Silesian University in Opava
Mathematical Institute in Opava

Petra Kordulová

Quasilinear hyperbolic equation with hysteresis

Abstract of the Ph.D. Thesis
October 2007

Mathematical Analysis

Slezská univerzita v Opavě
Matematický ústav v Opavě

Petra Kordulová

Kvazilineární hyperbolická rovnice s hysterezí

Autoreferát dizertační práce
Říjen 2007

Matematická analýza

Výsledky tvořící dizertační práci byly získány během doktorského studia oboru Matematická analýza na Matematickém ústavu Slezské univerzity v Opavě v letech 2004-2007. Výzkum byl podporován projekty MSM 4781305904, GD 201/03/H152, GP 201/02/P040.

Dizertant:	Mgr. Petra Kordulová Matematický ústav v Opavě Slezská univerzita v Opavě
Školitel:	Prof. RNDr. Štefan Schwabik, DrSc. [1] Matematický ústav AV ČR
Školící pracoviště:	Matematický ústav v Opavě Slezská univerzita v Opavě
Oponenti:	Prof. Ralph E. Showalter Oregon State University, USA Prof. RNDr. Jan Franců, CSc Vysoké učení technické, Brno

Autoreferát byl rozeslán dne 24.10. 2007.

Státní doktorská zkouška a obhajoba dizertační práce se konají dne 5.11. 2007 v 11 hod. v zasedací místnosti Ústavu matematiky a statistiky PřF MU, Janáčkovo nám. 2a, Brno. S dizertací je možno se seznámit v knihovně Matematického ústavu v Opavě, Na Rybníčku 1, Opava.

Předseda zkušební komise:	Prof. RNDr. Jaroslav Smítal, DrSc. Matematický ústav v Opavě Slezská univerzita v Opavě Na Rybníčku 1 746 01 Opava

[1] Konzultantka RNDr. Jana Kopfová, Ph.D.

Contents

1 Abstract

Thesis is devoted to the presentation of new results about a quasilinear hyperbolic equation of first order with hysteresis operator. The following is the main contribution of the thesis:

1. We show in detail that the equation with hysteresis term represented by generalized play operator can be transformed into a system of differential inclusions with m- and T-accretive operator. This implies the existence and uniqueness of an integral solution of our equation.
2. Using the result from the first part we derive a stability result for the solution in the L^1 space.
3. In the case that hysteresis term is represented by operators, whose loops are convex, with properties of local Lipschitz continuity and piecewise monotonicity, we prove the existence of a smooth solution of our problem.
4. We give an example of operator satisfying an existence theorem and a counterexample with a nonconvex hysteresis operator.

Concernig the first point we obtain an integral solution, which is a very weak notion of a solution, but the results can be extended to possibly discontinuous generalized Prandtl-Ishlinskii operators of play type. This includes the case of possibly discontinuous Preisach operators. A similar result about m- and T-accretivity was proved for parabolic equation in Visintin's book from 1994. He claimed the existence and uniqueness of the integral solution in L^1. He just outlined the argument, which is similar to that used for the linear second order elliptic operator in divergence form. The main difference is in the first part of the proof. Here the whole proof is made in detail.

The second point deals with asymptotic behaviour of solution. We apply the theorem of Wittbold together with the theorem about accretivity for hyperbolic operators with hysteresis to get the stability result for solution in L^1. There have been no known asymptotic results yet. The asymptotic results for parabolic equation were given by Kopfová in 1998.

Furthermore in the third point we introduce a weak formulation of our problem in Sobolev spaces. By investigation of the smoothness of solutions for continuous hysteresis operators, we get the exi-

stence of a smooth solution. The existence theorem is proved by a method based on an approximation of implicit time discretization scheme, a-priori estimates and passage to the limit. The smoothness result satisfies an entropy condition. We obtain an uniqueness result by nonlinear semigroup approach. The assumptions of the existence theorem are satisfied, e.g., by the generalized play operator and the Prandtl-Ishlinskii operator, whose hysteresis loops are convex. In Visintin's book one can find an existence results for quasilinear parabolic equations with memory.

In the fourth part we compute an exact solution for the classical play and a bit modified generalized play operator using the method of characteristics. We point out that convex hystersis operator substituted by classical play satisfies the assumptions of the existence theorem and we get continuous solution of our problem. In the case we assume a slightly modified nonconvex generalized play operator we find out that the characteristics with different values of solutions cross, i.e., we get a discontinuous solution.

2 Introduction

We study the quasilinear hyperbolic equation with hysteresis

$$\begin{aligned} \frac{\partial(u+v)}{\partial t} + \frac{\partial u}{\partial x} &= f, v = \mathcal{F}(u) \text{ in } \Omega \times [0,T], \qquad (2.1) \\ u(x,0) &= u_0(x), \\ u(\alpha,t) &= 0, \end{aligned}$$

where $\Omega = (\alpha, \beta)$, as a generic model for the transport and adsorption of a chemical of concentration $u(x,t)$ carried in a solution with constant unit velocity in a tube $x \in (\alpha, \beta)$ for $t > 0$. Here $\mathcal{F}(\cdot)$ is a rather general functional describing adsorption and desorption of the chemical on the particles of solid filling up the tube. In the general situation considered here, the adsorption-desorption functional $\mathcal{F}(\cdot)$ exhibits hysteresis, i.e., the relations between u and v for the case when u is increasing (adsorption) and decreasing (desorption) follow different curves. There is hysteresis represented by a type of a generalized play. The motivation for our study comes from applications in chemical and geological engineering.

Hysteresis is an exciting and mathematically challenging phenomenon that occurs in rather different situations: it can be a product of fundamental physical mechanisms (such as phase transitions) or the consequence of a degradation or imperfection (like the play in a mechanical system), or it is built deliberately into a system in order to monitor its behaviour, as in the case of the heat control via thermostats. The interplay between memory effects and the occurrence of hysteresis loops has the effect that hysteresis is a nonlinear phenomenon which is not easy to treat mathematically.

Hence it was only in the early seventies that the group of Russian scientists around Krasnosel'skii introduced the concept of hysteresis and started a systematic investigation of its properties which culminated in the fundamental monograph Krasnosel'skii-Pokrovskii (1983). From that moment many mathematicians have contributed to the mathematical study of hysteresis and important monographs have appeared, see Brokate and Sprekels [1], Krejčí [7], Mayergoyz [8] and Visintin [10].

The equation (2.1) was studied in [2], [3], [9], [10]. Visintin investigated the Cauchy problem for equation (2.1) with hysteresis functional represented by a possibly discontinuous generalized play operator by using the semigroup approach. He claimed the existence and uniqueness of the integral solution in L^1, but he just outlined the proof. These results can be extended to possibly discontinuous generalized Prandtl-Ishlinskii operators of play type, this includes the case of possibly discontinuous Preisach operators.

In Visintin's book [10] it was posed as an open problem whether the integral solution of (2.1) with hysteresis satisfies an entropy condition introduced by Kružkov. Such an entropy condition was derived by Kopfová [2]. Visintin also considered the Cauchy problem for completed relay operator and its regularization and proved the existence of a weak solution [11].

The results can be extended to the more general quasilinear hyperbolic equations of the form

$$\frac{\partial(u+v)}{\partial t} + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j u) + cu \quad = \quad f, \tag{2.2}$$

where let b_j and c be given smooth functions, [3], [10].

Showalter and Peszynska obtained the existence and uniqueness of differentiable solutions by the theory of nonlinear semigroups in a Hilbert space L^2. They assumed that hysteresis is represented by the classical play operator and also by more general case of convex adsorption-desorption hysteresis functional.

The goal is to present new results concerning quasilinear hyperbolic equation of first order with hysteresis models.

The first part is devoted to the theory of nonlinear semigroups. We transform our equation (2.1) into a system of differential inclusions containing an accretive operator. A main result of the section is an important theorem about m- and T-accretivity of the operator provided that $\mathcal{F}(\cdot)$ is generalized play operator. A theorem about existence and uniqueness of an integral solution follows. Such integral solution satisfies an entropy condition [2].

In the second part we investigate the smoothness of solutions of our problem coupled with a nonconvex generalized play operator and with suitably restricted class of hysteresis models, whose hysteresis loops are convex. This branch of hysteresis is represented by a generalized play operator, a generalized Prandtl-Ishlinski operator and a Preisach operator. We find out that they prevent the formation of shocks. On the other hand the nonconvex hysteresis operators cause a discontinuity of solution. This was shown throughout the investigation of a slightly modified nonconvex generalized play operator.

In the third part we investigate the asymptotic behaviour of the solution of a boundary value problem associated to the equation (2.1) with zero right-hand side. We apply the theorem of Wittbold together with the theorem about accretivity for hyperbolic operators with hysteresis to get the stability of the solution.

3 Hysteresis and semigroups

In this section we study the accretivity properties of the possibly discontinuous generalized play operator for the equation

$$\frac{\partial(u+v)}{\partial t} + \frac{\partial u}{\partial x} = f \quad \text{in} \quad \Omega \times [0,T]. \tag{3.1}$$

The theory of nonlinear semigroups is used to prove existence and uniqueness of integral solutions. Exact result is stated in Theorem 3.2. Theorem 3.4 proves that the integral solution satisfies the entropy condition.

We denote by $\mathbb{R}^2_1$ the Banach space of vectors $U := (u, v) \in \mathbb{R}^2$, endowed with the norm

$$||(u,v)||_{\mathbb{R}^2_1} := |u| + |v| \quad \forall (u,v) \in \mathbb{R}^2_1.$$

$L^1(\Omega; \mathbb{R}^2_1)$ is a Banach space endowed with the norm

$$\begin{aligned} ||U||_{L^1(\Omega;\mathbb{R}^2_1)} &:= \int_\Omega (|u(x)| + |v(x)|)\mathrm{d}x \\ \forall U &:= (u,v) \in L^1(\Omega; \mathbb{R}^2_1). \end{aligned} \tag{3.2}$$

The space $\mathcal{H}$ is defined as $\mathcal{H} := \{u \in H^1(\Omega), u(\alpha) = 0\}$. The norm in space $\mathcal{H}$ is denoted by $\| \cdot \|_{\mathcal{H}}$.

We transform the equation (3.1) containing possibly discontinuous generalized play hysteresis operator into a system of differential inclusions with accretive operators [10], Section VIII. The equation (3.1) with initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ is equivalent to the Cauchy problem:

$$\left\{ \begin{aligned} \tfrac{\partial U}{\partial t} + \mathcal{A}(U) + \Re(U) &\ni F, \quad \text{in } \Omega \times [0,T] \\ U(0) &= U_0, \end{aligned} \right. \tag{3.3}$$

where

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &:= \{U \in \mathbb{R}^2 : \inf \gamma_r(u) \le v \le \sup \gamma_l(u)\} \\ \mathcal{A}(U) &:= \{(\xi, -\xi) \in \mathbb{R}^2 : \xi \in \phi(U) \cap \mathbb{R}\} \\ & \quad\ \forall U \in \mathcal{D}(\mathcal{A}) \\ \mathcal{B}(u) &:= \frac{\partial u}{\partial x} \\ \Re(U) &:= (\mathcal{B}(u), 0) \\ \mathcal{D}(\Re) &:= \{U \in L^1(\Omega; \mathbb{R}^2_1) : \mathcal{B}u \in L^1(\Omega)\} \\ \mathcal{Q}(U) &:= \mathcal{A}(U) + \Re(U) \\ \mathcal{D}(\mathcal{Q}) &:= \{U := (u,v) \in L^1(\Omega; \mathbb{R}^2_1) : U \in \mathcal{D}(\mathcal{A}) \\ & \quad\ \text{a.e. in } \Omega,\ u \in W^{1,1}(\Omega), u(\alpha) = 0\} \end{aligned} \tag{3.4}$$

and by setting $\quad U := (u,v), \quad U_0 := (u_0, v_0), \quad F := (f, 0),$

$$\phi(u,v) = \begin{cases} \{+\infty\} & \text{if} \quad v < \inf \gamma_r(u), \\ \widetilde{\mathbb{R}}^+ & \text{if} \quad v \in \gamma_r(u) \setminus \gamma_l(u), \\ \{0\} & \text{if} \quad \sup \gamma_r(u) < v < \inf \gamma_l(u), \\ \widetilde{\mathbb{R}}^- & \text{if} \quad v \in \gamma_l(u) \setminus \gamma_r(u), \\ \{-\infty\} & \text{if} \quad v > \sup \gamma_l(u), \\ \widetilde{\mathbb{R}} & \text{if} \quad v \in \gamma_l(u) \cap \gamma_r(u), \end{cases} \tag{3.5}$$

where $\widetilde{\mathbb{R}} := [-\infty, +\infty]$, $\widetilde{\mathbb{R}}^+ := [0, +\infty]$, and $\widetilde{\mathbb{R}}^- := [-\infty, 0]$.
First, we present a general statement (see [10], Section VIII.2, Proposition 2.1), which we shall apply to hysteresis model (see [10], Section VIII.2, Theorem 2.2).

Proposition 3.1 *(Accretivity. General Case) Assume that the (possibly multivalued) function* $\widehat{\phi} : \mathbb{R}^2 \to \mathcal{P}(\widetilde{\mathbb{R}})$ *is such that*

$$\mathcal{D}(\widehat{\phi}) := \{(u,v) \in \mathbb{R}^2 : \widehat{\phi}(u,v) \cap \mathbb{R} \neq \emptyset\} \neq \emptyset,$$

and

$$\begin{cases} \forall (u_i, v_i) \in \mathcal{D}(\widehat{\phi}), \forall \xi_i \in \widehat{\phi}(u_i, v_i) (i = 1, 2), \\ \textit{if } u_1 < u_2 \textit{ and } v_1 > v_2, \textit{ then } \xi_1 \leq \xi_2. \end{cases} \tag{3.6}$$

Set

$$\begin{cases} \mathcal{D}(\hat{\mathcal{A}}) & := & \{U := (u,v) \in \mathbb{R}^2 : \widehat{\phi}(u,v) \cap \mathbb{R} \neq \emptyset\}, \\ \hat{\mathcal{A}}(U) & := & \{(\xi, -\xi) \in \mathbb{R}^2 : \xi \in \widehat{\phi}(u,v) \cap \mathbb{R}\} \quad \forall U \in \mathcal{D}(\hat{\mathcal{A}}). \end{cases} \tag{3.7}$$

Then $\hat{\mathcal{A}}$ *is T-accretive in* $\mathbb{R}^2_1$.
Moreover, if

$$\begin{cases} \exists \hat{a} > 0 : \forall z \in \mathbb{R}, G_z : u \mapsto \{v \in \mathbb{R} : v - z \in \hat{a}\widehat{\phi}(u,v)\} \textit{ is a} \\ \textit{maximal monotone (possibly multivalued) function in } \mathbb{R}^2, \end{cases} \tag{3.8}$$

then $\hat{\mathcal{A}}$ *is m-accretive in* $\mathbb{R}^2_1$.

Theorem 3.1 *(Accretivity. Rate Independent Case) Assume that* γ_l *and* γ_r *are such that* $\inf \gamma_r(u) \leq \sup \gamma_l(u)$ *and that* $\mathcal{D}(\mathcal{A}) \neq \emptyset$. *Then the operator* $\mathcal{A}$ *is T- and m-accretive in* $\mathbb{R}^2_1$.

Proof. Direct application of Proposition 3.1, as condition (3.8) is fulfilled for any $\hat{a} > 0$. In fact, for any $(u, z) \in \mathbb{R}^2$ and any $\hat{a} > 0$,

G_z defined in (3.8) as follows

$$\begin{cases} G_z(u) &= \gamma_r(u) & if\ z \leq \inf \gamma_r(u), \\ G_z(u) &= [z, \sup \gamma_r(u)] & if\ \inf \gamma_r(u) < z < \sup \gamma_r(u), \\ G_z(u) &= \{z\}, & if\ \sup \gamma_r(u) \leq z \leq \inf \gamma_l(u), \\ G_z(u) &= [\inf \gamma_l(u), z] & if\ \inf \gamma_l(u) < z < \sup \gamma_l(u), \\ G_z(u) &= \gamma_l(u) & if\ \sup \gamma_l(u) \leq z; \end{cases} \tag{3.9}$$

is a maximal monotone (possibly multivalued) function. □

Theorem 3.2 *Assume that* $\inf \gamma_r(u) \leq \sup \gamma_l(u)$ *and that* $\gamma_l(u)$, $\gamma_r(u)$ *are affinely bounded, that is, there exist constants* C_1, $C_2 > 0$, *such that* $\forall w \in \mathbb{R}$, $\forall z \in \gamma_h(w)$

$$|z| \leq C_1|w| + C_2, \quad (h = l, r). \tag{3.10}$$

Let $\mathcal{A}$ *and* $\Re$ *be defined as previously. Then the operator* $\mathcal{A} + \Re$ *is m- and T-accretive in* $L^1(\Omega; \mathbb{R}^2_1)$.

Proof. We only outline the proof in its main steps. In the first three steps we study our operator in $L^2(\Omega; \mathbb{R}^2_1)$. In the first step we use Yosida approximation of curves γ_r and γ_l and for any $n \in N$. We prove m-accretivity of the operator $\mathcal{Q}$. In the second one we show T-accretivity of the operator, by using the Heaviside graph, for any $n \in N$. In the next step we take the limit in n and get the same properties also for the limit operator. Finally it is studied in $L^1(\Omega; \mathbb{R}^2_1)$.
□

As we saw, the operator occuring in the Cauchy problem is m- and T-accretive. Here we apply some classical results of the theory of nonlinear semigroups, see [10], Section VIII.6, Theorem 6.3.

Theorem 3.3 *Let* Ω *be an open subset of* $\mathbb{R}$. *Let* $L^1(\Omega; \mathbb{R}^2_1)$ *be endowed with the norm (3.2). Define the operator* $\Re$ *as in (3.4). Let* $\mathcal{A}$ *be as above, and assume that (3.10) holds. Take any* $U_0 := (u_0, v_0) \in L^1(\Omega; \mathbb{R}^2_1)$ *such that* $U_0 \in \mathcal{D}(\phi)$ *a.e. in* Ω, *and* $f \in L^1(\Omega \times (0,T))$ *and set* $F := (f, 0)$, $\mathcal{Q} := \mathcal{A} + \Re$.

Then the Cauchy problem (3.3) has one and only one integral solution $U : [0,T] \to L^1(\Omega; \mathbb{R}^2_1)$, *which depends continuously on data* u_0, v_0, f. *Moreover, if* $f \in BV(0,T; L^1(\Omega))$ *and* $\Re u_0 \in L^1(\Omega)$, *then* U *is Lipschitz continuous.*

Let $\mathcal{H}_{r\gamma}$ denotes the hysteresis region, i.e., the subset of $\mathbb{R}^2$ of admissible pairs (u, v) such that $\inf \gamma_r(u) \leq v \leq \sup \gamma_l(u)$.

Theorem 3.4 *(Entropy condition) Let the assumptions of Theorem 3.2 hold. Let $\mathcal{A}_0 U = \mathcal{A}(U) + \mathcal{R}(U)$ on $\mathcal{D}(\mathcal{A}_0)$, and let $\mathcal{S}(t) = (u, v)$ be the corresponding semigroup of contractions.*
Let $w \in \overline{\mathcal{D}(\mathcal{A})}$ and $t \geq 0$. Then if $w = (u_0, v_0) \in L^\infty(\Omega) \times L^\infty(\Omega)$,

$$\int_0^T \int_\Omega (|u - k| + |v - \tilde{k}|)\psi_t(x, t) \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega |u - k| \psi_x(x, t) \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_0^T \int_\Omega \mathrm{sign}(u - k) f(x, t) \psi(x, t) \mathrm{d}x \, \mathrm{d}t \geq 0$$

for every $\psi(x, t) \in \mathcal{C}_0^\infty((0, T) \times \Omega)$ such that $\psi \geq 0$ and every $k, \tilde{k} \in \mathcal{H}_{r\gamma}$ and $T > 0$.

Theorem is proved in Kopfová [3].

The theory of nonlinear semigroups is used to prove existence and uniqueness of integral solutions. Exact result arises from Theorem 3.2. Theorem 3.4 proves that the integral solution satisfies the entropy condition.

Remark 3.1 *The latter result can be applied to the analogous problem corresponding to a possibly discontinuous generalized Prandtl-Ishlinskii operator of play type. This includes the case of possibly discontinuous Preisach operator.*

4 Classification of solutions

In this chapter we present results obtained in [5].

4.1 Existence result

Let us set $Q = \Omega \times (0, T)$. We assume that $u_0 \in L^2(\Omega)$ is a given initial condition, $f \in L^2(Q)$ is a given function. We set $v_0(x) = [\mathcal{F}(u(x, \cdot))](0)$ a.e. in Ω. Let

$$\mathcal{F} : \mathcal{M}(\Omega; C([0, T])) \to \mathcal{M}(\Omega; C([0, T]))$$

be a hysteresis operator, where we denote by $\mathcal{M}(\Omega; C([0,T]))$ the Fréchet space of measurable functions $\Omega \to C([0,T])$
We want to solve the following problem.

Problem 4.1 *We search for a function* $u \in \mathcal{M}(\Omega; C([0,T])) \cap L^2(Q)$ *such that* $\mathcal{F}(u) \in L^2(Q)$ *and*

$$\int_0^T \int_\Omega (u + \mathcal{F}(u)) \frac{\partial \psi}{\partial t} \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \frac{\partial u}{\partial x} \psi \, \mathrm{d}x \, \mathrm{d}t =$$

$$- \int_0^T \int_\Omega f \psi \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \psi(x, 0)[u_0(x) + v_0(x)] \, \mathrm{d}x \tag{4.1}$$

for any $\psi \in L^2(Q) \cap H^1(0,T; L^2(\Omega))$ *with* $\psi(\cdot, T) = 0$ *a.e. in* Ω.

Interpretation. The variational equation (4.1) yields

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] + \frac{\partial u}{\partial x} = f \quad \text{in } \mathcal{D}'(Q) (\text{in the sense of distribution}),$$

whence

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] = f - \frac{\partial u}{\partial x} \quad \text{in } L^2(Q).$$

Thus $u + \mathcal{F}(u) \in H^1(0,T; L^2(\Omega))$. Hence, integrating by parts in time in (4.1), we get

$$[u + \mathcal{F}(u)]|_{t=0} = u_0 + v_0 \quad \text{in } L^2(\Omega).$$

Now we are ready to state and prove the following existence result.

Theorem 4.1 *(Existence) Let us assume operator* $\mathcal{F}$, *whose hysteresis loops are convex, with properties of:*
local Lipschitz continuity, i.e.,

$$\begin{cases} \exists \mathcal{L} > 0 : \forall w \in \mathcal{M}(\Omega; C([0,T])), \forall [t_1, t_2] \subset [0,T], \\ \textit{if } w(x, \cdot) \textit{ is affine in } [t_1, t_2], \textit{a.e. in } \Omega, \textit{ then} \\ \big| [\mathcal{F}(w)](x, t_2) - [\mathcal{F}(w)](x, t_1) \big| \le \mathcal{L} |w(x, t_2) - w(x, t_1)|, \end{cases}$$

and piecewise monotonicity in the following sense

$$\begin{cases} \forall w \in \mathcal{M}(\Omega; C([0,T])), \forall [t_1, t_2] \subset [0,T], \\ \textit{if } w(x, \cdot) \textit{ is affine in } [t_1, t_2], \textit{ a.e. in } \Omega, \textit{ then} \\ \{[\mathcal{F}(w)](x, t_2) - [\mathcal{F}(w)](x, t_1)\}[w(x, t_2) - w(x, t_1)] \ge 0 \textit{ a.e. in } \Omega. \end{cases}$$

Moreover, $f \in W^{1,1}(0,T;L^2(\Omega))$*,* $u_0 \in L^2(\Omega)$*,* $v_0 \in L^2(\Omega)$*.*
Then Problem 4.1 has at least one solution such that

$$u \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(\Omega;H^1(0,T)) \cap L^\infty(0,T;H^1(\Omega)),$$

$$\mathcal{F}(u) \in H^1(0,T;L^2(\Omega)).$$

The proof is led through a method based on an approximation of implicit time discretization scheme, a-priori estimates and passage to the limit.

Remark 4.1 *The assumptions of Theorem 4.1 are satisfied e.g. by the generalized play operator and the Prandtl-Ishlinskii operator, whose hysteresis loops are convex.*

Remark 4.2 *The Problem 4.1 corresponding to a generalized play operator, a generalized Prandtl-Ishlinskii operator (which includes the case of a Preisach operator) can be set in the form of the Cauchy problem [10]. For such a system we dispose of the notion of an integral solution in the sense of nonlinear semigroup theory, see Section 3. As it is stated in the following theorem, our smooth solution coincides with the integral solution (see [10], Section IX.2, Theorem 2.6).*

Theorem 4.2 *Assume that* $\gamma_r, \gamma_l \in C(\mathbb{R})$ *are such that* $\gamma_r(u) \leq \gamma_l(u)$*,* $\forall u \in \mathbb{R}$ *and Lipschitz continuous, and* $\mathcal{F}$ *is generalized play. Then the weak solution* (u,v) *has the following regularity:*

$$u \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(\Omega;H^1(0,T)) \cap L^\infty(0,T;H^1(\Omega)),$$

$$v \in H^1(0,T;L^2(\Omega)).$$

Hence (u,v) *coincides with the strong solution of the Cauchy problem (3.3).*

Remark 4.3 *Weak solution* (u,v) *coincides with the strong solution therefore with the integral solution of the Cauchy problem which satisfies the entropy condition (Theorem 3.4).*

4.2 Continuity

We study the partial differential equation

$$\frac{\partial(u+v)}{\partial t}+\frac{\partial u}{\partial x}=0, \tag{4.2}$$

with the initial condition

$$u(x,0)=u_0(x),\ x\ \in(\alpha,\beta),$$

and boundary condition

$$u(\alpha,t)=0.$$

Here $v=\mathcal{P}_r(\cdot)$ is a classical play operator.
The equation (4.2) can be rewritten as

$$u_t+u_x+\left\{\begin{array}{lll} 0 & \text{if} & u-1<v<u+1 \\ u_t & \text{if} & v=u+1 \text{ decreasing} \\ u_t & \text{if} & v=u-1 \text{ increasing} \end{array}\right\}=0.$$

For an explicit example, let us take the initial condition

$$u(x,0)=u_0(x)\equiv\begin{cases} x & \text{for} \quad -3\le x\le 0 \\ -6-x & \text{for} \quad -6<x<-3 \\ 0 & \text{for} \quad x\le -6. \end{cases}$$

In order to compute the exact solution, we use the method of characteristics. If our original equation is $u_t+\kappa u_x=0$, then the solution subject to the above initial condition would preserve its shape and travel with speed κ and $u(x,t)=u_0(x-\kappa t)$. In our case, with different values of κ, the characteristics must cross. The solution itself remains continuous.

The computations of the solution along with the sketch of the characteristics are given in Figure 1.

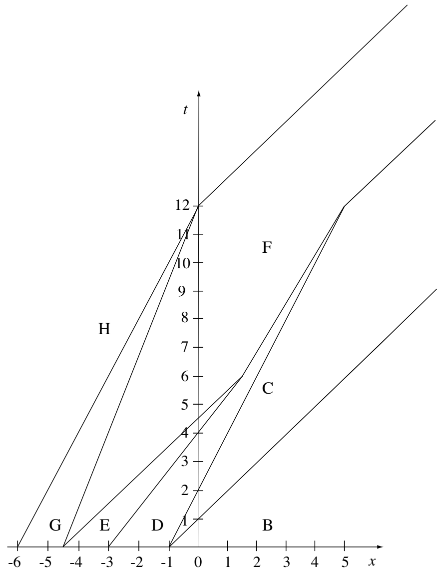

Figure 1: Regions of solution in $u(x,t)$ plane

region	*description*	$u(x,t)$	$v(x,t)$
B	$0 < t < x+1$	$x-t$	0
C	$x+1 < t < 2x+2, x \le 5$ $x+1 < t < x+7, x > 5$	-1	0
D	$2x+2 < t < \frac{4}{3}x+4, x \le \frac{3}{2}$ $2x+2 < t < \frac{12}{7}x + \frac{24}{7}, x > \frac{3}{2}$	$x - \frac{1}{2}t$	$x - \frac{1}{2}t + 1$
E	$\frac{4}{3}x+4 < t < \frac{9}{2}+x$	$-6-x+t$	$-1+\frac{1}{3}x$
F	$x+\frac{9}{2} < t < \frac{8}{3}x+12, x \le 0$ $x+\frac{9}{2} < t < x+12, 0 < x \le \frac{3}{2}$ $\frac{12}{7}x+\frac{24}{7} < t < x+12, \frac{3}{2} < x \le 5$ $x+7 < t < x+12, x > 5$	$\frac{1}{5}(t-x-12)$	$-1+\frac{1}{3}x$ $-1+\frac{1}{3}x$ $\frac{x}{7}-\frac{5}{7}$ 0
G	$\frac{8}{3}x+12 < t < 2x+12$	$-6-x+\frac{1}{2}t$	$-7-x+\frac{1}{2}t$
H	$2x+12 < t, x \le 0$ $x+12 < t, x > 0$	0	$-1, x \le 0$ $\frac{x}{3}-1, 0 < x \le \frac{3}{2}$ $\frac{x}{7}-\frac{5}{7}, \frac{3}{2} < x \le 5$ $0, x > 5$

4.3 Discontinuity

Now we consider a special example of a generalized play operator. The left hysteresis boundary curve is given by a function

$$\gamma_l(u) = \begin{cases} u+1 & \text{if} \quad \text{-}2 \le u \le 0, \\ 1 & \text{if} \quad u \ge 0, \\ -1 & \text{if} \quad u \le -2, \end{cases}$$

and the right boundary curve is given by

$$\gamma_r(u) = \begin{cases} u-1 & \text{if} \quad 0 \le u \le 2, \\ 1 & \text{if} \quad u \ge 2, \\ -1 & \text{if} \quad u \le 0. \end{cases}$$

Remark 4.4 *Notice that this operator does not satisfy the hypothesis of convexity of hysteresis loops.*

The initial condition is for simplicity:

$$u(x,0) = u_0(x) = x \text{ and } \quad v(x,0) = x-1, \;\; x \;\in (\alpha,\beta).$$

The equation (4.2) can be rewritten for this operator as

$$u_t + u_x + \left\{ \begin{array}{ccl} 0 & \text{if} & u-1 < v < u+1 \\ u_t & \text{if} & v = u+1 \text{ decreasing} \\ u_t & \text{if} & v = u-1 \text{ increasing} \\ 0 & \text{if} & v = 1 \\ 0 & \text{if} & v = -1 \end{array} \right\} = 0.$$

The initial condition is increasing for $x \in (-\infty, \infty)$.

A: We are here inside the hysteresis loop. This means $v = 0$, $v_t = 0$ and we have the equation $u_t + u_x = 0$, i.e., $\kappa = 1$ in the above computations. Therefore $t = x + k$ are characteristics and the solution is constant on them. The solution is determined by the initial condition $u(x,t) = u_0(x-t) = x-t$. This will be our solution until $-1 < u < 1$, because then we hit the right or left hysteresis boundary curve and therefore the equation will be changed. Thus $x - 1 < t < x + 1$.
The characteristics are $t = x + k$.

B: The same situation as above, but now we hit the left boundary curve of the play operator and stay there ($u = -1, v = 0$). We are above the line $t = x + 1$ and the solution is determined by the continuity of the solution.
The boundaries are determined by $x + 1 < t$.
In our cases A, B κ is equal to one. v remains constant. In these cases the play operator does not play any role yet.

C: Now we start considering the play operator. This means $v = u+1$, $v_t = u_t$ and we have the equation $u_t + \frac{1}{2}u_x = 0$, i.e., $\kappa = \frac{1}{2}$ in the above computations. Therefore $t = 2x + k$ are characteristics. The solution is determined by the initial condition: $u(x,t) = x - \frac{1}{2}t$ and it must hold $u < -1$. So $x - \frac{1}{2}t < -1 \Rightarrow 2x + 2 < t$. We move on the left hysteresis boundary curve of the play operator and so $v(x,t) = u(x,t) + 1 = x - \frac{1}{2}t + 1$. But v can be maximally equal to -1. So if we set $-1 = x - \frac{1}{2}t + 1$, then $t = 4 + 2x$ is the time when v reaches the value -1.
We firstly assume the equation with $\kappa = \frac{1}{2}$ so $u = x - \frac{1}{2}t$. Secondly, we consider the equation with $\kappa = 1$. So we have $u = x - t$, characteristics are $t = x + k$. For $x \in (-\infty, \infty)$ some values of $t = 2x + 4$ belong to interval $[0, \infty)$. If we try to find a line where both solutions coincide we find out $x = 0$, i.e., such a line does not exist. Thus $v(x,t)$ is equal to -1 in this interval. When we sketch the characteristics of our two equations ($t = 2x + k, t = x + k$), we find out that the second ones spread higher values of solution than the first ones and that they cross. Thus the solution must be discontinuous (see Figure 2).

The consequence of the nonconvexity of this type of the play operator is its discontinuity. So the convexity of hysteresis loop is broken down and shock arises. Hence it is necessary to presume convex hysteresis model to get a continuous solution.

5 Asymptotic behaviour

In the first subsection we recall results about the asymptotic behaviour of the equation

$$u_t + \mathcal{A}u \ni 0, \quad u(0) = u_0, \tag{5.1}$$

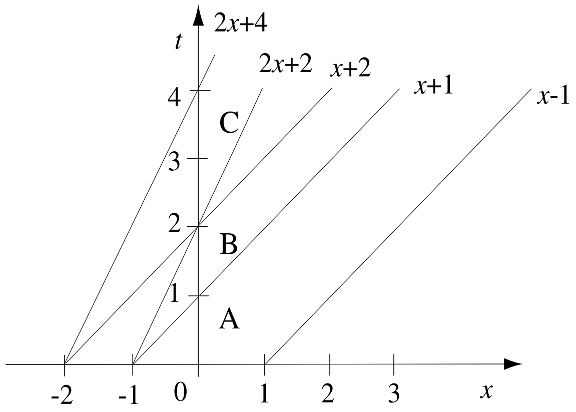

Figure 2: Characteristics intersect

where let $\mathcal{A}$ be an m- and T-accretive operator in $L^1(\Omega)$. In the second subsection we extend these results to equation (2.1) with hysteresis.

5.1 The sub/supersolution method

In this section we recall results about the asymptotic behaviour of the equation

$$u_t + \mathcal{A}u \ni 0, \quad u(0) = u_0, \tag{5.2}$$

where let $\mathcal{A}$ be an m- and T-accretive operator in $L^1(\Omega)$.

Definition 5.1 *A stationary supersolution of (5.2) is defined to be a function $v \in L^1(\Omega)$ satisfying*

$$u_0 \leq v, \qquad a.e. \ on \ \Omega, \tag{5.3}$$

and

$$(I + \lambda\mathcal{A})^{-1}v \leq v, \qquad a.e. \ on \ \Omega, \ \forall\lambda > 0. \tag{5.4}$$

A stationary subsolution of (5.2) is defined in the same way with reversed inequalities.

Remark 5.1 *Note that if $v \in \mathcal{D}(\mathcal{A})$ and if $\mathcal{A}$ is single valued, then (5.3) and (5.4) are equivalent to*

$$\begin{aligned} v &\geq u_0, \quad a.e. \ on \ \Omega, & (5.5)\\ \mathcal{A}v &\geq 0, \quad a.e. \ on \ \Omega. & (5.6) \end{aligned}$$

Also note that if $\mathcal{A}$ is an accretive operator, (5.5) and (5.6) imply (5.3) and (5.4).

Theorem 5.1 *(Wittbold [12]). Let $\mathcal{A}$ be an m- and T-accretive operator in $L^1(\Omega)$, i.e.,*

$$R(I + \lambda\mathcal{A}) = L^1(\Omega), \quad \forall\lambda \geq 0, \tag{5.7}$$

and

$$\|(J_\lambda^{\mathcal{A}}(u) - J_\lambda^{\mathcal{A}}(\tilde{u}))^+\|_1 \leq \|(u - \tilde{u})^+\|_1, \quad \forall\lambda \geq 0, u, \tilde{u} \in L^1(\Omega), \tag{5.8}$$

where

$$J_\lambda^{\mathcal{A}}(u) = (I + \lambda\mathcal{A})^{-1}(u), \text{ and } R \text{ denotes the range.} \tag{5.9}$$

Suppose that $\mathcal{A}^{-1}0 = \{0\}$ and let $u_0 \in \overline{\mathcal{D}(\mathcal{A})}$. Then the following holds: If there exist a stationary subsolution and a stationary supersolution of

$$u_t + \mathcal{A}u \ni 0, \quad u(0) = u_0,$$

in $L^1(\Omega)$, then the solution u of (5.2) is stable, i.e., $\|u(x,t)\|_1 \to 0$ as $t \to \infty$.

The existence of sub/supersolutions alone implies stability in $L^1(\Omega)$. In particular, we do not need the full strenght of accretivity of operators to apply this method, but only that resolvents are order preserving.

5.2 Stability result

In this section we mention two new lemmas and apply the above results to get the asymptotic behaviour of the solution of equation (2.1) with $f \equiv 0$, see [4].

Theorem 5.2 *Suppose all conditions of Theorem 3.2 are satisfied, i.e., the operator $\mathcal{Q} := \mathcal{A} + \Re$ is m- and T-accretive in $L^1(\Omega; \mathbb{R}_1^2)$. Suppose also that $u_0 \in \overline{\mathcal{D}(\mathcal{Q})}$. Then there exists $v_\infty(x)$ dependent on x only, such that for the solution*

$$U = \begin{pmatrix} u \\ v \end{pmatrix}$$

of

$$\frac{\partial U}{\partial t} + \mathcal{Q}(U) = 0, \tag{5.10}$$

$$U(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \tag{5.11}$$

the following holds

$$\begin{aligned} \lim_{t\to\infty} \|u(x,t)\|_1 &= 0, \\ \lim_{t\to\infty} \|v(x,t)\|_1 &= v_\infty(x). \end{aligned}$$

Before proving this theorem we state two lemmas which will be used in the proof of the theorem.

Lemma 5.1 *Let $\mathcal{Q}$ be m- and T-accretive operator in $L^1(\Omega; \mathbb{R}^2_1)$. Then the following holds:*

$$\mathcal{Q}^{-1}0 = \left\{ \begin{pmatrix} 0 \\ v(x) \end{pmatrix}, \text{ such that } \inf \gamma_r(0) \le v(x) \le \sup \gamma_l(0) \right\}.$$

Lemma 5.2 *If $u_0 \in L^\infty(\Omega)$, there exist a stationary supersolution and a stationary subsolution of the equation $\frac{\partial U}{\partial t} + \mathcal{Q}(U) \ni 0$. Moreover, those can be chosen so that they belong to $\overline{\mathcal{D}(\mathcal{Q})}$.*

Proof of the main Theorem. Suppose first that all conditions of Theorem 5.2 are satisfied, i.e., the operator $\mathcal{Q}$ is m- and T-accretive in $L^1(\Omega; \mathbb{R}^2_1)$,

$$R(I + \lambda\mathcal{Q}) = L^1(\Omega; \mathbb{R}^2_1),$$

and

$$\|(J_\lambda^{\mathcal{Q}}(U) - J_\lambda^{\mathcal{Q}}(\tilde{U}))^+\|_1 \le \|(U - \tilde{U})^+\|_1, \tag{5.12}$$

where

$$J_\lambda^{\mathcal{Q}}(U) = (I + \lambda\mathcal{Q})^{-1}(U)$$

and

$$\|U\|_1 = \int\limits_\Omega (|u(x)| + |v(x)|)\mathrm{d}x,$$

denotes the norm in $L^1(\Omega;\mathbb{R}^2_1)$.
Futhermore we suppose that $u_0 \in L^\infty(\Omega)$. We will write

$$J_\lambda^{\mathcal{Q}}(U_1) \leq J_\lambda^{\mathcal{Q}}(U_2),$$

if and only if $J_\lambda^{\mathcal{Q}_1}(U_1) \leq J_\lambda^{\mathcal{Q}_1}(U_2)$ and $J_\lambda^{\mathcal{Q}_2}(U_1) \leq J_\lambda^{\mathcal{Q}_2}(U_2)$ where

$$\begin{aligned} U_1 &= \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \\ \mathcal{Q}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \mathcal{Q}_1(u,v) \\ \mathcal{Q}_2(u,v) \end{pmatrix} = \begin{pmatrix} \xi + \frac{\partial u}{\partial x} \\ -\xi \end{pmatrix}. \end{aligned}$$

We will first prove that the resolvent $J_\lambda^{\mathcal{Q}}$ is order preserving in the previously defined sense: Suppose $U_1 \leq U_2$, i.e., $u_1 \leq u_2$ and $v_1 \leq v_2$, then

$$u_1 - u_2 \leq 0, \quad v_1 - v_2 \leq 0,$$

i.e.,

$$(u_1 - u_2)^+ = 0, \quad (v_1 - v_2)^+ = 0.$$

Therefore by (5.12) we have

$$\|(J_\lambda^{\mathcal{Q}}(U_1) - J_\lambda^{\mathcal{Q}}(U_2))^+\|_1 = \int_\Omega |(J_\lambda^{\mathcal{Q}_1}(U_1) - J_\lambda^{\mathcal{Q}_1}(U_2))^+| \mathrm{d}x$$

$$+ \int_\Omega |(J_\lambda^{\mathcal{Q}_2}(U_1) - J_\lambda^{\mathcal{Q}_2}(U_2))^+| \mathrm{d}x \leq 0,$$

from which it follows that we have $J_\lambda^{\mathcal{Q}_i}(U_1) - J_\lambda^{\mathcal{Q}_i}(U_2) \leq 0$, $i = 1, 2$, a.e. on Ω, i.e., $J_\lambda^{\mathcal{Q}}(U_1) \leq J_\lambda^{\mathcal{Q}}(U_2)$ and $J_\lambda^{\mathcal{Q}}$ is order preserving.

We may consider the solution of (5.10) corresponding to the initial value V, i.e., $S^{\mathcal{Q}}(.)V$, the semigroup motion through V. The resolvent identity,

$$(\tilde{\lambda} I + \mathcal{Q})^{-1} - (\tilde{\mu} I + \mathcal{Q})^{-1} = (\tilde{\mu} - \tilde{\lambda})(\mu I + \mathcal{Q})^{-1}(\tilde{\lambda} I + \mathcal{Q})^{-1},$$

gives us

$$\lambda(I + \lambda\mathcal{Q})^{-1} - \mu(I + \mu\mathcal{Q})^{-1} = \left(\frac{\lambda - \mu}{\lambda\mu}\right)\lambda\mu(I + \lambda\mathcal{Q})^{-1}(I + \mu\mathcal{Q})^{-1},$$

where we used the notation $\lambda = 1/\tilde{\lambda}$, $\mu = 1/\tilde{\mu}$, and the order preservation property that

$$\begin{aligned}(J_\lambda^{\mathcal{Q}})^n V &= J_\lambda^{\mathcal{Q}}((J_\lambda^{\mathcal{Q}})^{n-1} V) = J_\mu^{\mathcal{Q}}\left(\frac{\mu}{\lambda}(J_\lambda^{\mathcal{Q}})^{n-1} V + \frac{\lambda-\mu}{\lambda}(J_\lambda^{\mathcal{Q}})^n V\right)\\ &\geq J_\mu^{\mathcal{Q}}((J_\lambda^{\mathcal{Q}})^n V), \quad a.e. \text{ on } \Omega,\, \forall \mu, \lambda > 0,\, n \in N.\end{aligned}$$

Applying this estimate with $\lambda = t/n$ and passing to the limit as $n \to \infty$ yields

$$S^{\mathcal{Q}}(t)V \geq J_\mu^{\mathcal{Q}} S^{\mathcal{Q}}(t)V, \quad a.e. \text{ on } \Omega, \forall t > 0, \mu > 0.$$

If we iterate this last inequality n times, we obtain for $\mu = s/n$,

$$S^{\mathcal{Q}}(t)V \geq (J_{s/n}^{\mathcal{Q}})^n S^{\mathcal{Q}}(t)V, \quad a.e. \text{ on } \Omega, \forall t, s > 0,$$

and thus, in the limit $(n \to \infty)$,

$$S^{\mathcal{Q}}(t)V \geq S^{\mathcal{Q}}(s)S^{\mathcal{Q}}(t)V = S^{\mathcal{Q}}(t+s)V, \tag{5.13}$$

a.e. on $\Omega, \forall t, s > 0$. Futhermore, because $J_\lambda^{\mathcal{Q}}$ is order preserving,

$$V \geq U_0 \geq 0 \text{ implies } \quad J_\lambda^{\mathcal{Q}} V \geq J_\lambda^{\mathcal{Q}} U_0 \geq J_\lambda^{\mathcal{Q}} 0,$$

and also

$$S^{\mathcal{Q}}(t)V \geq S^{\mathcal{Q}}(t)U_0 \geq S^{\mathcal{Q}}(t)0 = \begin{pmatrix} 0 \\ v_0(x) \end{pmatrix} \geq 0. \tag{5.14}$$

The last estimate together with monotonicity in (5.13) implies that $V_\infty = \|.\|_1 - \lim_{t\to\infty} S^{\mathcal{Q}}(t)V$ exists and $V_\infty \in \mathcal{Q}^{-1}0$. However, as (5.14) gives us,

$$0 \leq S^{\mathcal{Q}}(t)U_0 \leq S^{\mathcal{Q}}(t)V, \quad a.e. \text{ on } \Omega, \forall t > 0,$$

it follows that $\|S^{\mathcal{Q}}(t)U_0\|_1$ exists and $S^{\mathcal{Q}}(t)u \to 0, S^{\mathcal{Q}}(t)v \to v(x)$ as $t \to \infty$, $v_0(x) \leq v(x)$. □

6 Publications concerning the Thesis

[1] P. Kordulová: Quasilinear hyperbolic equations with hysteresis. Journal of Physics: Conference series **55** (2006) 135-143

[2] P. Kordulová: Asymptotic behaviour of a quasilinear hyperbolic equation with hysteresis. Nonlinear Analysis: Real World Application **8** (2007) 1398-1409

[3] P. Kordulová: Continuity of solutions of a quasilinear hyperbolic equation with hysteresis. Accepted in Applications of Mathematics

7 Presentations

[1] Free Boundary Problems, Coimbra, Portugal, June 2005

[2] EQUADIFF 11, Bratislava, July 2005

[3] International Workshop on Multi-rate Processes and Hysteresis, Cork, Ireland, April 2006

[4] Aplimat, Bratislava, February 2007

[5] EQUADIFF 07, Vienna, Austria, August 2007

[6] ISAAC 2007, Ankara, Turkey, August 2007

8 Working visits

[1] WIAS Berlin, Germany, December 2004

[2] University College Cork, Cork, Ireland, October-December 2006, supported by grant of the Moravian-Silesian region

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[2] J. Kopfová: Entropy condition for a quasilinear hyperbolic equation with hysteresis. Differential Integral Equations **18** (2005) 451-467

[3] J. Kopfová: Hysteresis in a first order hyperbolic equation. Dissipative phase transitions, Ser. Adv. Math. Appl. Sci., **71**,

World Sci. Publ., Hackensack, NJ, (2006) 141-150

[4] P. Kordulová: Asymptotic behaviour of a quasilinear hyperbolic equation with hysteresis. Nonlinear Analysis: Real World Application **8** (2007) 1398-1409

[5] P. Kordulová: Continuity of solutions of a quasilinear hyperbolic equation with hysteresis. accepted in Applications of Mathematics

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