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Contents

1 Introduction

The Ph.D. thesis is based on two articles: *A new hyperbolic equation possessing a zero curvature representation* and *Recursion operator for the IGSG equation*, while the first one was already published, the second one was accepted to print. The common term of both papers is a zero-curvature representation for a system of nonlinear partial differential equations.

The first paper presents a previously unknown hyperbolic equation of pseudo-spherical type. It was found bz using a direct procedure to compute a zero curvature representation. The zero-curvature representation of the equation admits no parameter and is reducible to a proper subalgebra of $\mathfrak{s}l_2$.

In the second paper the inverse and direct recursion operator were found for the intrinsic generalized sine-Gordon equation in any number $n > 2$ of independent variables. Among the flows generated by the direct operator we identify a higher-dimensional analogue of the pmKdV equation.

2 Geometric theory of differential equations

Let E be a system of nonlinear partial differential equations

$$F^l(x^i, u^k, u^k_I) = 0, \tag{1}$$

where $x^i, i = 1, 2$ ($x^1 = x, x^2 = y$) are independent variables, u^k is arbitrary number ($k = 1, ..., N$) of unknown functions and $u^k_I = \dfrac{\partial^s u^k}{\partial x^{i_1} ... \partial x^{i_s}}$ are derivatives of these functions, with $I = i_1 ... i_s$ is multiindex. Say that every function F^l is smooth and depends on finite number of variables x, y, u^k, u^k_I. Besides these local variables is possible use also nonlocal variables or pseudopotentials z^p satisfying the system of equations

$$z^p_x = f^p, z^p_y = g^p, \tag{2}$$

where functions f^p, g^p depend on finite number of local variables and the same dependence is for pseudopotentials z^p. We postulate that system (2) is compatible as a consequence of (1) $z^i_{xy} = z^i_{yx}$.

Let X, U be manifolds, $r \geq 0$ an integer or ∞, $x \in X$ a $u \in U$ arbitrary points. Let $C^\infty(x, u)$ be the set of smooth mappings f defined at x, with values in U, such that $f(x) = u$. We say that two mappings $f, g \in C^\infty(x, u)$ are r-equivalent (at point x) if there exist a chart $(\tilde{X}, \varphi)$ on X and a chart $(\tilde{U}, \psi)$ on U such that $x \in \tilde{X}$ $u \in \tilde{U}$, and for each k, $0 \le k \le r$ or for each k when $r = \infty$:

$$D^k(\psi f \varphi^{-1})(\varphi(x)) = D^k(\psi g \varphi^{-1})(\varphi(x)).$$

The relation *r-equivalent at point x* is an equivalence relation on $C^\infty(x, u)$. An equivalence class with respect to this equivalence relation is called an r-jet with source x and target u. By $J^r_x f$ is denoted an r-jet with source x and target u, whose representative is a mapping f. The set of r-jets with source $x \in X$ and target $u \in U$ is denoted $J^r_{(x,y)}(X, U)$, and $J^r(X, U) = \bigcup J^r_{(x,y)}(X, U)$. The set $J^0(X, U)$ or only J^0 is canonically identified with $X \times U$. J^∞ is an infinite-dimensional jet space, equipped by local coordinates x, y, u^k, u^k_I (here $X = \mathbb{R}^2$). The functions F^l then may be interpreted as functions defined on J^∞.

A fibered manifold is a triple (U, π, X) in which U and X are manifolds and π is a surjective submersion of U onto X. A mapping $\gamma : V \to U$, where $V \subseteq X$ is an open set, is called a section of the fibered manifold (U, π, X), if $\pi \circ \gamma = id_V$. By $J^r_x \gamma$ is denoted r-jet whose representative is a section γ. Similarly $J^r\pi = \{J^r_x\gamma; x \in X, \gamma \text{ is a section}\}$.

In every domain of definition of the independent variables x, y, we have two distinguished commuting vector fields on J^∞

$$D_x = \frac{\partial}{\partial x} + \sum_{k,I} u^k_{Ix} \frac{\partial}{\partial u^k_I}, \quad D_y = \frac{\partial}{\partial y} + \sum_{k,I} u^k_{Iy} \frac{\partial}{\partial u^k_I}, \tag{3}$$

which are called total derivatives.

The equation manifold $\mathcal{E}$ associated with system (1) is defined to be the submanifold in J^∞ determined by the infinite system of equations $F^l = 0$ and $D_I F^l = 0$ for I running through all symmetric multiindices in x, y. The total derivatives D_x, D_y are vector fields on J^∞ tangent to $\mathcal{E}$, therefore they admit a restriction to $\mathcal{E}$. The restricted fields then generate the Cartan distribution $\mathcal{C}$ on $\mathcal{E}$. The pair $(\mathcal{E}, \mathcal{C})$, called a diffiety, is an invariant geometric object associated with system (1).

Let $(\mathcal{E}, \mathcal{C})$ and $(\mathcal{E}', \mathcal{C}')$ be two diffieties, the mappings from $(\mathcal{E}, \mathcal{C})$ onto $(\mathcal{E}', \mathcal{C}')$ that preserve the Cartan distributions are called morphisms of diffieties. A bijective morphism of equation manifolds is called isomorphism. A morphism maps solutions of the system to solutions.

3 A zero-curvature representation

Let G be a Lie group, let $\mathfrak{g}$ be the corresponding Lie algebra. A zero-curvature representation (ZCR) [7] for $\mathcal{E}$ with coefficients in $\mathfrak{g}$ we mean a $\mathfrak{g}$-valued one-form $\alpha = Adx + Bdy$ defined on $\mathcal{E}$ such that

$$D_y A - D_x B + [A, B] = 0 \tag{4}$$

holds as a consequence of system (1). A, B are $\mathfrak{g}$-valued differential functions, which depends on x, y, u^k, u^k_I and possibly on an essential (spectral) parameter λ.

Let G be the connected and simply connected matrix Lie group associated with $\mathfrak{g}$. Then for an arbitrary G-valued function S, the form $\alpha^S = A^S dx + B^S dy$, where

$$A^S = D_x S S^{-1} + SAS^{-1}, \quad B^S = D_y S S^{-1} + SBS^{-1}, \tag{5}$$

is another ZCR, which is said to be gauge equivalent to the former. A function Z is called a linear pseudopotential for (1) whenever it satisfies the linear system $D_x Z = AZ, D_y = BZ$.

We call a $\mathfrak{g}$-valued ZCR (A, B) reducible if either A, B or any of the gauge-equivalent pairs fall entirely into a subalgebra $\mathfrak{g}$. Otherwise it is called irreducible.

The pseudo-sphere (PSS) is a surface of $\mathbb{R}^3$, its Gaussian curvature is constant $K = -1$. A differential equation for a real-valued function $u(x, t)$ is said to describe a pseudo-spherical surface if it is the necessary and sufficient condition for the existence of smooth functions f_{ij}, $1 \leq i \leq 3$, $1 \leq j \leq 2$, depending on u and its derivatives, such that the 1-forms $w_1 = f_{11}dx + f_{12}dt$, $w_2 = f_{21}dx + f_{22}dt$, $w_{12} = f_{31}dx + f_{32}dt$, satisfy the structure equations of a surface of a constant Gaussian curvature equal to -1, that is: $dw_1 = w_{12} \wedge w_2$, $dw_2 = w_1 \wedge w_{12}$, $dw_{12} = w_1 \wedge w_2$.

The PSS property implies the existence of the ZCR with values in $\mathfrak{sl}_2$ but ZCR need not depends on the parameter. If ZCR depends on the spectral parameter λ then the equation is a soliton type.

4 A covering and Bäcklund transformation

A covering over an equation manifold $\mathcal{E}$ is a pair consisting of another equation manifold $\mathcal{E}'$ and a surjective morphism $p : \mathcal{E}' \to \mathcal{E}$. Two coverings $\mathcal{E}'$ and $\mathcal{E}''$ are isomorphic over $\mathcal{E}$ when there exists an isomorphism $\mathcal{E}' \cong \mathcal{E}''$ that commutes with the projections to $\mathcal{E}$.

A ZCR is trivial when it is gauge equivalent to zero. A covering $\mathcal{E}' \to \mathcal{E}$ is said to trivialize a ZCR $\alpha = Adx + Bdy$ if the pullback of α along the morphism $\mathcal{E}' \to \mathcal{E}$ is a trivial ZCR.

A Bäcklund transformation Φ from a diffiety $\mathcal{E}_1$ to a diffiety $\mathcal{E}_2$ is defined as a pair of coverings p over $\mathcal{E}_1$ and q over $\mathcal{E}_2$ with a common source $\tilde{\mathcal{E}}$.

5 Nonlocal symmetries

A $linearization$ of the system (1) is the system

$$l_{F^l}[U] = 0, \tag{6}$$

where

$$l_F[U] = \sum_k \sum_I \frac{\partial F}{\partial u_I^k} U_I^k. \tag{7}$$

Geometrically, the linearization can be introduced as the vertical vector bundle $V\mathcal{E} \to \mathcal{E}$ with respect to the projection $\mathcal{E} \to M$ on the base manifold M.

Let's define a symmetry of (1) as a real-valued, vector-valued or matrix-valued differential function U that satisfy (6) on solution manifold $\mathcal{E}$ of the system (1).

Morphisms $\mathcal{E} \to V\mathcal{E}$ that are section of the bundle $V\mathcal{E} \to \mathcal{E}$ corresponds to symmetries of $\mathcal{E}$.

A nonlocal symmetry corresponds to a morphism $\mathcal{E}' \to V\mathcal{E}$ over $\mathcal{E}$, where $\mathcal{E}'$ is a covering of the original equation.

6 The classification works

If we are interested in a problem of classification of integrable equations we know there are these basic methods:

a) Painlevé method [17]

b) symmetry analysis

c) zero-curvature representation.

A symmetry analysis explores which equations have enough rich algebra of symmetries. This method was used to complete classfications for example evolution equotions.

Recall that there are two kinds of integrability by integrable systems

a) S-integrable systems, which possess a ZCR with parameter,

b) C-integrable systems, which is possible to transform to linear system.

The symmetry analysis does not distinguish between S- and C-integrability. In full generality, classification of hyperbolic equations that could be solved by the inverse scattering method is still an open problem.

K. Tenenblat and her research group devoted a number of works to the classification of nonlinear equations of PSS type [3],[4]. The PSS property implies the existence of the ZCR with values in $\mathfrak{sl}_2$, hence indicates integrability by the inverse scattering method if the ZCR depends on the spectral parameter.

Other representative of ZCR method is M. Marvan who classified scalar second order evolution equations [9].

7 A new hyperbolic equation of pseudo-spherical type

A main motivation for the work with ZCR was an attempt to classify all hyperbolic equations

$$u_{xy} = F(x, y, u, u_x, u_y) \tag{8}$$

possessing a nongenerate ZCR with values in $\mathfrak{sl}_2$, by using the "direct method" of [8].

Consider a nonlinear hyperbolic equation (8) with $\mathfrak{sl}_2$-matrices A, B which satisfy a condition (4). Matrices A, B are not reducible to a solvable algebra.

To matrices A, B there exists so called characteristic matrix R and we restrict ourselves to the normal form J_r for R

$$J_r = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}. \tag{9}$$

The matrix A is supposed to be in the normal form with respect to the action of the stabilizer of the matrix J_r:

$$A = \begin{pmatrix} a_1 a_2 & a_2 \\ a_2 & -a_1 a_2 \end{pmatrix}, \text{while} \quad \mathrm{B} = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}. \tag{10}$$

For an arbitrary $\mathfrak{sl}_2$-valued function C on (8) let $\hat{D_x} C = D_x C - [A, C], \hat{D_y} C = D_y C - [B, C]$.

We consider the following system of differential rquations in total derivatives, consisting of 6 equations in 6 unknowns $(a_1, a_2, b_1, b_2, b_3, r)$:

$$D_y A - D_x B + [A, B] = 0$$

$$\sum_I \left(-\hat{D}\right)_I \left(\frac{\partial F}{\partial u_I} J_r\right) = \hat{D}_x \hat{D}_y J_r, \tag{11}$$

and parameter F.

For solution of (11) we used a software for differential calculus on jet spaces and diffieties [11].

During computation we went through a lot of branches but only one of them led to a usable result and so we found a new PSS equation, which is apparently missing in the literature. Using the following sequence of transformations 1) $(x, y, u) \rightarrow (x, y, w)$, kde $K = w$, 2) $(x, y, w) \rightarrow (x, y, a)$, where $w = 4a - \frac{c_1}{8}, b_{31} = -4a^2 + \frac{c_1^2}{16}$, 3)$(x, y, a) \rightarrow (x, y, v)$, where $v = au, a^2 = b$, we can introduce obtained PSS equations in the form:

$$v_{xy} = \Bigg(- v v_x v_y + \frac{v_x}{2}\left(\frac{\partial b}{\partial y} + 2\frac{\partial b}{\partial x} v\right) - v^2\left(\frac{\partial^2 b}{\partial x^2} + 8v^2 - 16b\right) + \\ + \frac{\partial^2 b}{\partial x^2} - \frac{1}{2}\left(\frac{\partial b}{\partial x}\right)^2 - 8b^2\Bigg) / (b - v^2). \tag{12}$$

Thus the resulting equation (12) comes out as a representant of a whole equivalence class of equations.

8 Recursion operator

In Olver's formalism [14], a recursion operator Ψ is a pseudodifferential operator $\sum_{i=-r}^{s} f_i D^i \circ h_i$ which maps symmetries to symmetries. This provides a convenient way to generate infinite families of symmetries. Pseudodifferential operators involve inverses D^{-1} of the total derivative operator $D = D_x$. Here D^{-1} is formally defined by identities $D \circ D^{-1} = id$, $D^{-1} \circ D = id$. But the latter identity is actually invalid.

A problem of inverting a recursion operator motivated Guthrie [2] to a generalization such that his nonlocalities are no longer limited to inverses D^{-1} of total derivatives. Equivalent definition of recursion operator formulated by M. Marvan [6] is as follows:

A recursion operator for an equation manifold $\mathcal{E}$ is a pair of coverings $K, L : \mathcal{E} \to \mathcal{VE}$ over linearization $\mathcal{VE}$ of $\mathcal{E}$ such that K and L commutes with projections of $\mathcal{VE}$ to $\mathcal{E}$.

From this point of view we can consider a recursion operator as a special case of Bäcklund autotransformation of linearized diffiety.

The hierarchy of symmetries generated by a recursion operator may be extended also to the opposite direction by inverting the operator Ψ. Thus obtained operator Ψ^{-1} generates analogic hierarchy of symmetries and is called inverse operator recursion.

9 Recursion operator for the IGSG equation

A classical wave equation and sine-Gordon equations were generalized to higher dimensions, by considering a pair of functions v, h of n independent variables $x = (x^1, ..., x^n)$, where $v(x) = (v^1(x), ..., v^n(x))$ is a unit vector field $\sum_i (v^i)^2 = 1$ in $\mathbb{R}^n$ and $h(x)$ is an off-diagonal $n \times n$ matrix satisfying:

$$v^i_j = v^j h^{ji}, \qquad j \neq i,$$

$$v^i_i = -\sum_{s \neq i} v^s h^{is},$$

$$h^{ij}_i = -h^{ji}_j - K v^i v^j - \sum_{s \neq i,j} h^{si} h^{sj}, \qquad i > j,$$

$$h^{ij}_j = -h^{ji}_i - \sum_{s \neq i,j} h^{is} h^{js}, \qquad i < j,$$

$$h_k^{ij} = h^{ik}h^{kj}, \qquad j \neq i \neq k \neq j. \tag{13}$$

This set of equations is called the intrinsic generalized wave equation (IGWE) when $K = 0$, and the intrinsic generalized sine-Gordon equation (IGSGE) when $K \neq 0$. Moreover this system is integrable in the sense of soliton theory.

Lie symmetries of IGSGE and IGWE were computed by Tenenblat and Winternitz [16] along with the corresponding invariant solutions. However no higher symmetries have been written yet.

The system (13) has a ZCR, which consists of sparse antisymmetric $2n \times 2n$ matrices $A_{(k)}$, $k = 1, ..., n$, which satisfy: $\mathbf{A}_{(k)x^l} - \mathbf{A}_{(l)x^k} + [\mathbf{A}_{(k)}, \mathbf{A}_{(l)}] = 0$.

Symmetries are determined by functions V^i, $H^{i,j}$ on the manifold $\mathcal{E}$, that satisfy

$$\begin{aligned}
&\sum_i v^i V^i = 0,\\
&V_j^i = v^j H^{ji} + h^{ji} V^j, \qquad j \neq i,\\
&V_i^i = -\sum_{s\neq i}(v^s H^{is} + h^{is} V^s),\\
&H_i^{ij} = -H_j^{ji} - K(v^i V^j + v^j V^i)\\
&- \sum_{s\neq i,j} (h^{si}H^{sj} + h^{sj}H^{si}), \qquad i > j,\\
&H_j^{ij} = -H_i^{ji} - \sum_{s\neq i,j} (h^{is}H^{js} + h^{js}H^{is}), \qquad i < j,\\
&H_k^{ij} = h^{ik}H^{kj} + h^{kj}H^{ik}, \qquad j \neq i \neq k \neq j,
\end{aligned} \tag{14}$$

where $V_j^i = D_j V^i$.

Following Guthrie [2] we interpret recursion operators as Bäcklund autotransformations for the linearized system (14). There exists a recursion operator that can be written in terms of an auxiliary system of equations

$$W_{x^k} = [A_{(k)}, W] + l_{A_{(k)}} W. \tag{15}$$

If V^i, H^{ij} are symmetries and W satisfies (15), then V'^i, H'^{ij} are also symmetries:

$$V'^i = 2zV^i - 2z \sum_{s=1 s\neq i}^{n} v^s W^{is},$$

$$H'^{ij} = Kv^j \sum_{s=1}^{n} v^s W^{s,n+i} - \left(\frac{1}{2}K + 2z^2\right) W^{j,n+i}. \tag{16}$$

Thus formulas (15) and (16) determine a family of inverse recursion operators $\mathcal{R}_{\mathcal{Z}}$ for the IGSG equation. To obtain a direct recursion operator is necessary to find an inversion of $\mathcal{R}_{\mathcal{Z}}$. So this direct operator $\mathcal{L}$ generates a local flow of third order, wchich are called the generalized pmKdV equation.

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10 Publications concerning the thesis

[1] Pobořil M., A new hyperbolic equation possessing a zero curvature representation, *Fundam. Prikl. Mat.* **10** (2004), no. 1, 239-241.

[2] Marvan M. and Pobořil M., Recursion operator for the IGSG equation, *Fundamental'naya i Prikladnaya Matematika* , to appear.

11 Presentations

[1] The 23rd Winter School Geometry and Physics, Srní, Czech Republic, January 18 - 25, 2003. Talk on: Hyperbolic equations possessing a non-abelian pseudo-potential.

[2] The 25th Winter School Geometry and Physics, Srní, Czech Republic, January 15 - 22, 2005. Talk on: New results on the generalized wave and sine-Gordon equation in dimension 3.