Silesian University in Opava

Mathematical Institute in Opava

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Berezin transforms on spaces of holomorphic and harmonic functions

Ph.D. Thesis

August, 2009

Supervised by Miroslav Engliš

Mathematical Analysis

Slezská univerzita v Opavě

Matematický ústav v Opavě

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Berezinova transformace na prostorech holomorfních a harmonických funkcí

Autoreferát dizertační práce

Srpen, 2009

Školitel Miroslav Engliš

Matematická analýza

Výsledky této dizertační práce vznikly v průběhu doktorského studia oboru Matematická analýza na Matematickém ústavu v Opavě Slezské univerzity v Opavě během let 2004–2009.

Výzkum byl podporován granty GD201/03/H152 a MSM4781305904.

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Autoreferát byl rozeslán dne 23. září 2009.
Státní doktorská zkouška a obhajoba dizertační práce se konají dne 15. října 2009 v 10 hodin 30 minut v budově Ústavu matematiky a fyziky Přírodovědecké fakulty MU v Brně, Kotlářská 2, před oborovou komisí doktorského studia Matematické analýzy.

S dizertací je možno se seznámit v Knihovně Matematického ústavu Slezské univerzity v Opavě, Na Rybníčku 1, Opava.

Předseda oborové rady:	Prof. RNDr. Miroslav Engliš, DrSc. Matematický ústav v Opavě Slezská univerzita v Opavě Na Rybníčku 1 746 01, Opava

Contents

1. Introduction

This thesis consists of two papers, [1] and [2]. The first paper considers the asymptotic behaviour of a certain integral transform associated to Hilbert spaces with reproducing kernels, the *Berezin transform*, and generalizes to spaces of harmonic functions a result about it which is standard for spaces of holomorphic functions and has well-known applications in mathematical physics (quantization). Namely, for the standard weighted Bergman spaces on the complex unit ball, the Berezin transform of a bounded continuous function tends to this function pointwise as the weight parameter tends to infinity. In [1] we show that this remains in force also in the context of harmonic, rather than holomorphic, Bergman spaces on the ball. This generalizes the recent result of C. Liu for the unit disc, in addition to extending the corresponding result for the holomorphic case.

The second paper proves a result about the ordinary (holomorphic) Berezin transform of *operators*, which relates this transform to some geometric quantities. Namely, we give estimate for higher-order covariant derivatives of the Berezin transform of bounded linear operators on any reproducing kernel Hilbert space of holomorphic functions. This extends, and puts into a wider perspective, a number of recent results due to L. Coburn, J. Xia, B. Li, M. Engliš and G. Zhang, H. Bommier-Hato, and others. The answer turns out to involve the curvature of the Bergman-type metric associated to the reproducing kernel.

In the next two sections, we present more details on each paper.

2. Berezin transform on the harmonic Bergman space of the ball

Let $\mathbb{B}^n$ be the ball in $\mathbb{R}^n$, $n \geq 2$, and dz the Lebesgue measure on $\mathbb{B}^n$. For $\alpha > -1$, consider the measure

$$dA_\alpha(z) := c_\alpha(1 - |z|^2)^\alpha dz,$$

where

$$c_\alpha = \frac{\Gamma(\alpha + \frac{n}{2} + 1)}{\pi^{\frac{n}{2}}\Gamma(\alpha + 1)}$$

is chosen so as to make dA_α a probability measure. For simplicity, we will usually assume that α is an integer.

The harmonic Bergman space $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ consists, by definition, of all harmonic functions in $L^2(\mathbb{B}^n, dA_\alpha)$. It is known that point evaluation functionals are continuous on the harmonic Bergman space, so it possesses a reproducing kernel, i.e. there exists a function $R_\alpha(x, y)$ on $\mathbb{B}^n \times \mathbb{B}^n$, harmonic in each variable, such that

$$f(x) = \int_{\mathbb{B}^n} f(y) R_\alpha(x, y)\ dA_\alpha(y) = \langle f, R_\alpha(\cdot, x)\rangle$$

for each $f \in L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ and $x \in \mathbb{B}^n$.

The Berezin transform of a bounded linear operator T on $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ is the function $\tilde{T}^\alpha(z)$ on $\mathbb{B}^n$ defined by

$$\tilde{T}^{(\alpha)}(z) = \frac{\langle T^{(\alpha)} R_{\alpha z}, R_{\alpha z}\rangle}{\langle R_{\alpha z}, R_{\alpha z}\rangle} = \frac{T^{(\alpha)} R_{\alpha z}(z)}{R_\alpha(z,z)},$$

where, for the sake of brevity, we have denoted $R_{\alpha z}(w) := R_\alpha(z,w)$.

Finally, for $f \in L^\infty(\mathbb{B}^n)$, the Toeplitz operator T_f with symbol f is the operator on $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ defined by

$$T_f g = Q_\alpha(fg),$$

where $Q_\alpha : L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha) \to L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ is the orthogonal projection. That is,

$$T_f g(z) = \int_{\mathbb{B}^n} g(x) f(x) R_\alpha(z,x)\ dA_\alpha(x).$$

It was shown by C. Liu [32] that if $n = 2$ (so that $\mathbb{B}^2$ is just the unit disc in the complex plane $\mathbb{C}$), then for $f \in C(\overline{\mathbb{B}^n})$

$$\tilde{T}_f^{(\alpha)} \quad \to \quad f \qquad \text{uniformly, and} \tag{1}$$

$$\|T_f^{(\alpha)}\| \quad \to \quad \|f\|_\infty \tag{2}$$

as $\alpha \to \infty$.

This extends the same result known previously for Toeplitz operators on Bergman spaces of holomorphic functions, which finds important applications in mathematical physics (quantization on Kähler manifolds, see e.g. [16]).

The aim of the paper [1] is to generalize Liu's result also to $n \geq 3$. To do this, we first establish a (reasonably) explicit formula for the kernels $R_\alpha(x,y)$ and our main result is a generalization of (1) and (2).

We remark that we actually obtain a somewhat stronger result than (1), namely, we show that for any $f \in BC(\mathbb{B}^n) := C(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n)$ we also have

$$\tilde{T}_f^{(\alpha)}(z) \to f(z)$$

as $\alpha \to \infty$ for all $z \in \mathbb{B}^n$. This gives a new piece of information even for the original case $n = 2$.

Our main results in [1] are the following:

Theorem 2.1. ([1, Theorem 4.2]) *If $f \in BC(\mathbb{B}^n)$, the space of all bounded continuous functions on $\mathbb{B}^n$, then for each $z \in \mathbb{B}^n$*

$$\tilde{T}_f^{(\alpha)}(z) \to f(z)$$

as $\alpha \to \infty$ through the integers.

Corollary 2.2. ([1, Corolary 4.4]) *For any $f \in C(\overline{\mathbb{B}^n})$*

$$\|T_f^{(\alpha)}\| \longrightarrow \|f\|_\infty \qquad \text{as } \alpha \to \infty.$$

3. Geometric properties of the holomorphic Berezin transform

For a domain $\Omega \subset \mathbf{C}^n$, denote by $\mathcal{O}(\Omega)$ the vector space of all holomorphic functions on Ω, and let $\mathcal{H} \subset \mathcal{O}(\Omega)$ be an arbitrary Hilbert space which has a reproducing kernel, i.e. such that the point evaluation functionals $f \mapsto f(z)$ are continuous from $\mathcal{H}$ into $\mathbf{C}$ for any $z \in \Omega$. The reproducing kernel $K(z,w)$ of $\mathcal{H}$ is then a function on $\Omega \times \Omega$, holomorphic in $z, \overline{w}$, which has the reproducing property

$$f(z) = \langle f, K_z \rangle \qquad \forall f \in \mathcal{H},$$

where $K_z = K(\cdot, z) \in \mathcal{H}$. We will assume throughout that $\|K_z\|^2 = K(z,z)$ satisfies

$$K(z,z) > 0 \qquad \forall z \in \Omega.$$

The formula

$$K(z,z) = \|K_z\|^2 = \sup\{|f(z)|^2 : f \in \mathcal{H}, \|f\| \leq 1\}$$

then exhibits $\log K(z,z)$ as a supremum of logarithms of moduli of holomorphic functions, implying that $\log K(z,z)$ is plurisubharmonic. In other words, the matrix of mixed second order derivatives

$$g_{j\overline{k}} := \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log K(z,z) \tag{3}$$

defines an Hermitian (semi-)Riemannian metric on Ω by

$$\|v\|_z^2 := \sum_{j,k} g_{j\overline{k}} v_j \overline{v}_k \tag{4}$$

for $v \in T_z\Omega \cong \mathbf{C}^n$ the tangent space at $z \in \Omega$, which in turn induces the (semi-) distance function $\beta(\cdot,\cdot)$ on Ω in the standard way [26] [27].

In his quantization program in 1970's, Berezin [9] introduced a general symbol calculus for linear operators on reproducing kernel spaces. More specifically, for $X \in \mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$, the *Berezin symbol* (or *Berezin transform*) of X is the function on Ω defined as

$$\widetilde{X} := \langle X k_z, k_z \rangle$$

where

$$k_z = K(z,z)^{-1/2} K(\cdot, z) = \frac{K_z}{\|K_z\|}$$

is the normalized kernel function at z. (This is also often called *coherent state* in physics literature.) It is immediate that $\widetilde{X}$ is real analytic and $\|\widetilde{X}\|_\infty \leq \|X\|$, and it is well-known that X is uniquely determined by $\widetilde{X}$.

The prototypes of the spaces $\mathcal{H}$ are the Bergman spaces $A^2(\Omega)$ of all holomorphic functions in $L^2(\Omega, dV)$ on a bounded domain $\Omega \subset \mathbf{C}^n$ with Lebesgue measure dV, or the Segal-Bargmann(-Fock) spaces $A^2(\mathbf{C}^n)$ of all entire functions in $L^2(\mathbf{C}^n, d\mu)$ for the Gaussian measure

$$d\mu(z) = (2\pi)^{-n} e^{-|z|^2/2}\, dV(z).$$

The reproducing kernel $K(z,w)$ is then just the original kernel function of Bergman [10] for Ω bounded, while $K(z,w)=e^{z\cdot\overline{w}/2}$ for $\Omega=\mathbf{C}^n$. Similarly, the metric (3) is the Bergman metric on $\Omega \subsetneqq \mathbf{C}^n$, and coincides (up to a constant factor) with the Euclidean metric for $\Omega=\mathbf{C}^n$. In both cases, Coburn [17] obtained a Lipschitz estimate for the Berezin symbol on $A^2(\Omega)$, namely,[1]

$$|\widetilde{X}(a)-\widetilde{X}(b)| \le 2\,\|X\|\,\beta(a,b) \tag{5}$$

for any $a,b\in\Omega$ and $X\in\mathcal{B}(A^2(\Omega))$. Furthermore, he showed in [18] that the above estimate is sharp in the sense that

$$\sup_{\substack{a,b\in\Omega, a\neq b,\\ 0\neq X\in\mathcal{B}(A^2(\Omega))}} \frac{|\widetilde{X}(a)-\widetilde{X}(b)|}{\|X\|\,\beta(a,b)} = 2. \tag{6}$$

It was subsequently noted by Xia (unpublished) that for $\Omega=\mathbf{C}^n$, the proof in [17] can even be used to provide a stronger result: namely, $\widetilde{X}$ and its partial derivatives of all orders are bounded. M. Engliš and G. Zhang [22] improved upon and extended Xia's result by showing that $L\widetilde{X}$ is bounded for any invariant linear differential operator L on Ω and any $X\in\mathcal{B}(\mathcal{H})$, when $\mathcal{H}$ is any one of the standard weighted Bergman spaces on a bounded symmetric domain Ω.

The proof in [22] relied on the homogeneity of Ω under its group of holomorphic automorphisms, and made it clear that the invariant geometry of Ω was, at least for bounded symmetric domains, the right context in which to view $\widetilde{X}$; for this reason, there was also stated a conjecture there to the effect that, for any $k=1,2,\dots,$

$$\sup_{z\in\Omega}\|\nabla^k\widetilde{X}(z)\|_z \le c_k\,\|X\| \qquad \forall X\in\mathcal{B}(A^2(\Omega)) \tag{7}$$

with some constants c_k, for any bounded domain $\Omega\subset\mathbf{C}^n$. Here $\nabla^k\widetilde{X}$ stands for the k-th covariant derivative of $\widetilde{X}$, and $\|\cdot\|_z$ for its (tensor) norm at $z\in\Omega$ with respect to the Bergman metric (3).

Our first result generalizes the directional derivative estimate proved for $\mathcal{H}=A^2(\Omega)$ in [19], and also implies the Lipschitz estimates.

Theorem 3.1. ([2, Theorem 2]) *For $T\in\mathcal{B}(\mathcal{H})$ and $v\in\mathbf{C}^n$,*

$$|D_v\widetilde{T}(z)| \le 2\,\|T\|\,\|v\|_z, \tag{8}$$

with $\|v\|_z$ as in (4).

In a different direction, Helene Bommier-Hato studied the case of $\mathcal{H}=A^2(\mathbf{C}^n,d\mu_m)$, the space of all entire functions on $\mathbf{C}^n$ square-integrable with respect to the "power-Gaussian" measures

$$d\mu_m(z)=e^{-|z|^m}\,dV(z) \tag{9}$$

[1] The constant 2 appears here instead of $\sqrt{2}$ in [17] due to a different normalization of the metric (4): the one used in [17] is twice our (4).

on $\mathbf{C}^n$, with an arbitrary $m > 0$. It was proved in [13] that $\widetilde{X}$ is locally Lipschitz, more specifically

$$|\widetilde{X}(a) - \widetilde{X}(b)| \leq C\, \|X\|\, |a|^{\frac{m}{2}-1}|b-a|$$

for $|a|$ large and b in a small neighbourhood of a. Similarly, in [14] it was shown that the directional derivatives satisfy

$$|D_v\widetilde{X}(a)| \leq C\, \|X\|\, |a|^{\frac{m}{2}-1}\, \|v\|,$$

implying that $\widetilde{X}$ is even globally Lipschitz for $m \leq 2$. These results from [13] are also covered by our Theorem 3.1.

The Lipschitz estimate (5) means, in particular, that $\widetilde{X}$ is uniformly continuous with respect to the Bergman metric; this was applied for $\Omega = \mathbf{D}$, the unit disc, by Suárez [39], and for Ω the unit ball of $\mathbf{C}^n$, $n > 1$, by Nam, Zheng and Zhong [36], in the study of Toeplitz algebras. Our Theorem 3.1 may have similar applications for more general domains.

Theorem 3.1 can be stated using the language of norms of first-order covariant derivatives (viewed as differential forms) with respect to the Riemannian metric (3).

Theorem 3.2. ([2, Theorem 7]) *For any $T \in \mathcal{B}(\mathcal{H})$ and $z \in \Omega$,*

$$\|\partial\widetilde{T}(z)\|_z \leq \|T\|, \qquad \|\overline{\partial}\widetilde{T}(z)\|_z \leq \|T\|.$$

Our second result are analogous estimates for the second-order covariant derivatives.

Theorem 3.3. ([2, Theorem 8]) *For $T \in \mathcal{B}(\mathcal{H})$ and $z \in \Omega$,*

$$\|\partial\partial\widetilde{T}(z)\|_z \leq \sqrt{S + n^2 + n}\, \|T\|,$$

where S is the scalar curvature. Similarly for $\|\overline{\partial}\overline{\partial}\widetilde{T}(z)\|_z$.

Theorem 3.4. ([2, Theorem 9]) *For $T \in \mathcal{B}(\mathcal{H})$ and $z \in \Omega$,*

$$\|\overline{\partial}\partial\widetilde{T}(z)\|_z = \|\partial\overline{\partial}\widetilde{T}(z)\|_z \leq 2\sqrt{n}\, \|T\|.$$

Corollary 3.5. ([2, Corolary 10]) *For $T \in \mathcal{B}(\mathcal{H})$ and $z \in \Omega$,*

$$\|\nabla^2\widetilde{T}(z)\|_z \leq 2\sqrt{2(S + n^2 + 5n)}\, \|T\|.$$

Furthermore, for $n = 1$ the result in the last corollary is also sharp, i.e. the left- hand side is in general unbounded if the right-hand side is. In particular, as there exist spaces for which the scalar curvature S blows up at the boundary, the above-mentioned "covariant differentiation conjecture" (7)in general *fails* for $k \geq 2$.

The proof also implies that

$$S \geq -n(n+1)$$

for any metric associated as in (3) to a reproducing kernel Hilbert space $\mathcal{H} \subset \mathcal{O}(\Omega)$. This contrasts with the fact that it is easy to devise Kähler metrics on Ω whose scalar curvature assumes arbitrarily large negative values. The inequality (3) must thus be something inherent to metrics coming from reproducing kernels.

Finally, we prove the following substitute for the covariant differentiation conjecture when $k \geq 2$.

Theorem 3.6. ([2, Theorem 12]) *For any $m \geq 2$, there exists a scalar quantity r_m on Ω, given by a polynomial expression involving the contravariant metric tensor $g^{\bar{j}k}$, the curvature tensor $R_{i\bar{j}k\bar{l}}$, and the latter's covariant derivatives of orders $\leq 2m-4$, such that*

$$\|\nabla^m \widetilde{T}(z)\|_z^2 \leq r_m(z)\ \|T\|^2$$

for any $z \in \Omega$ and $T \in \mathcal{B}(\mathcal{H})$.

Expressions of a similar kind as our r_k occur as coefficients of the asymptotic expansion of the heat kernel and related geometric quantities, see e.g. [23] or [8].

4. Conferences

(1) CMS Winter Meeting 2004, Montral, Qubec, December 11-13, 2004. Talk on "The simplest subspace of generators of matrix algebras".
(2) 7th International Conference on Clifford Algebras, Toulouse, France, May 20-29, 2005. Talk on "The simplest subspace of generators of matrix algebras".
(3) 6th International ISAAC Congress, Ankara, Turkey, August 13-18, 2007. Talk on "Reproducing Kernels and Toeplitz Quantization on Harmonic Bergman Spaces".

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