Silesian University in Opava

Mathematical Institute

Peter Sebestyén

On classification of irreducible zero curvature representations

Abstract of the Ph.D. Thesis

January 2008

Geometry and Global Analysis

Slezská univerzita v Opavě

Matematický ústav

Peter Sebestyén

Klasifikace ireducibilních reprezentací nulové křivosti

Autoreferát dizertační práce

Leden 2008

Geometrie a globální analýza

Výsledky tvořící tuto dizertační práci byly získány během doktorského studia oboru Geometrie a globální analýza na Matematickém ústavu Slezské univerzity v Opavě v letech 2003–2008. Výzkum byl podporován projekty MSM 192400002, MSM4781305904 a GAČR 201/04/0538. Prezentace výsledků byla podporována projektem FRVŠ 3563/2005/G6.

Dizertant:	RNDr. Peter Sebestyén Matematický ústav SU, Opava
Školitel:	Doc. RNDr. Michal Marvan, CSc. Matematický ústav v Opavě
Školící pracoviště:	Matematický ústav v Opavě
Oponenti:	Prof. RNDr. Ladislav Hlavatý, DrSc. FJFI ČVUT Praha Doc. RNDr. Artur Sergyeyev, Ph.D. Matematický ústav v Opavě

Autoreferát byl rozeslán dne: 31. ledna 2008

Státní doktorská zkouška a obhajoba dizertační práce se konají dne 5. března 2008 v 11 hodin, před oborovou komisí doktorského studia geometrie a globální analýzy v zasedací místnosti rektorátu Slezské univerzity v Opavě.

S dizertací je možno se seznámit v knihovně Matematického ústavu SU, Na Rybníčku 1, Opava.

Předseda oborové komise:	Prof. RNDr. Miroslav Engliš, DrSc. Matematický ústav v Opavě

Obsah

1. Anotation

Main results are published in two independent papers [1] and [2] appended to this thesis. The common subject is the classification of irreducible $\mathfrak{sl}_n$-valued zero curvature representations (ZCR), based on gauge transformation.

In the first paper we obtained the complete classification of irreducible $\mathfrak{sl}_3$-valued ZCRs.

The second paper gives a complete list of normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations with the characteristic element possessing a single Jordan cell.

2. Introduction

Historical information can be found in [P] and [W].

The very beginnig of the Soliton theory dates back to the year 1834 when John Scott Russell observed solitary wave in shallow water in a boat channel. His experience is described in the famous Report on waves of 1844. Korteweg and de Vries [K] gave analytical description of observed phenomena by the well-known KdV equation in 1895.

Surprisingly, the big growth of soliton theory in twentieth century begun with numerical experiments carried out by Fermi, Pasta and Ulam in 1954–55, on the Los Alamos MANIAC computer ([F], the reprint of the original report). Zabusky and Kruskal [ZK] (1965) subsequently studied the continuum limit of the Fermi-Pasta-Ulam lattice equations and found that certain solutions could be described in terms of the Korteweg-de Vries equation. They found that the solitary wave solutions had behavior similar to the superposition principle, despite the fact that the waves themselves were highly nonlinear. They denoted such waves *solitons* and described the basic behavior of them.

An important step in the solution of the KdV equation was provided by Gardner, Greene, Kruskal and Miura [G] (1967). They solved KdV by the methods of *Inverse Scattering Transform* (IST) and obtained so called *n-soliton solutions*. Later on, Lax [L] (1968) found out pairs of linear operators, known as *Lax pairs*, for which the KdV equation is isospectral integrability condition.

Ablowitz, Kaup, Newell and Segur [A] (1973) applied methods of IST to the wide class of nonlinear partial differential equations. Zakharov and Shabat [ZS] (1979) found out correspondence between integrability and existence of *Zero curvature representation* (ZCR) possesing a non-removable (spectral) parameter. Calogero denoted this class of nonlinear systems as *S–integrable*. The name of ZCR came from an observation that (2) coincides with zero curvature condition on an appropriate connection to be flat. Zero curvature representations are strongly related with Lax pairs. However, the notion of ZCR is more general then the notion of Lax pair (at least in dimension 1 + 1).

When a ZCR with spectral parameter for a given equation is known, one may obtain particular solutions of the equation (e.g., via Riemann–Hilbert problem or Bäcklund transformation), infinite series of conservation laws. One of the possible methods for finding ZCR for a given equation is so called *prolongation procedure* introduced by Wahlquist and Estabrook [WE] in 1976. In 1983, Dodd and Fordy [DF] made the prolongation procedure algorithmic for a wide class of equations. However, there still remained some cases for wich this algorithm does not work well. Some substantial problems were dealt with by Molino [M], followed by Finley and McIver [FM]. In the nineties Marvan proposed a direct procedure to compute ZCR for a given equation ([M1], [M2] and [M3]) which amounts to solving of certain system of equations in total derivatives.

Nevertheless, existing computational procedures are insufficient for solving general classification problems, unless in combination with methods based on different criteria of integrability. The most complete lists of integrable systems are obtained from the formal symmetry approach ([MSS], [MSY] and [MS]).

In the eighties Vinogradov ([Vi1] and [Vi2]) begun to develop the geometric theory of PDEs and introduced so called *$\mathcal{C}$-spectral sequence* and *category of diffieties*. Diffiety is a geometric object – an infinite dimensional submanifold of the *jet space* J^∞ – which completely describes the PDE and, in fact, is more general then PDE itself. Later on Krasil'shchik and Vinogradov [KV] introduced the general concept of a covering (in the category of diffieties) of which Wahlquist–Estabrook structures are a special case. There is an effective tool for computing isomorphism classes of coverings after unique representatives are found in each class.

Using the Vinogradov theory, Marvan [M1] introduced a *characteristic element* of a ZCR, which is a matrix that transforms by conjugation during gauge transformations of the ZCR. Independently Sakovich [S] defined the characteristic element for evolution equations, and developed the theory of so called *cyclic bases*.

In case of one PDE, the characteristic element together with the ZCR form a triple of matrices. Whereas the characteristic element transforms by conjugation during gauge transformation (see below) of the ZCR, one can transform the characteristic element to its Jordan normal form. The remaining gauge freedom then can be used for further reduction of one of the matrices constituing the ZCR. This method is used for classification of ZCRs in this thesis.

3. Preliminaries

3.1. **Jet bundle.** In this section we reproduce the definition of the Jet bundle from [B, Ch. 3. and Ch. 4.].

Let us have an m-dimensional *locally trivial bundle* $\pi : E \to M$ over an n-dimensional manifold M. A *section* of the bundle π is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. I.e., the map s takes a point $x \in M$ to some point of the *fiber* E_x. In a particular case of the trivial bundle $M \times N \to M$, sections are the maps $M \to N$. Let us assume that all bundles under consideration are vector bundles, i.e., their fibers are vector spaces and the gluing functions are linear transformations.

Let $\mathcal{U} \subset M$ be a neighborhood over which the bundle π becomes trivial, i.e., such that $\pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times \mathbb{R}_m$. If $\mathcal{U}$ is a coordinate neighborhood on the manifold M with local coordinates $x_1, \dots, x_n$, then any point of the fiber is determined by its projection to $\mathcal{U}$ and by its coordinates $u^1, \dots, u^m$ with respect to the chosen basis. The functions $x_1, \dots, x_n, u^1, \dots, u^m$ are coordinates in $\pi^{-1}(\mathcal{U})$ and are called *adapted coordinates*. I.e., any section is represented in adapted coordinates by a vector function $f = (f_1, \dots, f_m)$ in the variables $x_1, \dots, x_n$.

Two sections s_1 and s_2 of the bundle π will be called *tangent* with order k over the point $x_0 \in M$, if the vector functions $u = f_1(x)$, $u = f_2(x)$ corresponding to these sections have the same partial derivatives up to order k at the point x_0. This condition is equivalent to the fact that the k-th order Taylor series of the sections coincide. The tangency condition for $k = 0$ reduces to coincidence of $f_1(x_0)$ and $f_2(x_0)$, i.e., the graphs of the sections s_1 and s_2 must intersect the fiber E_{x_0} at the same point.

Tangency of sections with order k at a point x is an equivalence relation which will be denoted by $s_1 \sim_{k,x} s_2$. The set of equivalence classes of sections, i.e., the set of k-th order Taylor series, will be denoted by J^k_x and called the *space of k-jets* of the bundle π at the point x. The point of this space (the equivalence class of a section s) will be denoted by $[s]^k_x$. Thus, if $s_1 \sim_{k,x} s_2$, then $[s_1]^k_x = [s_2]^k_x$. The *space of k-jets of the bundle π* is the union of J^k_x over all points $x \in M$:

$$J^k(\pi) = \bigcup_{x \in M} J^k_x.$$

We define the projection $\pi_k : J^k(\pi) \to M$. Then for any point $\theta = [s]^k_x \in J^k(\pi)$ we have $\pi_k(\theta) = x$, and $\pi_k^{-1}(x) = J^k_x$.

For the case $k = 0$ we have $J^0(\pi) = \bigcup_{x \in M} E_x = E$, i.e., the space $J^0(\pi)$ coincides with the total space of the bundle π.

Let us define local coordinates on the space of k-jets of the bundle π. Let $x_1, \ldots, x_n, u^1, \ldots, u^m$ be an adapted coordinate system in the bundle π over a neighborhood $\mathcal{U}$ of the point $x \in M$. Consider the set $\pi_k^{-1} \subset J^k(\pi)$. Let us complete local coordinates $x_1, \ldots, x_n, u^1, \ldots, u^m$ by the functions p^j_σ defined by the formula

$$p^j_\sigma([s]^k_x) = \frac{\partial^{|\sigma|} s^j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \;, \quad j = 1, \ldots, m \,, \quad |\sigma| \leq k \,.$$

We call the coordinates p^j_σ *canonical* (or *special*) *coordinates* associated to the adapted coordinate system (x_i, u^j).

We define the projections

$$\pi_{t+1,t} : J^{t+1}(\pi) \to J^{t+1}(\pi) \,, \pi_{t+1,t}([s]^{t+1}_x) = [s]^t_x \,,$$

where $t = 0, \ldots, k$. Since the equivalence class $[s]^{t+1}_x \in J^{t+1}(\pi)$ determines the class $[s]^t_x \in J^t(\pi)$ uniquely, the projections $\pi_{t+1,t}$ are well defined.

Let us consider the chain of projections

$$M \xleftarrow{\pi} E \xleftarrow{\pi_{1,0}} J^1(\pi) \longleftarrow \cdots \longleftarrow J^k(\pi) \xleftarrow{\pi_{k+1,k}} J^{k+1}(\pi) \longleftarrow \cdots$$

For any point $x \in M$ let us choose a sequence of points $\theta_l \in J^l(\pi)$, $l = 0, 1, \ldots, k, \ldots$, such that the equalities $\pi_{l+1,l}(\theta_{l+1}) = \theta_l$, $\pi(\theta_0) = x$ are valid. Due to these equalities and using the definition of the spaces $J^l(\pi)$ one can choose a local section s of the bundle π such that $\theta_k = [s]^l_x$ for any l. Thus any point θ_l is determined by the partial derivatives up to order l of the section s at the point x, while the whole sequence of points $\{\theta_l\}$ contains information on all partial derivatives of the section s at x. Denote by $J^\infty(\pi)$ the set of all such sequences. Points of the space $J^\infty(\pi)$ may be understood as infinite Taylor series of these sections.

For any point $\theta_\infty = \{x, \theta_k\}_{k \in \mathbb{N}} \in J^\infty(\pi)$, let us set $\pi_{\infty,k}(\theta_\infty) = \theta_k$ and $\pi_\infty(\theta_\infty) = x$. Then for all $k \geq l \geq 0$ we have the following equalities $\pi_k \circ \pi_{\infty,k} = \pi_\infty$ and $\pi_{k,l} \circ \pi_{\infty,k} = \pi_{\infty,l}$ are valid. In addition, if s is a section of the bundle π, then the map $j_\infty(s) : M \to J^\infty(\pi)$ is defined by the equality $j_\infty(s)(x) = \{x, \theta_k\}_{k \in \mathbb{N}}$. One has the following identities: $\pi_{\infty,k} \circ j_\infty(s) = j_k(s)$ and $\pi_\infty \circ j_\infty(s) = \mathrm{id}_M$.

Let $\Gamma(\pi)$ denote the set of all sections of the bundle π. The section $j_\infty(s)$ of the bundle $\pi_\infty : J^\infty(\pi) \to M$ is called the *infinite jet of the section* $s \in \Gamma(\pi)$.

The set $J^\infty(\pi)$ is endowed with a natural structure of a smooth manifold, however, it is infinitedimensional. Local coordinates arising in $J^\infty(\pi)$ over a neighborhood $\mathcal{U} \subset M$ are $x_1, \ldots, x_n$ together with all functions p^j_σ, where $|\sigma|$ is of an arbitrary (but finite) value.

The bundle $\pi_\infty : J^\infty(\pi) \to M$ is called the *bundle of infinite jets*, while the space $J^\infty(\pi)$ is called the *manifold of infinite jets* of the bundle π.

3.2. **Cohomological Algebra.** The following section is based on the lecture notes from the VII., VIII., and IX. Italian summers schools in Santo Stefano del Sole held by G. Moreno, Ch. di Pietro and L. Vitagliano.

Let R be a commutative unitary ring. Let $\{\mathcal{A}^i\}_{i\in\mathbb{Z}}$ be a family of R–modules. The direct sum $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}^i$ is called a *$\mathbb{Z}$–graded module* (or, simply, a *graded module*). Let $a \in \mathcal{A}^i \subset \mathcal{A}$ for some $i \in \mathbb{Z}$. Then a is called a *homogeneous element of degree i*. If a is a homogeneous element, then its degree will be denoted by $|a|$. Let $\mathcal{A}, \mathcal{B}$ be graded R–modules. An R–module morphism $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is called a *graded homomorphism* of degree k if for any $i \in \mathbb{Z}$ we have $\phi(\mathcal{A}^i) \subset \mathcal{B}^{i+k}$. Together with graded homomorphisms, graded R–modules form a category which will be denoted by $Mod_R^{\mathbb{Z}}$ to underline that the gradings are elements in $\mathbb{Z}$.

Let $\mathcal{A}$ be a graded R–module. Moreover, suppose that $\mathcal{A}$ is an associative R–algebra, with internal multiplication $\wedge$. Let $a \in \mathcal{A}^i$ and $b \in \mathcal{A}^j$ be homogeneous elements. If for any such a, b we have $a \wedge b \in \mathcal{A}^{i+j}$, then $\mathcal{A}$ is called a *graded R–algebra*. Let $\mathcal{A}, \mathcal{B}$ be graded R–algebras. Then an algebra morphism $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is called a *graded homomorphism* of degree k if it is a degree k graded homomorphism of R–modules. Together with graded homomorphisms, graded R–algebras form a category which will be denoted by $Alg_R^{\mathbb{Z}}$ (i.e., gradings are elements in $\mathbb{Z}$).

Exactly, as in the nongraded cases, subalgebras and ideals of a given $\mathcal{A} \in Alg_R^{\mathbb{Z}}$ are defined. If $\mathcal{I} \subset \mathcal{A}$ is an ideal, then the quotient $\mathcal{A}/\mathcal{I}$ is defined and it belongs to $Alg_R^{\mathbb{Z}}$.

Let $\mathcal{A} \in Alg_R^{\mathbb{Z}}$. A differential calculus can be developed over $\mathcal{A}$ as in the case $\mathcal{A} \in Alg_R$ keeping in mind the heuristic law: *in any formula a sign $(-1)^{|\star||\star|'} must$ be added for any pair of graded homogeneous objects $\star, \star' which$ exchange their positions.* As an example let $\mathcal{A} \in Alg_R^{\mathbb{Z}}$. A graded R–module morphism

$$X : \mathcal{A} \longrightarrow \mathcal{A}$$

is said a graded $\mathcal{A}$–derivation of degree $|X| \in \mathbb{Z}$ if, for any pair of homogeneous elements $a, b \in \mathcal{A}$ the graded Leibnitz rule is satisfied,

$$X(a \wedge b) = (-1)^{(|a|+|X|)|b|} b \wedge X(a) + (-1)^{|a||X|} a \wedge X(b).$$

Similarly, the graded commutator of $a, b \in \mathcal{A}$ is defined as

$$[a, b] = a \wedge b - (-1)^{|a||b|} b \wedge a.$$

Let $\mathcal{P} = \bigoplus_{i\in\mathbb{Z}} \mathcal{P}^i$ be a graded R–module. A *sequence* of R–modules is a degree 1 graded endomorphism f of $\mathcal{P}$. In practice a sequence is a chain of R–module morphisms

$$\cdots \longrightarrow \mathcal{P}^{i-1} \xrightarrow{f^{i-1}} \mathcal{P}^i \xrightarrow{f^i} \mathcal{P}^{i+1} \xrightarrow{f^{i+1}} \cdots$$

and $f = \bigoplus_{i\in\mathbb{Z}} f^i$. Let $\mathcal{P}, \mathcal{Q}$ be graded R–modules and $f : \mathcal{P} \longrightarrow \mathcal{P}$ and $g : \mathcal{Q} \longrightarrow \mathcal{Q}$ be sequences. A morphism of the sequences f and g is a graded morphism $h : \mathcal{P} \longrightarrow \mathcal{Q}$ compatible with the sequences, i.e., such that $h \circ f = g \circ h$. We will often consider degree 0 morphisms of sequences, i.e., $h(\mathcal{P}^i) \subset \mathcal{Q}^i$ for any $i \in \mathbb{Z}$.

A sequence $f : \mathcal{P} \longrightarrow \mathcal{P}$ is called *short* if it consists of only two arrows, i.e., $f^i = 0$ for any $i \neq 0, 1$. A sequence $f : \mathcal{P} \longrightarrow \mathcal{P}$ is called *exact* in the i–th term (or in the term $\mathcal{P}^i$), if $\ker f^i = \operatorname{im} f^{i-1}$. A sequence is called *exact* if it is exact in each term, i.e., $\ker f = \operatorname{im} f$.

Let $\mathcal{P}$ be a graded R–module and $\delta : \mathcal{P} \longrightarrow \mathcal{P}$ a sequence. The pair $(\mathcal{P}, \delta)$ is a *complex* if $\delta \circ \delta = 0$, i.e., $\delta^i \circ \delta^{i-1} = 0$ for any i. Then δ is called a *differential*.

Note that $(\mathcal{P}, \delta)$ is a complex if $\ker \delta \supset \operatorname{im} \delta$, if $\ker \delta^i \supset \operatorname{im} \delta^{i-1}$ for any i. Elements in $\ker \delta^i$ are said *i–cocycles* and elements in $\operatorname{im} \delta^{i-1}$ are said *i–coboundaries*. We can take the quotient

$$H^i(\delta) \equiv \ker \delta^i / \operatorname{im} \delta^{i-1}$$

and it is called the i*–th cohomology module* (or *cohomology in the term* $\mathcal{P}^i$). The graded module

$$H(\delta) = \bigoplus_{i\in\mathbb{Z}} H^i(\delta) = \ker\delta/\operatorname{im}\delta$$

is called the *cohomology module.* $H(\delta)$ is sometimes denoted by $H(\mathcal{P})$ when this does not give rise to confusion. Let $p \in \ker\delta$, i.e., p is a cocycle. Its cohomology class will be denote by $[p]_\delta$ (or simply $[p]$ if this doesn't give rise to any confusion).

Let $(\mathcal{P},\delta)$ and $(\mathcal{Q},\varepsilon)$ be complexes. A *morphism of complexes* is nothing but a morphism of the sequences $\delta : \mathcal{P} \longrightarrow \mathcal{P}$ and $\varepsilon : \mathcal{Q} \longrightarrow \mathcal{Q}$. Complexes of R–modules, together with their morphisms form a category. Let $h : \mathcal{P} \longrightarrow \mathcal{Q}$ be a degree k morphism of the complexes $(\mathcal{P},\delta)$ and $(\mathcal{Q},\varepsilon)$.

For any i, it is well defined the R–linear map

$$H^i(h) : H^i(\delta) \ni [p]_\delta \longmapsto [h(p)]_\varepsilon \in H^{i+k}(\varepsilon).$$

The map

$$H(h) \equiv \bigoplus_{i\in\mathbb{Z}} H^i(h) : H(\delta) \longrightarrow H(\varepsilon)$$

is said to be induced by the complex morphism h and it is a degree k graded morphism of R–modules.

Note that H is a covariant functor from the category of complexes of R–modules to the category of graded R–modules.

3.3. **Diffieties.** A *diffiety* is a (possibly, infinite dimensional) smooth manifold X equipped with a finite dimensional Fröbenius distribution $\mathcal{C}$.

Let us assume an $(n+m)$-dimensional locally trivial bundle $\pi : E \to M$ over an m-dimensional manifold M. Let $x_1,\dots,x_m,u^1,\dots,u^n$ are coordinates in $\pi^{-1}(\mathcal{U})$ for a neighborhood $\mathcal{U}$ over which the bundle π becomes trivial. Let $J^\infty(\pi)$ be an infinite-dimensional jet space with the local jet coordinates (x_i, u^j, u^j_I). Note that $I = (i_1 \dots i_\kappa)$ denotes a symmetric multiindex (instead of σ, used in section 3.1). We have distinguished vector fields on $J^\infty(\pi)$

$$D_i = \partial/\partial x^i + \sum_{j,I} u^j_{I,i}\,\partial/\partial u^j_I,$$

which are called *total derivatives.*

Let us consider a finite system of finite-order PDE's

$$F^l(x^i, u^j, \dots, u^j_I, \dots) = 0, \tag{1}$$

$l = 1,\dots,n$, on unknowns u^j and their derivatives u^j_I. We assume that system (1), along with all its differential consequences $D_I F^l = 0$, determines a submanifold $\mathcal{E} \subseteq J^\infty(\pi)$. The total derivatives are tangent to $\mathcal{E}$, hence they have a well-defined action on smooth functions on $\mathcal{E}$. The restricted fields $D_i = D_i|_\mathcal{E}$ span the Cartan distribution $\mathcal{C}$ on $\mathcal{E}$. Then $(\mathcal{E},\mathcal{C})$ is obviously a diffiety.

3.4. **ZCR and the characteristic element.** In this section we reproduce the definition of the ZCR and the characteristic element by M. Marvan in [M4].

Let we have a system (1) of PDE's with the corresponding diffiety $\mathcal{E}$. On $\mathcal{E}$, we have the direct decomposition of the tangent bundle $T\mathcal{E}$ as $\mathcal{C} \oplus V\mathcal{E}$, the Cartan distribution and the vertical vector bundle with respect to the projection on π. Let $C^\infty\mathcal{E}$ denote the ring of C^∞ functions on $\mathcal{E}$, let $\Lambda^{1,0}\mathcal{E} = \operatorname{Ann}\mathcal{C}$ and $\Lambda^{0,1}\mathcal{E} = \operatorname{Ann} V\mathcal{E}$ denote the $C^\infty\mathcal{E}$-modules of contact 1-forms and horizontal 1-forms, respectively. We have the induced splittings

$\Lambda^r\mathcal{E} = \bigoplus_{p+q=r} \Lambda^{p,q}\mathcal{E}$ into $r+1$ direct summands $\Lambda^{p,q}\mathcal{E} = \bigwedge^p \Lambda^{1,0}\mathcal{E} \wedge \bigwedge^q \Lambda^{0,1}\mathcal{E}$. The exterior differential split into the sum $d = \bar{d} + \ell$ of the *horizontal* differential $\bar{d}\colon \Lambda^{p,q}\mathcal{E} \to \Lambda^{p,q+1}\mathcal{E}$ and the *vertical* differential $\ell\colon \Lambda^{p,q}\mathcal{E} \to \Lambda^{p+1,q}\mathcal{E}$.

Coordinate formulas for the differentials $\bar{d}, \ell$ are derived from their action on $\Lambda^{0,0}\mathcal{E} = C^\infty\mathcal{E}$, which is

$$\begin{aligned} \bar{d}f &= \sum_i D_i f\, dx^i, \\ \ell f &= \sum_{j,I} \frac{\partial f}{\partial u^j_I}\omega^j_I, \qquad \omega^j_I = du^j_I - \sum_i u^j_{i,I}\, dx^i. \end{aligned}$$

We denote by $\bar{\Lambda}\mathcal{E}$ the graded exterior algebra of horizontal forms on $\mathcal{E}$. Let us consider a finite-dimensional real or complex Lie algebra $\mathfrak{g}$. The tensor product with $\bar{\Lambda}\mathcal{E} = \bigoplus_q \bar{\Lambda}^q\mathcal{E}$ is a graded nonassociative algebra under the bracket $[\mu\otimes A, \nu\otimes B] = (\mu\wedge\nu)\otimes[A,B]$ for $A,B\in\mathfrak{g}$ and $\mu,\nu\in\bar{\Lambda}\mathcal{E}$. Then

$$[\rho,\sigma] = -(-1)^{rs}[\sigma,\rho], \qquad \bar{d}[\rho,\sigma] = [\bar{d}\rho,\sigma] + (-1)^r[\rho,\bar{d}\sigma]$$

for $\rho\in\bar{\Lambda}^r\mathcal{E}\otimes\mathfrak{g}$, $\sigma\in\bar{\Lambda}^s\mathcal{E}\otimes\mathfrak{g}$. Due the Ado's theorem we can assume that $\mathfrak{g}$ is a matrix algebra, i.e., that $\mathfrak{g}$ is a subalgebra in some $\mathfrak{gl}_n$. Then $\bar{\Lambda}\mathcal{E}\otimes\mathfrak{gl}_n$ is a graded associative algebra with respect to the multiplication $(\mu\otimes A)\cdot(\nu\otimes B) = (\mu\wedge\nu)\otimes(A\cdot B)$ induced by the ordinary matrix multiplication, while

$$[\rho,\sigma] = \rho\cdot\sigma - (-1)^{rs}\sigma\cdot\rho, \qquad \bar{d}(\rho\cdot\sigma) = \bar{d}\rho\cdot\sigma + (-1)^r\rho\cdot\bar{d}\sigma$$

for $\rho\in\bar{\Lambda}^r\mathcal{E}\otimes\mathfrak{gl}_n$, $\sigma\in\bar{\Lambda}^s\mathcal{E}\otimes\mathfrak{gl}_n$. Elements of $C^\infty\mathcal{E}\otimes\mathfrak{g}$ will be called $\mathfrak{g}$-*matrices*.

A $\mathfrak{g}$-valued *zero-curvature representation* (ZCR) for $\mathcal{E}$ is a horizontal 1-form $\alpha\in\bar{\Lambda}^1\mathcal{E}\otimes\mathfrak{g}$ satisfying

(2) $$\bar{d}\alpha = \tfrac{1}{2}[\alpha,\alpha].$$

If $\alpha = \sum_i A_i\, dx^i$, $A_i\in\mathfrak{g}$, then eq. (2) becomes $D_jA_i - D_iA_j + [A_i,A_j] = 0$, for pairs $\{i,j\}$ such that $i\neq j$.

Given a ZCR α, we consider operators

$$\bar{\partial}_\alpha = \bar{d} - \mathrm{ad}_\alpha\colon \bar{\Lambda}^q\mathcal{E}\otimes\mathfrak{g} \to \bar{\Lambda}^{q+1}\mathcal{E}\otimes\mathfrak{g},$$

where $\mathrm{ad}_\alpha\rho = [\alpha,\rho]$ for any $\rho\in\bar{\Lambda}\mathcal{E}\otimes\mathfrak{g}$. We have $\bar{\partial}_\alpha\circ\bar{\partial}_\alpha = 0$ as a consequence of (2), which gives the *horizontal gauge complex* (or 0-th linear gauge complex [M1])

$$0\to\bar{\Lambda}^0\mathcal{E}\otimes\mathfrak{g}\xrightarrow{\bar{\partial}_\alpha}\bar{\Lambda}^1\mathcal{E}\otimes\mathfrak{g}\xrightarrow{\bar{\partial}_\alpha}\bar{\Lambda}^2\mathcal{E}\otimes\mathfrak{g}\to\cdots\to\bar{\Lambda}^m\mathcal{E}\otimes\mathfrak{g}\to 0.$$

The groups

$$\bar{H}^q_\alpha(\bar{\Lambda}\mathcal{E}\otimes\mathfrak{g}) = \frac{\ker\bigl(\bar{\Lambda}^q\mathcal{E}\otimes\mathfrak{g}\xrightarrow{\bar{\partial}_\alpha}\bar{\Lambda}^{q+1}\mathcal{E}\otimes\mathfrak{g}\bigr)}{\mathrm{im}\bigl(\bar{\Lambda}^{q-1}\mathcal{E}\otimes\mathfrak{g}\xrightarrow{\bar{\partial}_\alpha}\bar{\Lambda}^q\mathcal{E}\otimes\mathfrak{g}\bigr)}$$

are called the *horizontal gauge cohomology groups* with respect to the ZCR α.

If $\alpha = \sum_i A_i\, dx^i$, $A_i\in\mathfrak{g}$, then for an arbitrary $\mathfrak{g}$-matrix $C\in C^\infty\mathcal{E}\otimes\mathfrak{g}$ we have

(3) $$\bar{\partial}_\alpha C = \sum_i \widehat{D}_i C\, dx^i, \qquad \widehat{D}_i C = D_i C - [A_i, C].$$

Operators $\widehat{D}_i$ commute whenever α is a ZCR. We define recursively $\widehat{D}_{Ii} = \widehat{D}_I\circ\widehat{D}_i$.

Likewise, for $p = 1$ the corresponding modules $\Lambda^{1,q} \otimes \mathfrak{g}$ form the so-called *first linear gauge complex*

$$0 \to \Lambda^{1,0}\mathcal{E} \otimes \mathfrak{g} \xrightarrow{\bar{\partial}_\alpha} \Lambda^{1,1}\mathcal{E} \otimes \mathfrak{g} \to \cdots \to \Lambda^{1,m}\mathcal{E} \otimes \mathfrak{g} \to 0.$$

The groups

$$H^{1,q}_\alpha(\Lambda\mathcal{E} \otimes \mathfrak{g}) = \frac{\ker\bigl(\Lambda^{1,q}\mathcal{E} \otimes \mathfrak{g} \xrightarrow{\bar{\partial}_\alpha} \Lambda^{1,q+1}\mathcal{E} \otimes \mathfrak{g}\bigr)}{\operatorname{im}\bigl(\Lambda^{1,q-1}\mathcal{E} \otimes \mathfrak{g} \xrightarrow{\bar{\partial}_\alpha} \Lambda^{1,q}\mathcal{E} \otimes \mathfrak{g}\bigr)}$$

are computable in principle. We refer to [M1] or [Ve] for details of the isomorphism

$$H^{1,q}_\alpha(\Lambda\mathcal{E} \otimes \mathfrak{g}) \cong \frac{\ker(\mathfrak{g} \otimes \widehat{P}_{m-q} \to \mathfrak{g} \otimes \widehat{P}_{m-q-1})}{\operatorname{im}(\mathfrak{g} \otimes \widehat{P}_{m-q+1} \to \mathfrak{g} \otimes \widehat{P}_{m-q})}.$$

Here the starting point is the "compatibility complex" $P_0 \to P_1 \to P_2 \to \cdots$, where P_i are modules of sections of appropriate vector bundles over $\mathcal{E}$ (see [Ve]). We only remark here that $\dim P_0 = h =$ the number of unknowns, $\dim P_1 = n =$ the number of equations in system (1), and the first arrow $P_0 \to P_1$ can be identified with the operator of *universal linearization*, while each of the subsequent arrows expresses the integrability conditions for the previous one. Always $P_j = 0$ for all $j > m$, while for non-overdetermined systems we have even $P_j = 0$ for all $j > 1$. The dual complex $\cdots \to \widehat{P}_2 \to \widehat{P}_1 \to \widehat{P}_0$ is composed of formally adjoint operators between dual modules $\widehat{P} := \operatorname{Hom}(P, \bar{\Lambda}^m\mathcal{E})$. Finally,

$$\cdots \to \mathfrak{g} \otimes \widehat{P}_2 \to \mathfrak{g} \otimes \widehat{P}_1 \to \mathfrak{g} \otimes \widehat{P}_0$$

is obtained by replacing each operator D_i with its covariant counterpart $\widehat{D}_i = D_i - \operatorname{ad}_{A_i}$.

The element $\ell(\alpha) \in \Lambda^{1,1}\mathcal{E} \otimes \mathfrak{g}$ is a cocycle by (2); the corresponding 1-st cohomology class $\operatorname{ch}(\alpha) = [\ell(\alpha)] \in H^{1,1}_\alpha(\mathcal{E}, \mathfrak{g})$ will be called the *characteristic class* of the ZCR α. The *characteristic element* χ_α we introduced in [M1] is the image of $\operatorname{ch}(\alpha)$ under the above isomorphism

$$H^{1,1}_\alpha(\Lambda\mathcal{E} \otimes \mathfrak{g}) \cong \frac{\ker(\mathfrak{g} \otimes \widehat{P}_{m-1} \to \mathfrak{g} \otimes \widehat{P}_{m-2})}{\operatorname{im}(\mathfrak{g} \otimes \widehat{P}_{m} \to \mathfrak{g} \otimes \widehat{P}_{m-1})}.$$

In [M1, Prop. 4.2] we proved that if $H^{2,0}_\alpha = 0$ and $\operatorname{ch}(\alpha) = 0$, then $\alpha = \bar{d}\theta \cdot \theta^{-1}$ for an appropriate G-matrix θ, where G is connected and simply connected matrix Lie group associated with $\mathfrak{g}$. Such α's are called *trivial*.

Assuming system (1) non-overdetermined, P_2 is zero. Then $H^{1,1}_\alpha(\Lambda\mathcal{E} \otimes \mathfrak{g})$ vanishes for $m > 2$ and every ZCR is trivial, while for $m = 2$ we have an isomorphism $H^{1,1}_\alpha(\Lambda\mathcal{E} \otimes \mathfrak{g}) \cong \ker(\mathfrak{g} \otimes \widehat{P}_1 \to \mathfrak{g} \otimes \widehat{P}_0)$ and characteristic elements are n-tuples of $\mathfrak{g}$-matrices χ_l defined on $\mathcal{E}$ and satisfying

$$\sum_{l,J} (-1)^{|J|} \widehat{D}_J \left(\frac{\partial F^l}{\partial u^k_J} \chi_l \right) = 0, \qquad \widehat{D}_i = D_i - \operatorname{ad}_{A_i}. \tag{4}$$

We also have an explicit formula for the characteristic element for α, first written by Sakovich [S] for evolution equations: if $\mathfrak{g}$-matrices C^J_l satisfy

$$\bar{d}\alpha - \tfrac{1}{2}[\alpha, \alpha] = \sum_{l,J} D_J F^l \cdot C^J_l,$$

then

$$\chi_l = \sum_J (-\widehat{D})_J C^J_l \bigr|_{\mathcal{E}}. \tag{5}$$

For a matrix function $S\colon \mathcal{E} \to G$, we have the conjugation $\mathrm{Ad}_S\colon \bar{\Lambda}^q\mathcal{E} \otimes \mathfrak{g} \to \bar{\Lambda}^q\mathcal{E} \otimes \mathfrak{g}$ defined by $\gamma \mapsto S \cdot \gamma \cdot S^{-1}$. For any ZCR α, the form

$$\alpha^S = \bar{d}S \cdot S^{-1} + S \cdot \alpha \cdot S^{-1}$$

is another ZCR; we call it *gauge equivalent* to α. One easily checks that

$$\bar{\partial}_{\alpha^S} \circ \mathrm{Ad}_S = \mathrm{Ad}_S \circ \bar{\partial}_\alpha, \tag{6}$$

so that Ad_S is a morphism of the horizontal gauge complexes. Since Ad_S is invertible (with inverse $\mathrm{Ad}_{S^{-1}}$), we have

$$\bar{H}^q_\alpha(\bar{\Lambda}\mathcal{E} \otimes \mathfrak{g}) \cong \bar{H}^q_{\alpha^S}(\bar{\Lambda}\mathcal{E} \otimes \mathfrak{g}). \tag{7}$$

Similarly to (7), we have the isomorphism $H^{1,q}_\alpha(\Lambda\mathcal{E} \otimes \mathfrak{g}) \cong H^{1,q}_{\alpha^S}(\Lambda\mathcal{E} \otimes \mathfrak{g})$ induced by the conjugation Ad_S. Hence, gauge equivalent ZCR's have conjugate characteristic elements. The converse is not true in general.

We call a $\mathfrak{g}$-valued ZCR α *irreducible* if neither of the gauge equivalent forms α^S falls into a proper subalgebra of $\mathfrak{g}$. Otherwise α is called *reducible*.

Since we can assume that $\mathfrak{g}$ is the subalgebra in $\mathfrak{gl}_n$, it is convenient to suppose that $\mathfrak{g}$ is $\mathfrak{sl}_n$, the algebra of traceless $n \times n$ matrices. Indeed, every $\mathfrak{gl}_n$-valued matrix can be decomposed to an $\mathfrak{sl}_n$-valued matrix and a trace (a multiple of the unity matrix, i.e., a coservation law).

4. Appendices

4.1. **Normal forms of irreducible $\mathfrak{sl}_3$-valued zero curvature representations.** In the paper [1] we gave the complete list of normal forms of irreducible $\mathfrak{sl}_3$-valued ZCRs. The main idea for finding the normal form is based on the fact, that the characteristic element transforms by conjugation during gauge transformations of the ZCR. Without loss of generality, we assume that the chracteristic element is in the Jordan normal form. In the case of $\mathfrak{sl}_3$ we have five different Jordan normal forms (up to conjugation) denoted by $J_1, \dots, J_5$. For each of them we compute the correspondings stabilizer subgroups (with respect to conjugation) of the group SL_3, denoted by $W_1, \dots, W_5$. Since gauge transformation is a group action, we compute the action of the stabilizer subgroups on the matrix A in the ZCR (A, B).

Certain Jordan normal forms (resp. stabilizer subgroups) are invariant with respect to action by permutation matrices possibly composed with the automorphism $A \mapsto -A^\top$. We use this observation for reduction of the number of normal forms.

We proved that if the matrix A in the ZCR (A, B) falls to a proper subalgebra of $\mathfrak{sl}_3$, then the ZCR is either reducible or the matrix A is gauge equivalent to zero.

We obtained eight normal forms such that they do not lie in the proper subalgebra of $\mathfrak{sl}_3$.

4.2. **On normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations.** The second paper [2] gives a complete list of normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations with the characteristic element possessing a single Jordan cell. Here the mentioned Jordan normal for is denoted by J and the corresponding stabilizer subgroup by H_J. We can use for reducing the number of normal forms conjugation with a permutation matrix P, composed with the automorphism $A \mapsto -A^\top$. It is used notation $A^* := -P \cdot A^\top \cdot P^{-1}$. We defined a relation $\sim$ as follows. The ZCR (A, B) is equivalent with a ZCR (C, D) and write $(A, B) \sim (C, D)$, if (A, B) is gauge equivalent with the ZCR (C, D) or with the ZCR (C^*, D^*). It is proved that this relation is an equivalence.

Firstly we identify cases, when the normal form can be achieved purely algebraically. Our construction of the general normal form is based on this knowledge. Finally, we gave the algorithm for computing the gauge matrix which sends the matrix A to the corresponding normal form.

As an example we listed all irreducible normal forms for $\mathfrak{sl}_7$-valued zero curvature representations.

Publications concerning the thesis

[1] P. Sebestyén, Normal forms of irreducible $\mathfrak{sl}_3$-valued zero curvature representations, *Rep. Math. Phys.* **55**, no. 3, 435–445 (2005).

[2] P. Sebestyén, On normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations, *Rep. Math. Phys.*, to appear.

Presentations

[3] Geometry in Odessa 2005. Differential Geometry and its Applications, Odessa, Ukraine, May 23–29, 2005. Talk on: "Normal forms of irreducible $\mathfrak{sl}_3$-valued zero curvature representations"

[4] Symmetry in Nonlinear Mathematical Physics, Kiev, Ukraine, June 20–26, 2005. Talk on: "Normal forms of irreducible $\mathfrak{sl}_3$-valued zero curvature representations"

Summer schools

[5] 6th Diffiety school, July 13 – 28, 2003, Santo Stefano del Sole, Italy.

[6] 7th Diffiety school, July 19 – 31, 2004, Santo Stefano del Sole, Italy.

[7] 8th Diffiety school, July 16 – August 1, 2005, Santo Stefano del Sole, Italy.

Long-term visits

[8] Universita degli Studi di Salerno, October 2005 – February 2006, Salerno, Italy. Socrates–Erasmus Programm

Reference

[A] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Nonlinear-evolution equations of physical significance, *Phys. Rev. Letters* **31**, 125–127 (1973).

[B] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. ,Khor'kova, I. S. Krasil'shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, A. M. Vinogradov *Symmetries and conservation laws for differential equations of mathematical physics.* Translations of Mathematical Monographs, 182. (Providence, RI: American Mathematical Society) (1999).

[DF] R. Dodd and A. Fordy, The prolongation structures of quasi-polynomial flows, Proc. R. Soc. London A **385**, 389–429 (1983).

[F] E. Fermi, J. Pasta and S. Ulam, Studies of Nonlinear Problems. I, in Nonlinear Wave motion, Lectures in Applied Math., **15**, AMS (1974).

[FM] J.D. Finley III and J.K. McIver, Prolongation to higher jets of Estabrook–Wahlquist coverings for PDE's, *Acta. Appl. Math.* **32**, 197–225 (1993).

[G] C. S. Gardner, C. S. Greene, M. D. Kruskal and R. M. Miura, Method for Solving the Korteweg-de Vries Equation, *Phys. Rev. Lett.* **19**, 1095-1097 (1967).

[L] P. Lax, Integrals of Nonlinear Evolution Equations and Solitary Waves, *Comm. Pure Appl. Math.* **21**, 467-490 (1968).

[K] D. J. Korteweg and F. de Vries, On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves, *Philos. Mag.* **39**, 422–443 (1895).

[KM] I.S. Krasil'shchik and M. Marvan, Coverings and integrability of the Gauss-Mainardi-Codazzi equations, *Acta Appl. Math.* **56**, 217–230 (1999).

[KV] I.S. Krasil'shchik and A.M. Vinogradov, Nonlocal symmetries and the theory of coverings, *Acta Appl. Math.* **2**, 79–96 (1984).

[M1] M. Marvan, On zero curvature representations of partial differential equations, in: *Differential Geometry and Its Applications* Proc. Conf. Opava, Czechoslovakia, 1992 (online ELibEMS http://www.emis.de/proceedings), 103–122 (1993).

[M2] M. Marvan, A direct procedure to compute zero-curvature representations. The case sl_2, in: *Secondary Calculus and Cohomological Physics*, Proc. Conf. Moscow 1997 (online ELibEMS http://www.emis.de/proceedings/SCCP97), pp. 10 (1998).

[M3] M. Marvan, Scalar second order evolution equations possessing an irreducible sl2-valued zero curvature representation, *J. Phys. A: Math. Gen.* **35**, 9431–9439 (2002).

[M4] M. Marvan, On the horizontal gauge cohomology and non-removability of the spectral parameter, *Acta Appl. Math.* **72**, 51–65 (2002).

[MSS] A.V. Mikhailov, A.B. Shabat and V.V. Sokolov, The symmetry approach to classification of integrable equations, in: V.E. Zakharov, ed., *What is Integrability?*, 115–184 (Springer, 1991).

[MSY] A.V. Mikhailov, A.B. Shabat and R.I. Yamilov, Extension of the module of invertible transformations. Classification of integrable systems, *Commun. Math. Phys.* **115**,1–19 (1988).

[MS] A.V. Mikhailov and A.B. Shabat, Symmetries—test of integrability, in: A.S. Fokas and V.E. Zakharov, eds., *Important Developments in Soliton Theory*, 355–374 (Springer, 1993).

[M] P. Molino, Simple pseudopotentials for the KdV-equation, in: *Geometric Techniques in Gauge Theories*, Proc. Conf. Scheveningen, 1981, *Lecture Notes in Math.* **926**, 206–219 (Springer, 1982).

[P] R. S. Palais, The Symmetries of Solitons, *Bull. Amer. Math. Soc. (N.S.)* **34** 339–403 (1997).

[S] S. Yu. Sakovich, On zero-curvature representations of evolution equations, *J. Phys. A: Math. Gen.* **28**, 2861–2869 (1995).

[SK] J. Satsuma and D.J. Kaup, A Bäcklund transformation for a higher order Korteweg-de Vries equation, *J. Phys. Soc. Japan* **43**, no. 2, 692 (1977).

[Ve] A.M. Verbovetsky, Notes on the horizontal cohomology, in: M. Henneaux, J. Krasil'shchik and A. Vinogradov, eds., *Secondary Calculus and Cohomological Physics,* Proc. Conf. Moscow, 1997, Contemporary Mathematics 219 (Amer. Math. Soc., Providence) pp. 211–231 (1998).

[Vi1] A.M. Vinogradov, Local symmetries and conservation laws, *Acta Appl. Math.* **2**, 21–78 (1984).

[Vi2] A.M. Vinogradov, The C-spectral sequence, Lagrangian formalism, and conservation laws, II. The nonlinear theory *J. Math. Anal. Appl.* **100** 41–129 (1984).

[WE] H.D. Wahlquist and F.B. Estabrook, Prolongation structures and nonlinear evolution equations I, II, *J. Math. Phys.* **16**, 1–7 (1975); **17**, 1293–1297 (1976).

[W] http://en.wikipedia.org/wiki/Soliton

[ZK] N. J. Zabusky and M. D. Kruskal, Interaction of Solitons in a Collisionless Plasma and the Recurrence of Initial States, *Phys. Rev. Let.* **15**, 240–243 (1965).

[ZS] V.E. Zakharov and A.B. Shabat, Integrirovanie nelinejnykh uravnenij matematicheskoj fiziki metodom obratnoj zadachi rasseyaniya. II, *Funkc. Anal. Prilozh.* **13**, no. 3, 13–22 (1979).