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Contents

1. INTRODUCTION

The thesis concerns three independent papers [1]–[3]. All these papers consider dynamical systems generated by a continuous function on compact metric spaces. In the first paper the space is the compact interval, in the rest two papers the space is the circle.

The first paper gives an example of a continuous chaotic function on the interval which is not transitive. This disproves a conjecture of Bruckner and Hu [BH].

The second paper is devoted to extension of the notion of distributional chaos also for continuous mappings of the circle.

The last paper is devoted to developing of maximal ω-limit sets and spectral decomposition of continuous mappings of the circle.

2. BASIC TERMINOLOGY AND NOTATION

Throughout this abstract the set of continuous maps from a compact metric space (X, dist) into itself will be denoted by $C(X,X)$. Denote by $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ the circle. Recall that the *trajectory* of a point $x \in X$ under a map $f \in C(X,X)$ is the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where f^n is the n-th iteration of f. A point $x \in X$ is called a *periodic point* if there is a $k \geq 1$ such that $f^k(x) = x$, smallest such k is called the *period* of x. The set of all periods of all periodic points is denoted by $\mathrm{P}(f)$. The set of limit points of the trajectory of x is called *ω-limit set* and we denote the set by $\omega_f(x)$. The map f is *transitive* (or *nomadic*) if for any two open sets $U, V \neq \emptyset$ there is a positive integer n such, that $f^n(U) \cap V \neq \emptyset$.

For $x, y \in X$ and $n \in \mathbb{N}$ denote by $\delta_{xy}(n) = \mathrm{dist}(f^n(x), f^n(y))$ the distance of iterations. We say that f is *d-chaotic* in the sense Li&York if there is an uncountable set $S \subset X$ such that for any $x, y \in S$, $x \neq y$

$$\liminf_{n\to\infty} \delta_{xy}(n) = 0$$

and

$$\limsup_{n\to\infty} \delta_{xy}(n) = d > 0.$$

Set S is called a *scrambled set* for f, we say that f is extremely chaotic if f is d-chaotic for $d = \mathrm{diam} X$.

For f in the space $C(X,X)$, $x, y \in X$, real t, and any positive integer n define

$$\xi(x,y,t,n) = \sum_{i=0}^{n-1} \chi_{[0,t)}(\delta_{xy}(i)) = \#\{i;\ 0 \leq i < n \text{ and } \delta_{xy}(i) < t\}, \tag{1}$$

$$F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x,y,t,n), \tag{2}$$

and

$$F_{xy}(t) = \liminf_{n\to} \frac{1}{n}\xi(x,y,t,n), \tag{3}$$

where χ_A is the characteristic function of the set A.

Clearly both F^*_{xy}, F_{xy} are nondecreasing functions such that $F^*_{xy}(t) = F_{xy}(t) = 0$ for $t < 0$, and $F^*_{xy}(t) = F_{xy}(t) = 1$ for $t > \mathrm{diam} X$. We identify any two nondecreasing functions that coincide everywhere except at a countable set, and adopt the convention to

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chose functions F^*_{xy}, F_{xy} as left-continuous. Functions F^*_{xy}, F_{xy} are called the *upper* and *lower distribution function* of x and y, respectively. A function f *exhibits distributional chaos* if there are points $x, y \in \mathbb{S}$ such that $F^*_{xy}(t) = 1$ for all $t > 0$ and there is a point $s \in (0, \operatorname{diam} X)$ such that $F^*_{xy}(s) > F_{xy}(s)$.

Distributional chaos on the interval is supported by *basic sets*, i.e., maximal infinite ω-limit sets containing a periodic point. The properties of basic sets on the interval are well-known. Many of them were discovered by A. N. Sharkovsky (cf., e.g., [S1]–[S3]). Detailed study of basic sets on the interval, extending and improving the Sharkovsky's result is in [Bl].

Points $x, y \in \mathbb{S}$ are *synchronous* if the sets $\omega_f(x)$ and $\omega_f(y)$ are contained in the same maximal ω-limit set ω and if, for any periodic interval J such that its orbit $\operatorname{Orb} J$ contains ω, there is a $j \geq 0$ such that $f^j(x), f^j(y) \in J$. The *spectrum* $\Sigma(f)$ of f is the set of minimal elements of the set $D(f) = \{F_{xy};\ x \text{ and } y \text{ are synchronous}\}$. And the *weak spectrum* $\Sigma_w(f)$ of f is the set of minimal elements of the set $D_w = \{F_{xy};\ \liminf_{i\to\infty} \delta_{xy}(i) = 0\}$.

For more terminology see standard books like [Al] or [Bc].

3. Counterexample of an extremely chaotic function

Bruckner and Hu in paper [BH] stated the following result *Under Continuum hypothesis a continuous function F of compact interval is chaotic if and only if f^2 is nomadic.* In the paper [1] we conclude that:

Theorem A. *If a continuous function of the compact interval has a basic set with a portion of diameter d then it is d-chaotic.*

As a corollary of this theorem the function $f : [0, 1] \to [0, 1]$, given by

$$f(x) = \begin{cases} 3x & \text{if } x \in \left[0, \frac{1}{3}\right]; \\ 1 & \text{if } x \in \left(\frac{1}{3}, \frac{2}{3}\right); \\ 3 - 3x & \text{if } x \in \left[\frac{2}{3}, 1\right] \end{cases}$$

is extremely chaotic. On the other hand since f is constant on an open interval, neither f nor f^2 cannot be transitive. This disproves the conjecture of Bruckner and Hu.

4. Distributional chaos for continuous mappings of the circle

In the paper [ScSm] Schweizer and Smítal introduced the notion of distributional chaos for continuous functions of compact metric spaces. This kind of chaos was studied by several authors on different spaces (cf. e.g., [ScSm]). Main theorem in the paper [2] gives a list of four properties which are equivalent to distributional chaos.

Theorem B. *For $f \in C(\mathbb{S}, \mathbb{S})$, the following conditions are equivalent.*

(i) *Function f has positive topological entropy.*
(ii) *Function f^n has horseshoe for some $n \in \mathbb{N}$.*
(iii) $\mathrm{P}(f^n) = \mathbb{N}$ *for some $n \in \mathbb{N}$.*
(iv) *Function f exhibits distributional chaos.*
(v) *Function f has a basic set.*

5. Distributional chaos and spectral decomposition of dynamical systems of the circle

One-dimensional systems have many properties in common, it is natural to ask if result obtained by Schweizer and Smítal in [ScSm] holds also for dynamical systems on the circle. The paper [3] is devoted to solve this problem. Answer is generally positive, but there are some very natural exceptions. Since the theory of basic sets on the circle is not so developed as in the case of interval maps (see [BBHS] or [Bl], for more details on basic sets on the interval) and since distributional chaos is focused on basic sets. It was necessary to extend theory of basic sets also for continuous mappings of the circle. One of main results of this work is the following theorem which summarize obtained properties of basic sets on the circle (similar theorem can be proved also for basic sets on the interval).

Theorem C. *Let $f \in C(\mathbb{S}, \mathbb{S})$, $x \in \mathbb{S}$ and let $\tilde{\omega}$ be a basic set. Then*

(i) *$\tilde{\omega}$ is perfect;*
(ii) *if $\omega_f(x) \subset \tilde{\omega}$, then $\{y \in \tilde{\omega};\ \omega_f(y) = \omega_f(x)\}$ is dense in $\tilde{\omega}$;*
(iii) *if J is an interval such that $J \cap \tilde{\omega}$ is infinite then $\tilde{\omega} \cap J$ contains a periodic point;*
(iv) *the system of basic sets of f is countable;*
(v) *if $\tilde{\omega}_1 \neq \tilde{\omega}_2$ are indecomposable basic sets and $U = \mathrm{Env}(\tilde{\omega}_1)$, $V = \mathrm{Env}(\tilde{\omega}_2)$, then $U \cap V = \emptyset$, or U and V have at most two points in common, or $U \subset \mathrm{int}(V)$, or $V \subset \mathrm{int}(U)$; in particular, $U \neq V$;*
(vi) *if $\tilde{\omega}$ is indecomposable then, for every compact interval K contained in the interior of $\mathrm{Env}(\tilde{\omega})$, and every compact interval J such that $J \cap \tilde{\omega}$ is infinite, there is a $k \in \mathbb{N}$ such that $f^k(J) \supset K$. We describe this situation saying that $f|_{\tilde{\omega}}$ is strongly transitive.*

The following theorem (the main theorem in the paper [3]) is spectral decomposition of dynamical system on the circle (it is similar to the theorem for dynamical system on the interval up to some necessary modifications, cf. [ScSm]).

Theorem D. *Let $f \in C(\mathbb{S}, \mathbb{S})$.*

(A) *If the topological entropy of f is zero, then $\Sigma(f) = \Sigma_w(f) = \{\chi_{(0,\infty)}\}$.*
(B) *If the topological entropy of f is positive, then:*
  (B1) *Both the spectrum $\Sigma(f)$ and the weak spectrum $\Sigma_w(f)$ are finite and nonempty. Specifically $\Sigma(f) = \{F_1, \ldots, F_m\}$ for some $m \geq 1$, and $\Sigma \setminus \Sigma_w(f) = \{F_{m+1}, \ldots, F_n\}$ where $n \geq m$. Furthermore, for each i there is an $\varepsilon_i > 0$ such that $F_i(\varepsilon_i) = 0$.*
  For any positive integer $k \leq n$, let π_k be the system of sets P such that $\#P \geq 2$ and for any distinct u, v in P, $F_k = F_{uv} < F^*_{uv} = \chi_{(0,\infty)}$.
  (B2) *If $k \leq m$, then π_k contains a nonempty perfect set P_k.*
  (B3) *If, on the other hand, $m < k \leq n$ then π_k is nonempty and any P in π_k contains two or three points.*
  (B4) *If S is a scrambled set for f (or more generally if, for any u, v in S, $\liminf_{i\to\infty} \delta_{uv}(i) = 0$), then there are integers i, j, $k \leq m$ and a decomposition $S = S_i \cup S_j \cup S_k$ such that $F_{uv} \geq F_l$ if $u, v \in S_l$, for $l \in \{i, j, k\}$.*

6. Publications concerning the thesis

[1] M. Málek, *A counterexample of an extremely chaotic function*, Real Analysis Exchange, **23** (1) (1997/1998), 325–327.

[2] M. Málek, *Distributional chaos for continuous mappings of the circle*, Annales Mathematicae Silesianae, **13** (1999), 205–210.

[3] M. Málek, *Distributional chaos and spectral decomposition on the circle*, Topology and its Applications, to appear.

7. Quotations by other authors

[4] J. Cánovas, *Distributional chaos on tree maps, the star case*, Comment. Math. Univ. Carolin., **42** 3 (2001), 583–590. (cf. [2])

[5] J. Cánovas, R. Hric, *Distributional chaos of tree maps*, Topology and its Applications, to appear. (cf. [2])

8. Presentations

[6] 26th Winter School in Abstract Analysis, Křišťanovice, Czech Republic, January, 23–29, 1998. Talk on: "A counterexmalple concerning an extremely chaotic function."

[7] 2nd Czech-Slovak Conference on Dynamical Systems, Liptovský Trnovec, Slovakia, May 7–10, 1998. Talk on: "A counterexmalple concerning an extremely chaotic function."

[8] European Conference on Iteration Theory — ECIT 98, Muszyna, Poland, August 30–September 5, 1998. Invitation. Talk on: "Distributional chaos for continuous maps of the circle."

[9] 3rd Czech-Slovak Conference on Dynamical Systems, Liptovský Trnovec, Slovakia, September 23–29, 1999. Talk on: "Strong shifts and shift equivalence problem."

[10] 28th Winter School in Abstract Analysis, Křišťanovice, Czech Republic, January 23–29, 2000. Talk on: "Strong shifts and shift equivalence problem."

[11] 4th Czech-Slovak Conference on Dynamical Systems, Praděd, Czech Republic, June 22–28, 2000. Talk on: "Distributional chaos for continuous maps of the circle."

[12] 29th Winter School in Abstract Analysis, Lhota nad Rohanovem, February 3–10, 2001. Talk on: "ω-limit sets for continuous maps of the circle."

[13] 5th Czech-Slovak Conference on Dynamical Systems, Praděd, Czech Republic, June 18–23, 2001. Talk on: "Distributional chaos on circles."

[14] 6th Czech-Slovak Conference on Dynamical Systems, Praděd, Czech Republic, June 9—16, 2002. Talk on: "Distributional chaos and spectral decomposition on the circle."

References

[Al] L. Alsedà, J. Llibre, M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, World Scientific Publ., Singapore 1993.

[Bc] L. S. Block, W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math., vol. **1513**, Springer, Berlin 1992.

[Bl] A. Blokh, *The "spectral decomposition" for one-dimensional maps*, Dynam. Report. Expositions Dynam. Systems (N.S.), **4** (1995), 1–59.

[BBHS] A. Blokh, A. M. Bruckner, P. D. Humke and J. Smítal, *The space of ω-limit sets of a continuous map of the interval*, Trans. Amer. Math. Soc., **348 (1996)**, 1357–1372.

[BH] A. Bruckner, T. Hu, *On scrambled sets for chaotic functions*, Transactions of the American Mathematical Society, **301** (1987), 289–297.

[S1] A. N. Sharkovsky, *The partially ordered system of attracting sets,* Soviet Math. Dokl., **7** (1966), 1384–1386.

[S2] A. N. Sharkovsky, *The behavior of a map in neighborhood of an attracting set*, Ukrajin. Mat. Ž., **18** (1966), 60–83. (Russian)

[S3] A. N. Sharkovsky, *Continuous mappings on the set of ω-limit sets of iterated sequences,* Ukrajin. Mat. Ž., **18** (1966), 127–130. (Russian)

[ScSm] B. Schweizer, J. Smítal, *Measures of chaos and spectral decomposition of dynamical systems of the interval*, Trans. Amer. Math. Soc., **344** (1994), 737–754.