per}(f)$ of its periodic points. This disproves a statement published in several books and papers, e.g., by V.V. Fedorenko, S. F. Kolyada, A. N. Sharkovsky, A. G. Sivak and J. Smítal.

Theorem A. *No $f \in \mathcal{F}$ is Lyapunov stable on* $\mathrm{Per}(f)$. *On the other hand, $\mathcal{F}$ consists of mappings with zero topological entropy and contains a C^{∞} map.*

Theorem B. *Let $f \in C(I, I)$. If f is Lyapunov stable on* $\mathrm{Per}(f)$ *then f has zero topological entropy.*

Remark. A different counterexample to the problem is also given in [Si0]. A map f of the unit interval with zero topological entropy and such that f is not Lyapunov stable on the set of its periodic points is obtained from the Feigenbaum map. The method of blowing up orbits introduced by A. Denjoy [D] is used in the construction.

4 Periodic points not a G_δ set

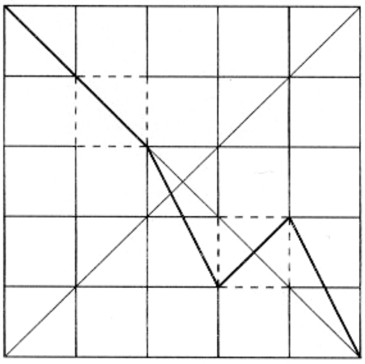
In Section 6 of the thesis we exhibit an example of a continuous map of the interval of type 2^∞ and hence, by a theorem of Misiurewicz, of zero topological entropy, for which the set of periodic points is not a G_δ set. This disproves a conjecture by Sharkovsky from 1965. Unfortunately, this conjecture has been incorrectly quoted as a true statement by other authors in many papers and books. The desired map is a limit of a sequence of maps starting with the well-known Feigenbaum map.

Theorem C. *There is a map $f \in C(I, I)$ with zero topological entropy and such that* $\mathrm{Per}(f)$ *is not a G_δ set.*

5 Recurrent points not a closed and not an F_σ set

We construct in the thesis in Section 7 a continuous map χ of the unit interval into itself of type 2^∞ which has a trajectory disjoint from the set $\mathrm{Rec}(\chi)$ of recurrent points of χ, but contained in the closure of $\mathrm{Rec}(\chi)$. In particular, $\mathrm{Rec}(\chi)$ is not closed. A function ψ of type 2^∞, with non-closed set of recurrent points, was found by [ChX]. However, there is not a trajectory in their construction contained in $\overline{\mathrm{Rec}(\psi)} \setminus \mathrm{Rec}(\psi)$, since any point in $\overline{\mathrm{Rec}(\psi)}$ is eventually mapped into $\mathrm{Rec}(\psi)$. Moreover, our construction is simpler.

The map χ is constructed as the uniform limit of a sequence of maps starting with the continuous map on the picture bellow and by the method of blowing up orbits by A. Denjoy [D].



Theorem D. *There is a map $\chi \in C(I, I)$ of type 2^∞ such that*

(i) *χ has a unique infinite maximal ω-limit set $\tilde{\omega} = R \cup P$, where R is a Cantor set, and $P = \{v_n\}_{n=-\infty}^{\infty}$ an infinite set of points isolated in $\tilde{\omega}$ such that $\chi(v_n) = v_{n+1}$, for any n;*

(ii) *$P \subset \overline{\mathrm{Per}(\chi)} \setminus \mathrm{Per}(\chi)$;*

(iii) *the set $\mathrm{Rec}(\chi)$ is not closed;*

(iv) *any point in $\mathrm{Per}(\chi)$ is isolated in $\omega(\chi)$, and repelling;*

(v) *χ is monotone in a neighborhood of any $p \in P$.*

We use χ to show that there is a continuous map of the interval of type 2^∞ for which the set of recurrent points is not an F_σ set. This example disproves a conjecture of A. N. Sharkovsky et al., from 1989.

Theorem E. *There is a map $f \in C(I, I)$ with zero topological entropy such that $\mathrm{Rec}(f)$ is not an F_σ set.*

Theorem F. *Let $f \in C(I, I)$. If $\mathrm{Rec}(f)$ is an F_σ set then f has zero topological entropy or, equivalently, is of type $\leq 2^\infty$.*

We provide also another application of χ for triangular maps of the unit square I^2.

6 On a problem concerning ω-limit sets of triangular maps in I^3

In the last Section 8 of the thesis we shortly show that there is a continuous triangular map of the unit cube I^3 into itself with the set of ω-limit points $\omega(F) = \{0\} \times I^2 = \omega_F(x, y, z)$ for any $(x, y, z) \in I^3$ such that $x \neq 0$. This map is of the form $F(x, y, z) = (f(x), g(x, y), h(x, z))$, where the maps $g(x, .)$ and $h(x, .)$ are non-decreasing. This solves a problem by F. Balibrea, L. Reich, and J. Smítal. The same problem was solved independently by [BGC].

Theorem G. *There is a triangular map $F \in C(I^3, I^3)$ with $\omega(F) = \{0\} \times I^2 = \omega_F(x, y, z)$, for any $(x, y, z) \in I^3$ such that $x \neq 0$. This map has a special form $F(x, y, z) = (f(x), g_x(y), h_x(z))$.*

7 Publications

[1] P. Šindelářová, A counterexample to a statement concerning Ljapunov stability, *Preprint MA* 16/2000, Mathematical Institute, Silesian University at Opava, 2000.

[2] P. Šindelářová, Counterexamples to Sharkovsky's conjectures concerning maps with zero topological entropy, *Real Analysis Exchange* **27** (1) 2001/2002, 25th Summer Symposium Conference Report, 47-50. (Abstract of the talk at Summer Symposium on Real Analysis, Ogden Utah, 2001).

[3] P. Šindelářová, A counterexample to a statement concerning Lyapunov stability, *Acta Math. Univ. Comen.* **70** (2001) (2), 265-268.

[4] P. Šindelářová, A zero topological entropy map for which periodic points are not a G_δ set, *Ergod. Th. & Dynam. Sys.* **22** (2002) (3), 947–949.

[5] P. Šindelářová, On a problem concerning ω-limit sets of triangular maps in I^3, *Preprint MA* 34/2002, Mathematical Institute, Silesian University at Opava, 2002.

[6] P. Šindelářová, A zero topological entropy map with recurrent points not F_σ, *Proc. Amer. Math. Soc.* **131** (2003) (7), 2089–2096.

8 Presentations

Conferences, schools

3rd Czech–Slovak Workshop on Discrete Dynamical Systems, Liptovský Trnovec, Slovakia, September 23–29, 1999.

28th Winter School in Abstract Analysis, Křišťanovice, Czech Republic, January 23–29, 2000. Talk on [1].

4th Czech–Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, June 22–28, 2000. Talk on [1].

September 2000–February 2001, Semester at Mathematical Institute of University Würzburg, Germany. Talk on [1].

29th Winter School in Abstract Analysis, Lhota nad Rohanovem, Czech Republic, February 3–10, 2001. Talk on [4].

Summer Symposium in Real Analysis XXV, Weber State University, Ogden, Utah, May 22–26, 2001. Talk on [2].

5th Czech–Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, June 18–23, 2001. Talk on [4] and [6].

School and Workshop on Dynamical Systems, Trieste, Italy, July 30–August 17, 2001.

30th Winter School in Abstract Analysis, Lhota nad Rohanovem, Czech Republic, January 19–26, 2002. Talk on [6].

Research Trimester on Dynamical Systems, Pisa, Italy, February 1–28, 2002. Talk on [4] and [6].

6th Czech–Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, June 9–16, 2002. Talk on [5].

Summer Symposium in Real Analysis XXVI, Washington and Lee University, Lexington, Virginia, June 25–29, 2002. Talk on [6].

New Directions in Dynamical Systems, Ryukoku University and Kyoto University, Kyoto, Japan, August 5–15, 2002. Talk on [4] and [6].

14th European Conference on Iteration Theory (ECIT 2002), Évora, Portugal, September 1–7, 2002. Talk on [6].

February 2003–March 2004, Mathematics Institute, University of Warwick, United Kingdom, Marie Curie Postgraduate Studentships.

Holomorphic Dynamics Workshop of the Warwick Dynamics Symposium 2002-2003, University of Warwick, Coventry, UK, April 4–12, 2003.

Dynamical Systems, University of North Texas, Denton, Texas, May 25–29, 2003. Talk.

Summer Symposium in Real Analysis XXVII, Silesian University in Opava, Czech Republic, June 23–29, 2003. Talk.

Symbolic Dynamics and Ergodic Theory Workshop of the Warwick Dynamics Symposium 2002-2003, University of Warwick, Coventry, UK, July 7–18, 2003.

Geometric Aspects of Dynamical Systems, Fourth Meeting of the Warwick Dynamics Symposium 2002-2003, University of Warwick, Coventry, UK, July 21–25, 2003.

Recent Trends in Nonlinear Science, Winter School, Palma de Mallorca, Spain, February 1–6, 2004.

Since May 2004, Mathematics Department, Auburn University, Auburn, Alabama.

Summer Symposium in Real Analysis XXVIII, Slippery Rock University, Pennsylvania, June 8–13, 2004. Talk: "Topological properties of maps with zero topological entropy".

AIMS' Fifth International Conference on Dynamical Systems and Differential Equations, Department of Mathematics and Statistics, California State Polytechnic University, Pomona, California, June 16–19, 2004. Invited talk in the section of Low Dimensional Dynamics: "On Maps with Zero Topological Entropy".

Projects

Participation at Cooperation project AKTION Österreich 2003/21, Iterative functional equations and their applications, Univ. Graz, Austria, May 6–9, 2003.

Awards

Second prize in the Mathematical competition of students of Czech universities SVOČ, May 2000. Talk on [1].

First prize in the Mathematical competition of students of Czech universities SVOČ, May 2001. Talk on [4] and [6].

Third prize in the 18th Annual GSC Research Forum, Auburn University, March 2005. Talk.

First prize in the Student Paper Competiton of SIAM Chapter, Auburn University, March 2005. Talk.

References

[ACS] L. Alsedà, M. Chas and J. Smítal, On the structure of the ω-limit sets for continuous maps of the interval, Discrete dynamical systems, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 9 (1999), no. 9, 1719–1729. MR **2000i:**37047

[BGC] F. Balibrea, J. L. García Guirao, and J. I. Muñoz Casado, On ω-limit sets of triangular maps on the unit cube, *J. Difference Equ. Appl.* 9 (2003), no. 3-4, 289–304. MR **2004e:**37029

[BRS] F. Balibrea, L. Reich, and J. Smítal, Iteration theory: Dynamical systems and functional equations, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **13** (2003), no. 7, 1627–1647. MR **2004m:**37014

[B] L. Block, Stability of periodic orbits in the theorem of Šarkovskii, *Proc. Amer. Math. Soc.* **81** (1981), 333–336. MR **82b:**58071

[BC] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math. 1513, Springer, Berlin - Heidelberg - New York, 1992. MR **93g:**58091

[BS] A. M. Bruckner and J. Smítal, A characterization of ω-limit sets of maps of the interval with zero topological entropy, *Ergod. Th. & Dynam. Sys.* **13** (1993), 7-9. MR **94k:**26006

[ChX] H. Chu and J. Xiong, A Counterexample in Dynamical Systems of the Interval, *Proc. Amer. Math. Soc.* **97** (1986), 361–366. MR **87i:**58140

[CE] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems,* Birkhauser: Boston, 1980. MR **82j:**58078

[D] A. Denjoy, Sur les courbes définies par les équations differentielles à la surface du tore, *J. Math. Pures Appl.* **11** (1932), 333-375. MR **21**#180

[F] V. V. Fedorenko, Classification of simple one-dimensional dynamical systems, *Akad. Nauk Ukrain. SSR Inst. Mat. Preprint* 1991, no. 5, 30 pp. (Russian). MR **92m:**58037

[FSS] V. V. Fedorenko, A. N. Sharkovsky and J. Smítal, Characterization of weakly chaotic maps of the interval, *Proc. Amer. Math. Soc.* **110** (1990), 141–148. MR **91a:**58148

[Fu] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, 1981. MR **82j:**28010

[J] L. Jonker, Periodic orbits and kneading invariants, *Proc. London Math. Soc.* **39** (3) (1979), 428-450. MR **81g:**58029

[K] S. F. Kolyada. On dynamics of triangular maps of the square, *Ergod. Th. & Dynam. Sys.* **12** (1992), 749–768. MR **93m:**58036

[KS] S. F. Kolyada and A. N. Sharkovsky, *On topological dynamics of triangular maps of the plane*, European Conference on Iteration Theory, Proceedings of the conference (ECIT '89) held in Batschuns, September 10-16, 1989. World Sci. Publishing Co., Inc., River Edge, NJ, 1991, 177-183. MR **93f:**58196

[MS] W. de Melo, S. van Strien, *One-Dimensional Dynamics*, Springer, Berlin, 1993. MR **95a:**58035

[M1] M. Misiurewicz, Absolutely continuous measures for certain maps of an interval, *Inst. Hautes Études Sci. Publ. Math*, No. 53, (1981), 17–51. MR **83j:**58072

[M2] M. Misiurewicz, Horseshoes for mappings of the interval, *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **27** (1979), 167-169. MR **81b:**58033.

[MSm] M. Misiurewicz and J. Smítal, Smooth chaotic maps with zero topological entropy, *Ergod. Th. & Dynam. Sys.* **8** (1988), 421-424. MR **90a:**58118

[Sh1] A. N. Sharkovsky, A mapping with zero topological entropy that has a continuum of Cantor minimal sets, *Dynamical systems and turbulence*, (1989), 109–117, iv, *Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev.* (Russian). MR **91e:**58150

[Sh2] A. N. Sharkovsky, Attracting sets containing no cycles. *Ukrain. Mat. Zhur.* **20**, No. 1, (1968), 136–142. (Russian) MR **37**#908

[Sh3] A. N. Sharkovsky, *Classification of one-dimensional dynamical systems*, European Conference on Iteration Theory (Caldes de Malavella, 1987), 42–55, World Sci. Publishing, Teaneck, NJ, 1989. MR **92a:**58075

[Sh4] A. N. Sharkovsky, On cycles and the structure of a continuous mapping. *Ukrain. Mat. Zhurn.* **17**, No.3, (1965), 104–111. (Russian) MR **32**#4213

[Sh5] A. N. Sharkovsky, On some properties of discrete dynamical systems, *Iteration theory and its applications (Toulouse, 1982)*, 153–158, Colloq. Internat. CNRS, 332, *CNRS, Paris*, (1982) MR **86k:**58058

[SKSF1] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak and V. V. Fedorenko, *Dynamics of One-dimensional Mappings*, Naukova Dumka, Kijev, 1989, 216 pp. (Russian) MR **91k:**58065

[SKSF2] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak and V. V. Fedorenko, *Dynamics of One-dimensional Maps*, Kluwer Academic Publishers Group, Dordrecht, 1997. MR **98k:**58083

[SMR] A. N. Sharkovsky, Yu. L. Maistrenko and E. Yu. Romanenko, *Difference equations and their applications*, Kluwer Academic Publishers Group, Dordrecht, 1993. MR **95j:**39001

[Si0] P. Šindelářová, A counterexample to a statement concerning Ljapunov stability, *Preprint MA* 16/2000, Mathematical Institute, Silesian University at Opava, 2000.

[Si1] P. Šindelářová, A counterexample to a statement concerning Lyapunov stability, *Acta Math. Univ. Comen.* **70** (2001) (2) 265-268.

[Si2] P. Šindelářová, A zero topological entropy map for which periodic points are not a G_δ set, *Ergod. Th. & Dynam. Sys.* **22** (2002), 947–949.

[Si3] P. Šindelářová, A zero topological entropy map with recurrent points not F_σ, *Proc. Amer. Math. Soc.* **131** (2003) (7), 2089–2096.

[S] D. Singer, Stable orbits and bifurcations of maps of the interval. *SIAM J.-Appl. Math.* **35** (1978), 260–267. MR **58**#13206