Silesian University in Opava
Mathematical Institute in Opava

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Solution of open problems in low-dimensional dynamics

Abstract of the Ph.D. Thesis

August 2009

Mathematical analysis

Slezská univerzita v Opavě
Matematický ústav v Opavě

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Solution of open problems in low-dimensional dynamics

Autoreferát dizertační práce

Srpen 2009

Mathematická analýza

Výsledky tvořící dezertační práci byly získány během doktorského studia oboru Matematická analýza na Matematickém ústavu Slezské univerzity v Opavě v letech 2005 – 2009.

Výzkum byl podporován granty MSM4781305904, GA201/03/1153, GD201/03/H152 a GA201/06/0318.

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Státní doktorská zkouška a obhajoba dizertační práce se konají dne 23. září 2009 v 11 hodin, před oborovou komisí doktorského studia matematické analýzy v zasedací místnosti rektorátu Slezské univerzity v Opavě.

S dizertací je možno se seznámit v knihovně Matematického ústavu Slezské univerzity v Opavě, Na Rybníčku 1, Opava.

Předseda oborové rady:	Prof. RNDr. Miroslav Engliš, DrSc. Matematický ústav Slezská univerzita v Opavě Na Rybníčku 1 Opava

Contents

1. Introduction

The main aim of this thesis is to solve open problems concerning dynamical systems generated by triangular, or skew-product maps of the square, and by continuous maps of general one-dimensional compact metric spaces like topological graphs, trees and dendrites. Essential parts of the thesis are contained in the papers [1], [2], [3] and [4]; we attach them as supplement.

The results were obtained in 2005 – 2009 at the Mathematical Institute of the Silesian University in Opava. The research was supported, in part, by projects MSM4781305904 from the Czech Ministry of Education, and GA201/03/1153, GD201/03/H152 and GA201/06/0318 from the Czech Science Foundation. The support of these institutions is highly appreciated.

2. Basic Terminology and notation

Throughout this abstract, $I = [0,1]$ is the unit compact interval, X a compact metric space with a metric ρ, and $C(X)$ the class of continuous maps of X into itself. For $f \in C(X)$, f^n denotes the *n-th iterate* of f, and a sequence $\{f^n(x)\}_{n=0}^{\infty}$ the *trajectory* of a point $x \in X$.

Recall that the set of accumulation points of the trajectory of a point $x \in X$ under f is the *ω-limit set* of x; it is denoted by $\omega_f(x)$. If $\omega_f(x) = M$ for every $x \in M$ then M is a *minimal set*. We denote by ω_f the *set of ω-limit points* of f. By $\mathrm{Fix}(f)$ we mean the *set of fixed points* of f, by $\mathrm{Per}(f)$ the *set of periodic points* of f and by $\mathrm{Rec}(f)$ the *set of recurrent points* of f, i.e. the set of all $x \in X$ such that $x \in \omega_f(x)$. The closure of $\mathrm{Rec}(f)$ is called the *centre* of f and is denoted by $\mathrm{C}(f)$.

A set $A \subset X$ is *(n,ε)-separated* if, for any distinct points $x_1, x_2 \in A$, there exists i such that $0 \le i < n$ and $\rho(f^i(x_1), f^i(x_2)) > \varepsilon$. For $Y \subset X$, denote by $s_n(\varepsilon, Y, f)$ the maximum possible number of points in an (n,ε)-separated subset of Y. The *topological entropy of f with respect to Y* and the *topological entropy of the map f* are defined by

$$\mathrm{h}(f|Y) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, Y, f), \text{ and } h(f) = \mathrm{h}(f|X),$$

respectively. Other terminology is given when needed.

3. Triangular maps

For a continuous map of the interval there is a long list of properties equivalent to zero topological entropy. About 40 of them are applicable to triangular maps (i.e. $F \in C(I^2), F(x,y) = (f(x), g_x(y)))$ but only few of them are equivalent in this more general setting. In the eighties, A. N. Sharkovsky proposed the problem of classification of the triangular maps of the square with respect to such properties. About 30 conditions were already considered, cf., e.g., [Ko], [BS1], [BS2], [BSS], [FPS1], [FPS2], [FPS3], [K1], [K2], [K3], [PS1], [PS2], [S], [SSp] and [SSt]. It turns out that these conditions belong to 17 equivalence classes, numbered 1 – 17; the properties belonging to the same class of equivalence, are distinguished by letters, like 1a – 1f, or 2a and 2b, etc.

In [1] and [2] we contribute to the solution of this problem by adding six other properties, 18 – 23. It appears that these conditions are mutually non-equivalent. We exhibit the relations between them and five other properties, 1a, 4b, 5, 13b and 14, that already have been studied. In this thesis we show all relations that follow from the previous results, and we also add a new, still unpublished result, Lemma 3.2. These results are summarized in Theorem 3.3.

We consider the following 32 properties of $f \in C(X)$ belonging to 23 different classes of equivalence (used symbols and notions are explained later).

(1) (a) $\mathrm{h}(f) = 0$;
 (b) $\mathrm{h}(f|\,\mathrm{CR}(f)) = 0$;
 (c) $\mathrm{h}(f|\Omega(f)) = 0$;
 (d) $\mathrm{h}(f|\omega(f)) = 0$;
 (e) $\mathrm{h}(f|\,\mathrm{C}(f)) = 0$;
 (f) $\mathrm{h}(f|\,\mathrm{Rec}(f)) = 0$;

(2) (a) $\mathrm{h}(f|\,\mathrm{UR}(f)) = 0$;
 (b) there is no minimal set with positive topological entropy;

(3) $\mathrm{h}(f|\,\mathrm{AP}(f)) = 0$;

(4) (a) $\mathrm{h}(f|\,\mathrm{Per}(f)) = 0$;
 (b) the period of any cycle of f is a power of two;
 (c) every cycle is simple;

(5) f has no homoclinic trajectory;

(6) $f|\,\mathrm{CR}(f)$ is non-chaotic;

(7) $f|\Omega(f)$ is non-chaotic;

(8) $f|\omega(f)$ is non-chaotic;

(9) $f|\,\mathrm{C}(f)$ is non-chaotic;
(10) $f|\,\mathrm{Rec}(f)$ is non-chaotic;
(11) $f|\,\mathrm{UR}(f)$ is non-chaotic;
(12) $\mathrm{UR}(f) = \mathrm{Rec}(f)$;
(13) (a) no infinite ω-limit set contains a cycle;
(b) any ω-limit set either is a cycle or contains no cycle;
(14) any ω-limit set contains a unique minimal set;
(15) f is not DC1;
(16) f is not DC2;
(17) f is not DC3;
(18) trajectory of any point can be strongly approximated by trajectories of closed connected periodic sets;
(19) trajectory of any point can be weakly approximated by trajectories of closed connected periodic sets;
(20) if $\omega_f(x) = \omega_{f^2}(x)$ then $\omega_f(x)$ is a fixed point;
(21) there is no infinite countable ω-limit set;
(22) trajectories of any two points are correlated;
(23) for any closed invariant set A and any $m \in \mathbb{N}$, the map $f^m|A$ cannot be topologically almost conjugate to the shift.

By $\mathrm{CR}(f)$, $\Omega(f)$, $\mathrm{UR}(f)$ and $\mathrm{AP}(f)$ we denote the *set of chain recurrent, non-wandering, uniformly recurrent* and *almost periodic points* of f, respectively.

Recall that a point $x \in X$ is called

- *chain recurrent* if for any $\varepsilon > 0$ there is a sequence of points $\{x_i\}_{i=0}^n$ with $x_0 = x = x_n$ and $\rho(x_{i+1}, f(x_i)) < \varepsilon$, for $i \in \{0, \ldots, n-1\}$.
- *non-wandering* if for any neighbourhood U of x, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$.
- *uniformly recurrent* if for any neighbourhood U of x, there exists $n \in \mathbb{N}$ such that if $f^m(x) \in U$ where $m \geq 0$, then $f^{m+i}(x) \in U$ for some i with $0 < i \leq n$.
- *almost periodic* if for any neighbourhood U of x, there is an $n \in \mathbb{N}$ such that $f^{in}(x) \in U$, for any $i \in \mathbb{N}$.

A map f is *chaotic (in the sense of Li and Yorke)* if there is an *f-chaotic pair*, i.e. if there exist $x, y \in X$ such that

$$0 = \liminf_{n\to\infty} \rho(f^n(x), f^n(y)) < \limsup_{n\to\infty} \rho(f^n(x), f^n(y)).$$

For any pair x, y of points in X and any $n \in \mathbb{N}$, define a *distribution function* $\Phi_{xy}^{(n)} : (0, \operatorname{diam} X] \to I$ by

$$\Phi_{xy}^{(n)}(t) = \frac{1}{n}\{0 \leq i \leq n-1; \rho(f^i(x), f^i(y)) < t\}.$$

Put

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t), \quad \text{and} \quad \Phi_{xy}^*(t) = \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t).$$

If there is a pair x, y of points in X such that

$$\begin{aligned}
&\Phi_{xy}^* \equiv 1 \ \text{ and } \ \Phi_{xy}(t) = 0, \quad \text{for some } t > 0, \quad \text{or} \\
&\Phi_{xy}^* \equiv 1 \ \text{ and } \ \Phi_{xy} < \Phi_{xy}^*, \quad \text{or} \\
&\Phi_{xy}(t) < \Phi_{xy}^*(t) \ \text{ for all } t \text{ in some non degenerate interval,}
\end{aligned}$$

then we say that f exhibits *distributional chaos of type 1-3*, briefly DC1, DC2, DC3, respectively.

Let $\varepsilon > 0$. Trajectory of a point $x \in X$ can be *strongly ε-approximated* by the trajectory of a set A if there exists $i \in \mathbb{N}_0$ such that $\operatorname{diam}(f^i(A)) < \varepsilon$ and

$$\lim_{n\to\infty} \rho(f^n(x), f^n(A)) = 0$$

and it can be *weakly ε-approximated* by the trajectory of a set A if there exist $i, n_0 \in \mathbb{N}_0$ such that $\operatorname{diam}(f^i(A)) < \varepsilon$ and, for any $n \geq n_0$,

$$\rho(f^n(x), f^n(A)) < \varepsilon.$$

The trajectory of a point $x \in X$ can be *strongly* (resp. *weakly*) *approximated* if it can be strongly (resp. weakly) ε-approximated for any $\varepsilon > 0$.

Let $x \in \operatorname{Fix}(f)$, and let x_n, $n = 1, 2, \ldots$, be distinct points in X such that $f(x_{n+1}) = x_n$, for any n, $f(x_1) = x$, and $\lim_{n\to\infty} x_n = x$. Then $\{x_n\}_{n=1}^{\infty}$ is a *homoclinic trajectory related to the point* x. A homoclinic trajectory related to a periodic orbit is defined similarly, cf., e.g., [BC].

Trajectories of the points $x, y \in X$ are *correlated*, if either $\omega_f(x)$ or $\omega_f(y)$ is a fixed point or

$$\omega_{f\times f}(x, y) \neq \omega_f(x) \times \omega_f(y),$$

where the map $f \times f : X \times X \to X \times X$ is given by $(x, y) \mapsto (f(x), f(y))$.

Denote by (Σ, σ) the *shift* of the space of sequences of two symbols. Thus, $\Sigma = \{0, 1\}^{\mathbb{N}}$, and $\sigma : x_1x_2\ldots \mapsto x_2x_3\ldots$. A map $f \in C(X)$ is *topologically almost conjugate* to the shift if there exists a continuous surjective map $\psi : X \to \Sigma$, such that $\psi \circ f = \sigma \circ \psi$ and any point from Σ has at most two preimages in X.

We emphasize that these 32 properties are mutually equivalent if f is a continuous map of the interval, cf., e.g., [BC], [SKSF]. We study this problem in the class of triangular maps.

The *triangular map* is a continuous map $F : I^2 \to I^2$ of the form

$$F(x, y) = (f(x), g_x(y))$$

where the map $f : I \to I$ is called the *base* of F, and $g_x : I_x \to I$ maps the *fibre* $I_x = \{x\} \times I$ into I. We denote the class of triangular maps by $C_\Delta(I^2)$.

In [1] there are studied the relations between properties 1a, 4b, 5, 13b, 14 and 20 – 23 of triangular maps $F \in C_\Delta(I^2)$. In [2] are furthermore added properties 18 and 19. The following theorem sumarize obtained results.

Theorem 3.1. *The relations between properties 1a, 4b, 5, 13b, 14 and 18 – 23 of a triangular map $F \in C_\Delta(I^2)$ are displayed by the following scheme, see Figure 1. There are no other implications except for these following by the transitvity.*

$$\begin{tikzcd}
 & & & 18 \arrow[d] & \\
 & & & 19 \arrow[d] & \\
 & 1a \arrow[d] & & 14 \arrow[ll] \arrow[d] \arrow[dr] & \\
 & 4b \arrow[dl] \arrow[dr] & & 13b \arrow[dl] & 20 \arrow[d] \\
5 \arrow[dr] & & 21 \arrow[dl] & & 22 \\
 & 23 & & &
\end{tikzcd}$$

FIGURE 1

Lemma 3.2. *There is a map $F \in C_\Delta(I^2)$ with the following properties:*

(i) $F|\,\mathrm{CR}(F)$ *is non-chaotic;*
(ii) *there exist two points whose trajectories are not correlated;*
(iii) F *is not* DC3.

Using the results obtained by other authors it is now possible to find the position of any of these 32 conditions, for the class of triangular maps. Properties denoted by the same number, i.e., 1a – 1f, or 2a – 2b, or 4a – 4c, or 13a – 13b are mutually equivalent in $C_\Delta(I^2)$. For the proof, see [BC], [K1] and [K3].

The following theorem summarizes obtained results.

Theorem 3.3. *All known relations between the properties* $1 - 23$ *of triangular maps are displayed below, see Table 1. Relations that have been already known are labeled by the corresponding reference. Relations proved in this thesis are indicated by dark grey, relations following from them by transitivity by light grey. Empty boxes indicate open problems; they are also listed in the next Problem.*

Problem 3.4. *The following relations between the properties of triangular map* $F \in \mathrm{C}_\Delta(I^2)$ *are not known:*

$3 \overset{?}{\Rightarrow} 2$ *Does the property* $\mathrm{h}(F|\,\mathrm{AP}(F)) = 0$ *imply* $\mathrm{h}(F|\,\mathrm{UR}(F)) = 0$*?*

$15 \overset{?}{\Rightarrow} 3$ *Does positiveness of topological entropy on the set of almost periodic points imply* DC1*?*

$16, 17 \overset{?}{\Rightarrow} 1 - 3, 11, 12$ *and* $17 \overset{?}{\Rightarrow} 6 - 10$ *Which relations are between:*

- *positive topological entropy;*
- *existence of homoclinic trajectory;*
- *chaos in the sense of Li and Yorke;*
- $\mathrm{UR}(F) \neq \mathrm{Rec}(F)$

and distributional chaos of type 2 and 3?

$18, 19 \overset{?}{\Rightarrow} 2, 3, 11, 15$ *Does the property that trajectory of any point can be strongly (resp. weakly) approximated by trajectories of closed connected periodic sets imply any of the properties:*

- $\mathrm{h}(F|\,\mathrm{UR}(F)) = 0$*;*
- $\mathrm{h}(F|\,\mathrm{AP}(F)) = 0$*;*
- $F|\,\mathrm{UR}(F)$ *is non-chaotic;*
- F *is not* DC1*?*

For better imagination we put the following scheme (see Figure 2) with all known implications. Missing arrow means that either the particular implication does not hold (except the implications that follow by transitivity) or this relation is not known.

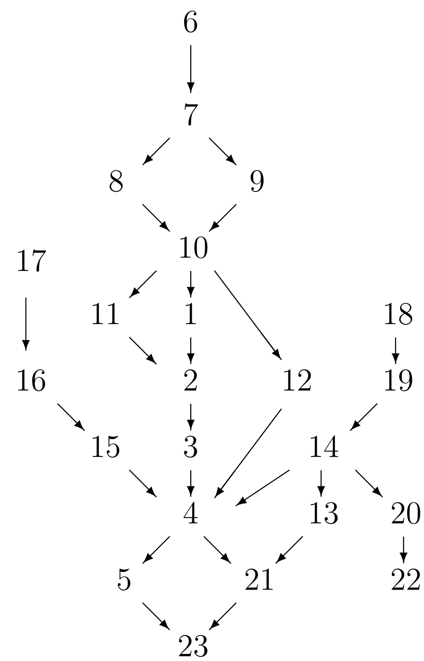

FIGURE 2

	1	2	3	4	5	6	7	8	9	10	11
1	$\bullet$	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	$\Rightarrow$	[K2] $\not\Rightarrow$	[K2] $\not\Rightarrow$	[K2] $\not\Rightarrow$	$\not\Rightarrow$	[K2] $\not\Rightarrow$	[FPS2],[K2] $\not\Rightarrow$
2	[Ko] $\not\Rightarrow$	$\bullet$	def. $\Rightarrow$	def. $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
3	[Ko] $\not\Rightarrow$		$\bullet$	def. $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
4	[Ko] $\not\Rightarrow$	$\not\Rightarrow$	[SSp] $\not\Rightarrow$	$\bullet$	[K1] $\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[FPS3] $\not\Rightarrow$	$\not\Rightarrow$
5	[K1] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[K1] $\not\Rightarrow$	$\bullet$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
6	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\bullet$	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$
7	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	[K2] $\not\Rightarrow$	$\bullet$	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$
8	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	[K2] $\not\Rightarrow$	$\bullet$	[K2] $\not\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$
9	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[K2] $\not\Rightarrow$	$\bullet$	def. $\Rightarrow$	def. $\Rightarrow$
10	[BGKM] $\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\bullet$	def. $\Rightarrow$
11	[Ko] $\not\Rightarrow$	[BGKM] $\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[FPS2],[K2] $\not\Rightarrow$	$\bullet$
12	[SSp] $\not\Rightarrow$	$\not\Rightarrow$	[SSp] $\not\Rightarrow$	[K2] $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[K2] $\not\Rightarrow$	[FPS3] $\not\Rightarrow$	$\not\Rightarrow$
13	[SSp] $\not\Rightarrow$	$\not\Rightarrow$	[SSp] $\not\Rightarrow$	[FPS3] $\not\Rightarrow$	[BS1] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
14	[K1] $\not\Rightarrow$	$\not\Rightarrow$	[SSp] $\not\Rightarrow$	[K1] $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[K2] $\not\Rightarrow$	[FPS2],[K2] $\not\Rightarrow$
15	[SSt] $\not\Rightarrow$	[PS1] $\not\Rightarrow$		[PS2] $\Rightarrow$	[PS2] $\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[PS1] $\not\Rightarrow$
16				$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[PS1] $\not\Rightarrow$	
17				$\Rightarrow$	$\Rightarrow$						
18	[2] $\not\Rightarrow$			$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	
19	$\not\Rightarrow$			$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	
20	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
21	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
22	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
23	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$

TABLE 1. - Part 1

	12	13	14	15	16	17	18	19	20	21	22	23
1	[FPS1] ⇏	[K1] ⇏	[K1],[Ko] ⇏	[FPS2] ⇏	⇏	⇏	⇏	⇏	[1] ⇏	⇒	[1] ⇏	⇒
2	[Ko] ⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇒	⇏	⇒
3	[Ko] ⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇒	⇏	⇒
4	⇏	⇏	[FPS2] ⇏	⇏	[SSt] ⇏	⇏	⇏	⇏	⇏	[1] ⇒	⇏	⇒
5	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	[1] ⇏	⇏	[1] ⇒
6	⇒	[K2] ⇏	[K2] ⇏	[K3] ⇏	⇏	[BSS] ⇏	⇏	⇏	⇏	⇒	3.2 ⇏	⇒
7	⇒	⇏	⇏	⇏	⇏	[BSS] ⇏	⇏	⇏	⇏	⇒	⇏	⇒
8	⇒	⇏	⇏	⇏	⇏	[BSS] ⇏	⇏	⇏	⇏	⇒	⇏	⇒
9	⇒	⇏	⇏	⇏	⇏	[BSS] ⇏	⇏	⇏	⇏	⇒	⇏	⇒
10	[K3] ⇒	⇏	⇏	⇏	⇏	[BSS] ⇏	⇏	⇏	⇏	⇒	⇏	⇒
11	[Ko] ⇏	⇏	[FPS2],[K2] ⇏	⇏	⇏	[BSS] ⇏	⇏	⇏	⇏	⇒	⇏	⇒
12	•	[K2] ⇏	[K2] ⇏	[K3] ⇏	⇏	⇏	⇏	⇏	⇏	⇒	⇏	⇒
13	[FPS2],[K2] ⇏	•	[Ko],[K1] ⇏	⇏	⇏	⇏	⇏	⇏	⇏	[1] ⇒	[1] ⇏	⇒
14	[K2] ⇏	[K1] ⇒	•	[BS2] ⇏	⇏	⇏	⇏	[2] ⇏	[1] ⇒	⇒	⇒	⇒
15	[SSt],[Ko] ⇏	⇏	⇏	•	[SSt] ⇏	⇏	⇏	⇏	⇏	⇒	⇏	⇒
16		⇏	⇏	def. ⇒	•	[PS1] ⇏	⇏	⇏	⇏	⇒	⇏	⇒
17		[PS1] ⇏	[PS1] ⇏	def. ⇒	def. ⇒	•	⇏	⇏	⇏	⇒	3.2 ⇏	⇒
18	[Ko],[2] ⇏	⇒	⇒		[SSt],[2] ⇏	⇏	•	def. ⇒	⇒	⇒	⇒	⇒
19	⇏	⇒	[2] ⇒		⇏	⇏	[2] ⇏	•	⇒	⇒	⇒	⇒
20	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	•	⇏	[1] ⇒	[1] ⇏
21	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	•	⇏	[1] ⇒
22	⇏	[1] ⇏	⇏	⇏	⇏	⇏	⇏	⇏	[1] ⇏	⇏	•	⇏
23	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	⇏	•

Table 1. - Part 2

4. Trees, graphs and dendrites

In this section we generalize some notions and results from the theory of discrete dynamical systems on the unit interval to the case of trees, graphs and dendrites.

Let *arc* be any topological space homeomorphic to the compact unit interval. A *graph* is a continuum (a nonempty compact connected metric space) which can be written as a union of finitely many arcs any two of which can intersect only in their endpoints. Let $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ be the *circle*. A *tree* is a graph containing no subset homeomorphic to the circle. A *dendrite* is a locally connected continuum containing no subset homeomorphic to the circle.

In [3], we study the properties of ω-limit sets, recurrent points and centre in these various cases of continuums.

It is known that, for compact interval I the set of all ω-limit points of f is closed, and contains the centre of f (cf. [Sh3], [BC]). In [Bl1], and recently in [MSu], it is proved that it holds also for any graph. The first main result in [3] shows that it is not true in the case of dendrites.

Theorem 4.1. *There is a continuous self-map f of a dendrite such that ω_f is not closed, and it is a proper subset of* $\mathrm{C}(f)$.

Another known fact concerning ω-limit sets is the following. By [Sh2], if $(\omega_k)_{k=1}^{\infty}$ is a sequence of ω-limit sets of a continuous interval map f such that $\omega_k \subset \omega_{k+1}$, for every $k \in \mathbb{N}$, then the closure of their union is also an ω-limit set of f. This fact can be obtained also as a consequence of a more general result proved in [BBHS]. Moreover, it holds also for graphs. By [MSh], the set of all ω-limit sets of a continuous graph map endowed with the Hausdorff metric is compact. Therefore, since an increasing (with respect to inclusion) sequence of ω-limit sets of a graph map f converges (with respect to the Hausdorff metric) to the closure of their union, the closure is also an ω-limit set of f. Again, this is not true for dendrites. In [3], we find an example of a continuous map of dendrite having strictly increasing sequence of ω-limit sets which is not contained in any maximal one. Consequently, the space of ω-limit sets of a continuous dendrite map endowed with the Hausdorff metric need not to be compact.

Theorem 4.2. *There is a sequence of ω-limit sets $(\omega_k)_{k=1}^{\infty}$ of a continuous self-map f of a dendrite such that $\omega_k \subset \omega_{k+1}$, for every $k \in \mathbb{N}$, but there is no ω-limit set of f containing every ω_k.*

It is also known that the positiveness of topological entropy, the existence of a horseshoe and the existence of a homoclinic trajectory are mutually equivalent, for interval maps (cf., e.g., [ALM], [BC], [SKSF]). The aim of [4] is to investigate the relations between these properties for continuous maps of trees, graphs and dendrites. We consider three different definitions of a horseshoe and two different definitions of a homoclinic trajectory.

Suppose that there are disjoint compact sets $A, B \subset X$ such that

$$f(A) \cap f(B) \supset A \cup B. \tag{1}$$

Then usually one says that f has a *horseshoe* or that A, B form a *horseshoe* for f. We call it a *strict general horseshoe.*

When X is a compact interval I, it is often said that f has a horseshoe if there are closed subintervals A, B (as the only possible kind of connected subcontinua) with disjoint interiors satisfying (1).

Generalizing this definition of a horseshoe for graphs and dendrites, we get the following three definitions.

Let X be a tree, a graph or a dendrite. Suppose that there are subsets A, B satisfying (1). If, moreover, A, B are

- arcs which are either disjoint or intersect only in their endpoints, then we say that A, B form an *arc horseshoe* for f.
- subcontinua which are either disjoint or intersect only in their endpoints, then we say that A, B form an *endpoint intersection horseshoe* for f.
- subcontinua which are either disjoint or intersect only in finitely many points, then we say that A, B form a *finite intersection horseshoe* for f.

It is clear that every arc horseshoe is an endpoint intersection horseshoe, and every endpoint intersection horseshoe is a finite intersection horseshoe.

Since any graph consists of finitely many arcs, sets A, B in the definitions of an arc horseshoe, a finite intersection horseshoe, an endpoint intersection horseshoe can intersect in finitely many points each one of finite order. Thus, always an iteration can be found for which there exists a strict general horseshoe. Together with results from [LM] we get that these three types of horseshoes are equivalent for graph maps. Thus, in this case we will call all these three notions just a *horseshoe.*

Finally, we define a homoclinic trajectory. For a diffeomorphism of a smooth manifold, a homoclinic point is defined to be a point which is in both the stable manifold and the unstable manifold of some hyperbolic periodic point.

For interval maps, one says that a point x is homoclinic (in the sense of Poincaré) if there is a periodic point p, not containing x in its orbit but in its unstable manifold and such that $p \in \omega_f(x)$. For example, L. Block in [B] defines a homoclinic point in the following more restrictive way. A point x is homoclinic if there is a periodic point p with period k, $x \neq p$, x belongs to the unstable manifold of p under f^k, and $f^{nk}(x) = p$ for some n.

We decided to adopt the following definition for continuous maps of compact metric spaces. Let $\alpha = \{p_1, p_2, \dots, p_k\}$ be a k-cycle, and $(x_n)_{n=-\infty}^{\infty}$ be a sequence such that $x_0 \notin \alpha$ and $f(x_n) = x_{n+1}$. If for every $j \in \{1, 2, \dots, k\}$, $\lim_{n\to-\infty} x_{nk+j} = \lim_{n\to\infty} x_{nk+j} = p_j$ then we say that $(x_n)_{n=-\infty}^{\infty}$ is a *homoclinic trajectory of f related to the cycle α* (or to the fixed point p if $\alpha = \{p\}$). If moreover $f(x_1) = p_1$, then we call this trajectory an *eventually periodic homoclinic trajectory.*

The first main result in [4] shows the relations between the positiveness of topological entropy, the existence a horseshoe and the existence of a homoclinic trajectory one of two mentioned kinds in the case of trees and graphs.

Theorem 4.3. *Let f be a continuous map of a graph. The following two properties are equivalent:*

(1) $h(f) > 0$*;*

(2) *there is an n such that f^n has a horseshoe.*

Each of them implies that

(3) *f has an eventually periodic homoclinic trajectory;*

which implies that

(4) *f has a homoclinic trajectory.*

There is no other implication between these four properties (except the implications that follow by transitivity).

If the graph is a tree then all these properties are mutually equivalent.

The situation is far more interested in the case of dendrites.

Theorem 4.4. *Let f be a continuous map of a dendrite. The relations between the properties*

(1) $h(f) > 0$*;*

(2) *there is an iteration of f which has an arc horseshoe;*
(3) *there is an iteration of f which has an endpoint intersection horseshoe;*
(4) *there is an iteration of f which has a finite intersection horseshoe;*
(5) *f has an eventually periodic homoclinic trajectory;*
(6) *f has a homoclinic trajectory;*

are described by the following scheme (see Figure 3,) where every arrow means the implication, arrows with question mark mean open problems, a missing arrow means that the implication does not hold except the implications that follow by transitivity.

$$\begin{tikzcd}
& 2 \arrow[dl] \arrow[d] \arrow[dr] & \\
1 & 3 \arrow[r, dashed, "?"] \arrow[d] \arrow[dr, dashed, "?"] & 5 \arrow[d] \\
& 4 & 6
\end{tikzcd}$$

FIGURE 3

As follows from Theorem 4.4 there are still open problems concerning the relations between the properties under consideration.

Problem 4.5. *Does the existence of an endpoint intersection horseshoe for a dendrite map f imply the existence of a homoclinic trajectory of f (or even an eventually periodic homoclinic trajectory)?*

5. PUBLICATIONS CONCERNING THE THESIS

[1] V. Kornecká, *A classification of triangular maps of the square*, Acta Math. Univ. Comen. **75** (2006), 241 – 252.

[2] V. Kornecká, *On a problem of Sharkovsky concerning the classification of triangular maps*, Grazer Math. Berichte **351** (2007), 91 – 99.

[3] Z. Kočan, V. Kornecká-Kurková, M. Málek, *On the centre and the set of ω-limit points of continuous maps on dendrites*, Topology and its Applications (2009), doi:10.1016/j.topol.2009.02.008.

[4] Z. Kočan, V. Kornecká-Kurková, M. Málek, *Entropy, horseshoes and homoclinic trajectories on trees, graphs and dendrites*, Ergodic Theory & Dynam. Syst. - accepted August, 2009.

6. Presentations on conferences

[5] 10th Czech-Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, June 25 – July 1, 2006.

[6] European Conference on Iteration Theory, Gargnano, Italy, September 10 – 16, 2006.

[7] Visegrad Conference Dymanical Systems, High Tatras 2007, Štrbské Pleso, Slovakia, June 17 – 23, 2007.

[8] 22nd Summer Conference on Topology and its Applications, Castellón, Spain, July 24 – 27, 2007.

[9] Conference and Summer School in Honor of Allan Peterson, Monastery of Novacella, Italy, July 28 – August 2, 2007.

[10] IIV Iberoamerican Conference on Topology and its Applications CITA 2008, Valencia, Spain, June 25 – 28, 2008.

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