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On the worst scenario method and its application to uncertain differential equations

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Contents

1 Abstract

The submitted Thesis is devoted to the worst scenario method and its application to particular problems with uncertain nonlinear differential equations. At first, a theoretical framework for solving a class of worst scenario problems is proposed. The existence of the worst scenario is proved through the convergence of a sequence of approximate worst scenarios. This theoretical framework is applied to problems with quasilinear elliptic equations with uncertain coefficients (the problem for an ordinary and a partial differential equation is solved). Furthermore, it is shown that the Galerkin approximation of the state solution can be calculated by means of the Kachanov method as the limit of a sequence of solutions to linearized problems. On the top of it, some illustrative numerical examples concerning a one-dimensional problem are presented.

2 Introduction

A great many problems in natural, technical, and social sciences can be solved by means of suitable mathematical models. By using such models, we are able to predict results of processes in real world. Nevertheless, modeling of the real world is encumbered with various sorts of uncertainty. Since the input data of mathematical models is uncertain, the output values are also encumbered by uncertainty. It is our goal to evaluate the uncertainty of output data if the uncertainty of input data is somehow specified.

In this work, we are concerned especially with models described by differential equations with boundary conditions. For instance, let us consider the following boundary value problem: Find an unknown function u such that

$$-(a(u)u')' = f \quad \text{in } (0,1), \tag{2.1}$$

$$u(0) = u_1, \qquad u(1) = u_2. \tag{2.2}$$

This example represents the mathematical model of one-dimensional steady heat conduction. The right-hand side function f characterizes internal heat sources, the coefficient a is the heat conductivity and depends on the temperature u, the values u_1 and u_2 are given boundary temperatures.

In this steady heat conduction model, we can be interested, for instance, in the temperature at a selected point of the heated body. The problem (2.1)–(2.2), and consequently the temperature at any point, depend on the coefficient a. However, coefficients are obtained through experimental measurements and are not known exactly. It is not uncommon that a set of inputs is given. Consequently, the a-dependent temperature must be considered uncertain. For a more detailed mathematical treatment of this problem, see [1].

It is possible to consider the situation above more generally. The problem (2.1)–(2.2) is a concrete example of so called state problem, the function u is called a state solution. So, consider a state problem (represented by a boundary value problem, for instance) whose input data is uncertain. This uncertainty is represented by $\mathcal{U}_{\mathrm{ad}}$, a given set of admissible input data. Since the state solution u depends on the input parameter $a \in \mathcal{U}_{\mathrm{ad}}$, we obtain a set of state solutions. As a rule, we are concentred with a real-valued quantity of interest related to the state solution and represented by a criterion functional $\Phi = \Phi(a, u(a))$, generally directly dependent on a. Due to the uncertainty of the state solution, we obtain a set of values of the criterion functional.

In practice, there exists a number of approaches to treatments of uncertainty in mathematical models. The choice of an acceptable approach depends largely on the amount of available information about the input data. If a probability characterization of input values is available, then stochastic methods can be applied at least in the form of the popular Monte Carlo method. If, however, the uncertainty in inputs cannot be described in terms of probability, other approaches, for instance fuzzy sets or the worst scenario method, can be applied. In some cases, it is suitable and efficient to combine various approaches to uncertainty. More detailed information can be found, e.g., in [13].

The stochastic methods as well as the fuzzy set approach assume certain additional information related to the input data of a mathematical model. Nevertheless, such information does not have to be always available, only the set of admissible input data can be known, and we wish to derive the corresponding set of outputs.

In engineering applications, mainly large values of the quantity of interest (e.g. temperature at a selected point of a heated body, or local stress at a point of a loaded body) are important . To illustrate this, we return to problem (2.1)–(2.2). It can be requested that, independently of $a \in \mathcal{U}_{\rm ad}$, the temperature at the selected point musn't exceed certain given value. Therefore, we search for an input parameter $a^0 \in \mathcal{U}_{\rm ad}$ such that the quantity of interest is maximal, i.e. we search for the worst (case) scenario responsible for the highest temperature at the observed point. Generally and more precisely, if we consider the criterion functional Φ mentioned above and if we set $\Psi(a) = \Phi(a, u(a))$, the goal is to find a parameter $a^0 \in \mathcal{U}_{\rm ad}$ such that the value $\Psi(a^0)$ is maximum. More detailed mathematical treatment of the worst scenario method is included in Section 3, see also [1], [2], [8], [9], [10], [11], [12], [13]. In practice, we are usually interested rather in the value $\Psi(a^0)$ than the worst scenario a^0.

We observe that the worst scenario method provides rather pessimistic conclusions. The realization of the worst scenario might be rather rare. It is suitable to use the worst scenario method only in such situations where, except for $\mathcal{U}_{\mathrm{ad}}$, no additional information is available. On the other hand, the worst scenario method appears, for instance, in the output uncertainty analysis of fuzzy set based models, see, e.g., [3] and [6].

3 Main results

The goal of this section is to explicate the theoretical framework of the worst scenario method and to sumarize new results that appeared on this subject in [8] and [9].

3.1 General theoretical framework of the worst scenario method

The general abstract scheme of the worst scenario method has been proposed by I. Hlaváček in [11, 13]. He considers an abstract state problem $\mathcal{P}(A, u)$, where A denotes input data and u represents a state variable. The state problem $\mathcal{P}(A, u)$ can stand for a differential equation, an integral equation, or a system of linear equations, for instance. In this work, we concentrate our attention to problems described by differential equations with uncertain coefficients. In the following part we introduce a general mathematical framework of the worst scenario method.

At first, we define the worst scenario problem. Let V be a real, separable, and reflexive Banach space and let V^* be its dual space. We assume that the state problem $\mathcal{P}(A, u)$ takes the form of an operator equation

$$\mathcal{A}u = b, \qquad u \in V, \tag{3.1}$$

where $\mathcal{A} : V \to V^*$, $b \in V^*$. The operator $\mathcal{A}$ depends on an input parameter A that belongs to an admissible set $\mathcal{U}_{\mathrm{ad}} \subset U$, where U is a Banach space. Consequently, the solution u of equation (3.1) depends on the parameter A, so that $u \equiv u(A)$. Furthermore, the state solution $u(A)$ is evaluated by a criterion functional $\Phi : \mathcal{U}_{\mathrm{ad}} \times V \to \mathbb{R}$ that can also directly evaluate the value of the input parameter A. The goal is to solve the following worst scenario problem: Find $A^0 \in \mathcal{U}_{\mathrm{ad}}$ such that

$$A^0 = \arg\max_{A \in \mathcal{U}_{\mathrm{ad}}} \Phi(A, u(A)). \tag{3.2}$$

The aim of the theoretical analysis is to prove the existence of the solution to problem (3.2) as well as to suggest a way to find, at least approximately, the worst scenario A^0.

As shown in [13, Theorem 3.1], the problem (3.2) has a solution under the following assumptions:

(a) the set $\mathcal{U}_{\mathrm{ad}}$ is compact in U;

(b) a unique solution $u(A)$ of the state problem (3.1) exists for any parameter $A \in \mathcal{U}_{\mathrm{ad}}$;

(c) if $A_n \to A$ in U, then $u(A_n) \to u(A)$ strongly or at least weakly in V;

(d) if $A_n \in \mathcal{U}_{\mathrm{ad}}$, $A_n \to A$ in U and $v_n \to v$ strongly or at least weakly in V, then

$$\limsup_{n\to\infty} \Phi(A_n, v_n) \le \Phi(A, v).$$

In a concrete practical problem, it can be difficult to show that this assumptions ensuring the solvability of the worst scenario problem (3.2) are fulfilled. In problems occurring in [8] and [9], see also Section 3.2, a difficulty arrises with the assumption (c) above (more detailed information can be found in [8]). For that reason, another approach is applied in [8] and the existence of the solution to problem (3.2) is proved via the convergence of the solutions to approximate worst scenario problems. The approximation is based on replacing the admissible set $\mathcal{U}_{\mathrm{ad}}$ by its finite-dimensional approximation $\mathcal{U}_{\mathrm{ad}}^M \subset \mathcal{U}_{\mathrm{ad}}$, and the space V by its finite-dimensional subspace V_h. The symbols h and M stand for the relevant discretization parameters. If we replace the space V in (3.1) by its subspace V_h, we obtain Galerkin approximation $u_h(A) \in V_h$ of the state solution $u(A)$. If we consider an approximation of (2.1)–(2.2), then the elements of $\mathcal{U}_{\mathrm{ad}}$ (some real, Lipschitz continuous functions) are substituted by the elements of $\mathcal{U}_{\mathrm{ad}}^M$ that can be continuous, piecewise linear functions, defined by values at nodal points x_i, $i \in \{1, \dots, M\}$. For more detailed description of the sets $\mathcal{U}_{\mathrm{ad}}$ and $\mathcal{U}_{\mathrm{ad}}^M$ see the following section. The finite-dimensional space V_h usually is a finite element space. The approximate worst scenario problem, dependent on the choice of $\mathcal{U}_{\mathrm{ad}}^M$ and V_h, is defined in the following way: Find $A_h^{M0} \in \mathcal{U}_{\mathrm{ad}}^M$ such that

$$A_h^{M0} = \arg \max_{A^M \in \mathcal{U}_{\mathrm{ad}}^M} \Phi(A^M, u_h(A^M)). \tag{3.3}$$

This approach also provides a way to calculate, at least approximately, the worst scenario A^0.

To prove that the solution of problem (3.2) can be obtained as the limit of a sequence of solutions to the problems (3.3), we establish the following assumptions:

(i) the sets $\mathcal{U}_{\text{ad}}$ and $\mathcal{U}_{\text{ad}}^M$ are compact in U;

(ii) for any parameter $A \in \mathcal{U}_{\text{ad}}$ there exists a unique state solution $u(A)$ of equation (3.1), and furthermore, a unique Galerkin approximation $u_h(A) \in V_h$ of the state solution $u(A)$;

(iii) the solution of the approximated state problem (that is, a finite-dimensional parallel to (3.1)) depends continuously on the input parameter, i.e., if $A_n \in \mathcal{U}_{\text{ad}}$ and $A_n \to A$ in U as $n \to \infty$, then $u_h(A_n) \to u_h(A)$ in V_h;

(iv) if $\{V_{h_n}\}$, where $h_n \to 0$ as $n \to \infty$, is a sequence of finite-dimensional subspaces of V, and if $A_n \in \mathcal{U}_{\text{ad}}$, $A_n \to A$ in U as $n \to \infty$, then $u_{h_n}(A_n) \to u(A)$ in V;

(v) it is possible to approximate any $A \in \mathcal{U}_{\text{ad}}$ with an arbitrary accuracy by an element $A^M \in \mathcal{U}_{\text{ad}}^M$ if M is a sufficiently large number, i.e. there exists a sequence $\{A^M\}$, $A^M \in \mathcal{U}_{\text{ad}}^M$, such that $A^M \to A$ in U as $M \to \infty$;

(vi) the criterion functional is continuous, i.e. if $A_n \in \mathcal{U}_{\text{ad}}$, $A_n \to A$ in U and $v_n \to v$ in V as $n \to \infty$, then

$$\Phi(A_n, v_n) \to \Phi(A, v).$$

We note that, except for (iv), these assumptions can also be found in [13, Chapter II].

It is not difficult to show that under assumptions (i) – (vi) the approximate worst scenario problem (3.3) has a solution. We can proceed as follows: For each $A \in \mathcal{U}_{\text{ad}}^M$, we define

$$\Psi_h(A) := \Phi(A, u_h(A)).$$

Let $A_n \in \mathcal{U}_{\text{ad}}^M$ and $A_n \to A$ in U as $n \to \infty$. It follows from (iii) that $u_h(A_n) \to u_h(A)$ in V_h. By virtue of assumption (vi) we get

$$\Phi(A_n, u_h(A_n)) \to \Phi(A, u_h(A)),$$

which means that

$$\Psi_h(A_n) \to \Psi(A).$$

Thus, the functional Ψ is continuous on $\mathcal{U}_{\text{ad}}^M$. Since, according to assumption (i), the set $\mathcal{U}_{\text{ad}}^M$ is compact in U, the functional Ψ_h has a maximum in $\mathcal{U}_{\text{ad}}^M$, i.e., an element $A_h^{M0} \in \mathcal{U}_{\text{ad}}^M$ solves problem (3.3). $\square$

Now, we are prepared to present the main theoretical result concerning the existence of a solution to problem (3.2) and inspired by [13, Theorem 3.4]. If the assumptions (i) – (vi) are fulfilled, then, according to [8, Theorem 3.1], there exists a sequence $\{A_{h_k}^{M_k 0}\}$ of solutions to the approximate worst scenario problems (3.3), $A_{h_k}^{M_k 0} \in \mathcal{U}_{\text{ad}}^{M_k}$, such that $h_k \to 0$ and $M_k \to \infty$ as $k \to \infty$, and

$$A_{h_k}^{M_k 0} \to A^0 \qquad \text{in } U, \tag{3.4}$$

$$u_{h_k}(A_{h_k}^{M_k 0}) \to u(A^0) \qquad \text{in } V, \tag{3.5}$$

$$\Phi\big(A_{h_k}^{M_k 0}, u_{h_k}(A_{h_k}^{M_k 0})\big) \to \Phi\big(A^0, u(A^0)\big) \tag{3.6}$$

as $k \to \infty$, where $A^0 \in \mathcal{U}_{\text{ad}}$ solves problem (3.2) and $u(A^0)$ is the corresponding solution to problem (3.1).

Remark We observe that it is possible to modify some of the asumptions (i) – (vi) above. To be specific, if we replace the strong convergence $u_{h_n}(A_n) \to u(A)$ in (iv) and $v_n \to v$ in (vi) by the weak convergence, then the conclusions (3.4) – (3.6) will be valid if we replace the strong convergence in (3.5) by the weak convergence. This latter modification of [8, Theorem 3.1] is applied to problem with a partial differential equation examined in [9], see also the folowing section.

3.2 Application to quasilinear elliptic differential equations with uncertain coefficients

In this part, we apply the theoretical framework from Section 3.1 to problems with quasilinear elliptic differential equations with uncertain coefficients. We suppose that the coefficients depend on the squared gradient (derivative) of the state solution. Such equations can describe some electromagnetic phenomena, fluid flow phenomena, and the elastoplastic deformation of a body, see [15, page 212]. At first, we examine a problem with a partial differential equation, after that we will give a result related to a problem with an ordinary differential equation.

Consider the following state problem (the weak formulation of a boundary value problem, see [9]): Find $u \in H_0^1(\Omega)$ such that

$$\iint_\Omega A(|\nabla u|^2)\nabla u \cdot \nabla v \,\mathrm{d}x\mathrm{d}y = \iint_\Omega fv \,\mathrm{d}x\mathrm{d}y \qquad \forall v \in H_0^1(\Omega), \tag{3.7}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open domain with a polygonal boundary, $H_0^1(\Omega)$ is the usual Sobolev space on Ω with vanishing traces on $\partial\Omega$, $A = (a_{ij})_{i,j=1}^2$ is a diagonal matrix, a_{ii}, $i \in \{1, 2\}$, are Lipschitz continuous functions on $\mathbb{R}_0^+$ (nonnegative real numbers), and $f \in L^2(\Omega)$.

Now, we define the admissible set in more detail. To be able to ensure the validity of assumptions mentioned in Section 3.1, we have to select suitable admissible coefficients. We will consider positive, increasing, and bounded Lipschitz continuous functions defined on $\mathbb{R}_0^+$. More precisely, the admissible set $\mathcal{U}_{\text{ad}}$ is defined in the following way: Let $\mathcal{U}_{\text{ad}}^i$, $i \in \{1,2\}$, be a set of Lipschitz continuous functions a_{ii} defined on $\mathbb{R}_0^+$ and such that

$$
\begin{aligned}
0 < c_{\min,i} \leq \frac{\mathrm{d}a_{ii}}{\mathrm{d}x} &\leq C_{\mathrm{L},i} \qquad \text{a.e. in } [0, x_C], && (3.8)\\
a_{ii}(x) &= a_{ii}(x_{\mathrm{C}}) \qquad \text{for} \quad x \geq x_{\mathrm{C}}, && (3.9)\\
0 < a_{\min,i} \leq a_{ii}(x) &\leq a_{\max,i} \qquad \forall x \in \mathbb{R}_0^+, && (3.10)
\end{aligned}
$$

where $C_{\mathrm{L},i},\ c_{\min,i},\ a_{\min,i},\ a_{\max,i},\ x_{\mathrm{C}}$ are positive constants. The admissible set $\mathcal{U}_{\text{ad}}$ is defined as the Cartesian product $\mathcal{U}_{\text{ad}}^1 \times \mathcal{U}_{\text{ad}}^2$. It is obvious that the elements of $\mathcal{U}_{\text{ad}}$ can be represented by diagonal matrices of functions. We observe that $\mathcal{U}_{\text{ad}}$ is a subset of the Cartesian product $U \times U$, where U is the Banach space of functions continuous on $\mathbb{R}_0^+$ and constant for $x \geq x_{\mathrm{C}}$, for more details see [9].

The operator equation (3.1) arrises from (3.7) if we set $V := H_0^1(\Omega)$ and define $\mathcal{A} : V \to V^*$ and $b \in V^*$ by

$$
\langle \mathcal{A}u, v \rangle := \iint_\Omega [a_{11}(|\nabla u|^2) u_x v_x + a_{22}(|\nabla u|^2) u_y v_y] \mathrm{d}x \mathrm{d}y \qquad (3.11)
$$

and

$$
\langle b, v \rangle := \iint_\Omega f v \, \mathrm{d}x \mathrm{d}y,
$$

where $u, v \in V$ and where u_x, v_x, u_y, v_y denote the partial derivatives of u and v.

As mentioned earlier, the existence of the solution to problem (3.2) is proved via the convergence of the approximate worst scenarios. Therefore, we define the admissible set $\mathcal{U}_{\text{ad}}^M$ and a finite-dimensional subspace V_h of V. Let x_j, $j \in \{1, \ldots, M\}$, be equally spaced points in $[0, x_C]$, $x_1 = 0$ and $x_M = x_C$. For $i \in \{1, 2\}$, we define the set $\mathcal{U}_{\text{ad}}^{M,i} \subset \mathcal{U}_{\text{ad}}^{i}$ of functions $a \in \mathcal{U}_{\text{ad}}^{i}$ such that $a|_{[x_j, x_{j+1}]} \in P_1([x_j, x_{j+1}])$, $j \in \{1, \ldots, M-1\}$, where $P_1([x_j, x_{j+1}])$ denotes the linear polynomials on the interval $[x_j, x_{j+1}]$. The admissible set $\mathcal{U}^M$ is defined as the Cartesian product $\mathcal{U}_{\text{ad}}^{M,1} \times \mathcal{U}_{\text{ad}}^{M,2}$. To define the space V_h, we introduce a triangulation $\mathcal{T}_h = \{T_1, \ldots, T_N\}$ of Ω. Let V_h be the space of functions $v_h \in V$, continuous on $\overline{\Omega}$ and such that $v_h|_{T_j}$ is a linear polynomial on the element T_j, $j \in \{1, \ldots, N\}$. We assume that the diameter of any triangle T_j, $j \in \{1, \ldots, N\}$, does not exceed h.

On the condition that the assumption (vi) from the previous section is fulfilled (the concrete criterion functional is appropriately selected), to be able to apply [8, Theorem 3.1], we have to verify assumptions (i) – (v), mentioned in Section 3.1.

By the Arzelà–Ascoli theorem, see [21, page 35], the sets $\mathcal{U}_{\text{ad}}^{i}$, $\mathcal{U}_{\text{ad}}^{M,i}$, $i \in \{1, 2\}$, are compact in U, and, consequently, the admissible sets $\mathcal{U}_{\text{ad}}$ and $\mathcal{U}_{\text{ad}}^{M}$ are compact in $U \times U$. It means that the asssumption (i) is fulfilled.

The existence of the solution of problem (3.7) is guaranteed by [22, Theorem 2.K.]. It is sufficient to verify that the operator $\mathcal{A}$ defined by (3.11) is monotone, continuous, and coercive on V. The proof of the continuity and the coercivity of $\mathcal{A}$ is not too difficult, see [9, Lemma 2.2 and Lemma 2.3]. The proof of the monotonicity of the operator $\mathcal{A}$ is a more challenging problem, see [9, Lemma 2.1]. To ensure the monotonicity of $\mathcal{A}$, we add the requierement (see [9] for the details)

$$4x_{\text{C}} C_{\text{L}}^{\max} \leq a_{\min} \tag{3.12}$$

to the admissible set $\mathcal{U}_{\text{ad}}$, where $a_{\min} := \min_{1 \leq i \leq 2} \{a_{\min,i}\}$, $C_{\text{L}}^{\max} :=$

$\max\limits_{1\leq i\leq 2}\{C_{\mathrm{L},i}\}$. Therefore, if the condition (3.12) is fulfilled, the state problem (3.7) has a solution. In addition, according to [9, Lemma 2.4], the operator $\mathcal{A}$ is strictly monotone and by virtue of [22, page 93, Corollary 1] the uniqueness of the solution to the problem (3.7) is guaranteed. The existence of the unique Galerkin approximation can be proved similarly, see [9, Theorem 2.2]. So that, the assumption (ii) is fulfilled.

The condition (3.12) ensuring the existence and uniqueness of the state solution and its Galerkin approximation is also used to verify the assumptions (iii) and (iv).

The assumption (iii) is fulfilled, see [9, Theorem 2.4].

To verify the assumption (iv), we have to introduce an appropriate sequence of finite-dimensional subspaces of V. To this end, let $\{\mathcal{T}_h\}$, $h \to 0$, be a regular family of triangulations of Ω, and $\{\mathcal{T}_{h_n}\} \subset \{\mathcal{T}_h\}$, $h_n \to 0$ as $n \to \infty$, be a sequence of these triangulations. Let $\{V_{h_n}\}$ be the corresponding sequence of the finite-dimenional spaces defined above. Then, according to [8, Lemma 4.4] and [9, Theorem 2.5], the assumption (iv) is fulfilled, if we replace the strong convergence $u_{h_n}(A_n) \to u(A)$ by the weak convergence.

By [9, Lemma 2.5], the assumption (v) is also fulfilled.

Thus, under the condition (3.12) and Remark (see Section 3.1), the worst scenario problem (3.2) with the operator state equation given by (3.7) has a solution $A^0 \in \mathcal{U}_{\mathrm{ad}}$. In addition, according to [8, Theorem 3.1], this solution is the limit of a sequence of solutions to approximate worst scenario problems (3.3). As will be shown in the folloving section, to solve the problem (3.3) with given $\mathcal{U}_{\mathrm{ad}}^M$

and V_h requires, among others, to find the Galerkin approximation $u_h(A^M) \in V_h$, where $A^M \in \mathcal{U}_{\text{ad}}^M$, to the solution of the nonlinear problem (3.7). This Galerkin approximation can be determined by means of the Kachanov method as the limit of a sequence of the solutions to linearized problems, see [9, Theorem 2.3]. To this end, we define the following finite dimensional parallel to (3.7):
Find $u_h \in V_h$ such that

$$\iint_\Omega A(|\nabla u_h|^2)\nabla u_h \cdot \nabla v \, \mathrm{d}x\mathrm{d}y = \iint_\Omega f v \, \mathrm{d}x\mathrm{d}y \qquad \forall v \in V_h. \tag{3.13}$$

We show that under certain condition (see below) the Kachanov method applied to problem (3.13) converges for any $A \in \mathcal{U}_{\text{ad}}$. The folloving part summarizes the results concerning the application of the Kachanov method to problem (3.13).

To be able to formulate a sufficient condition for the convergence of the Kachanov method, see (3.14) below, let us introduce some necessary constants.

It follows from the equivalence of the norm $\|\cdot\|_{H^1(\Omega)}$ and the semi-norm $|\cdot|_{H^1(\Omega)}$ in $H_0^1(\Omega)$ that

$$|v|_{H^1(\Omega)} \geq C_1 \|v\|_{H^1(\Omega)} \qquad \forall v \in V,$$

where $C_1 > 0$.

In the following, we will use the equivalence of norms on the finite-dimensional space V_h. To this end, we fix a triangulation $\mathcal{T}_h$ and the corresponding space V_h. Let $V_{h,c}$ be the space of functions on Ω that are constant on each triangle $T_j \in \mathcal{T}_h$, $j \in \{1, \ldots, N\}$. It follows from the equivalence of norms on $V_{h,c}$ that

$$\|v\|_{L^\infty(\Omega)} \leq C_2 \|v\|_{L^2(\Omega)} \qquad \forall v \in V_{h,c}\,,$$

where $C_2 > 0$.

It holds

$$\|u_x - v_x\|_{L^2(\Omega)} + \|u_y - v_y\|_{L^2(\Omega)} \leq C_3 \, \|u - v\|_V \qquad \forall u, v \in V_h \,,$$

where $C_3 > 0$.

Now, assume that

$$\frac{2C_2C_3C_L^{\max}\|f\|_{L^2(\Omega)}\sqrt{x_{\rm C}}}{C_1^4 a_{\min}^2} < 1. \tag{3.14}$$

Then, according to [9, Theorem 2.3], the Galerkin approximation $u_h \equiv u_h(A) \in V_h$ of the solution to problem (3.7) can be calculated by means of the Kachanov method:

Let $u^0 \in V_h$ be arbitrary. If $u^k \in V_h$ is known, the following iteration $u^{k+1} \in V_h$ is determined so that

$$\iint_\Omega [a_{11}(|\nabla u^k|^2)u_x^{k+1}v_x + a_{22}(|\nabla u^k|^2)u_y^{k+1}v_y]\mathrm{d}x\mathrm{d}y = \iint_\Omega fv \, \mathrm{d}x\mathrm{d}y$$

for all $v \in V_h$. Then

$$\|u_h - u^k\|_V \to 0 \qquad \text{as} \qquad k \to \infty.$$

In the end of this section, we turn our attention to the state problem examined in [8] and motivated by a boundary value problem with ordinary differentially equation: Find $u \in H_0^1(0,1)$ such that

$$\int_0^1 a(u'^2)u'v'\mathrm{d}x = \int_0^1 fv \, \mathrm{d}x \qquad \forall v \in H_0^1(0,1), \tag{3.15}$$

where the function $a \in \mathcal{U}_{\rm ad}$ is an admissible coefficient.

The admissible set $\mathcal{U}_{\rm ad}$ can be defined in the same way as $\mathcal{U}_{\rm ad}^i$ except for (3.8) that can be weakened to

$$0 \leq \frac{\mathrm{d}a}{\mathrm{d}x} \leq C_{\rm L} \qquad \text{a.e. in } [0, x_C],$$

where C_L is a positive constant. The set $\mathcal{U}_{\rm ad}^M$ is defined analogously as $\mathcal{U}_{\rm ad}^{M,i}$. For more detailed information see [8]. By introducing $x_0 = 0 < x_1 < \ldots < x_{N+1} = 1$, we define $V_h \subset H_0^1(0,1)$, the space of functions continuous on $[a,b]$, linear on the interval $[x_j, x_{j+1}]$, $j = 0, \ldots, N$, and with vanishing value at 0 and 1.

As well as in the two-dimensional case, the solution of the problem (3.2) can be obtained as the limit of a sequence of the solutions to approximate worst scenario problems (3.3). However, we obtained rather stronger results in one-dimensional case.

It is possible to verify the assumptions (i) – (vi) analogously. In this case, in contrast to the 2D–problem, a condition like (3.12) can be omitted. In addition, we are able to prove that the corresponding nonlinear operator defined by the left hand side of (3.15) is even strongly monotone, see [8, Lemma 4.2]. This implies that we can prove the strong convergence $u_{h_n}(A_n) \to u(A)$ in the assumption (iv), see [8, Theorem 4.4], and, consequently, select the criterion functional satisfying the assumption (vi) with the strong convergence $v_n \to v$. It makes possible to use a larger class of criterions. Furthermore, we obtain the strong convergence in (3.5).

4 Numerical examples

In this section, we will show a procedure to find, at least approximately, a solution of problem (3.3). For computational simplicity reasons, we confine ourselves to the one-dimensional problem examined in [8]. We will consider examples of equations with symmetric and non-symmetric right-hand side. The computations were performed in MATLAB.

Let $\mathcal{U}_{\rm ad}^M$ be the finite-dimensional admissible set and V_h the finite-dimensional subspace of Sobolev space $H_0^1(0,1)$ concerning the one-dimensional problem defined at the end of Section 3.2. At first, we set $\Psi(a) = \Phi(a, u(a))$, so that we will examine a-dependent functional Ψ defined on $\mathcal{U}_{\mathrm{ad}}^M$. Furthermore, the finite-dimensional admissible set $\mathcal{U}_{\mathrm{ad}}^M$ can be identified with a compact subset $\widehat{\mathcal{U}}_{\mathrm{ad}}^M$ of the Euclidean space $\mathbb{R}^M$, if we define

$$\begin{aligned}\widehat{\mathcal{U}}_{\mathrm{ad}}^M = \{\alpha \in \mathbb{R}^M : \exists a \in \mathcal{U}_{\mathrm{ad}}^M \quad \alpha &= (\alpha_1, \ldots, \alpha_M)\\ &= (a(x_1), \ldots, a(x_M))\},\end{aligned}$$

see also [1]. In this sense, the functional Ψ is, as a matter of fact, a real function $\widehat{\Psi} = \widehat{\Psi}(\alpha)$, where $\alpha = (\alpha_1, \ldots, \alpha_M) \in \widehat{\mathcal{U}}_{\mathrm{ad}}^M$. To obtain the value of function $\widehat{\Psi}$ at any point $\alpha \in \widehat{\mathcal{U}}_{\mathrm{ad}}^M$, it is necessary to solve the following nonlinear problem (a finite-dimensional analogy to (3.15)): Find $u_h \in V_h$ such that

$$\int_0^1 a(u_h'^2) u_h' v' \mathrm{d}x = \int_0^1 f v \, \mathrm{d}x \qquad \forall v \in V_h, \tag{4.1}$$

where $a \in \mathcal{U}_{\mathrm{ad}}^M,\ a(x_i) = \alpha_i,\ i = 1, \ldots, M$. The solution of the state problem (4.1) is obtained by using the Kachanov method (see previous section). Subsequently, the criterion functional Φ is applied. Summarily, we solve the following global optimization problem arising from (3.3): Find $\alpha^0 \in \widehat{\mathcal{U}}_{\mathrm{ad}}^M$ such that

$$\alpha^0 = \arg \max_{\alpha \in \widehat{\mathcal{U}}_{\mathrm{ad}}^M} \widehat{\Psi}(\alpha).$$

To find the element α^0 at least approximately, we use the Nelder-Mead simplex method, see, e.g., [16]. This method is implemented by the standard MATLAB function *fminsearch*. By means of this algorithm, we can find a local extreme of a real function defined on the entire space $\mathbb{R}^M$. The Nelder-Mead algorithm starts from an arbitrarily selected initial $(M+1)$-simplex. It is sufficient to enter an initial point, the *fminsearch* algorithm determines the remaining vertices of the initial simplex. The values at vertices are evaluated and, subsequently, the simplex is transformed. If this procedure is repeated, a non-increasing real sequence of function values is generated. This process continues until a termination criterion is met.

We have to solve a global optimization problem on the bounded set $\widehat{\mathcal{U}}_{\rm ad}^M \subset \mathbb{R}^M$. To solve this problem by the unconstrained optimization routine *fminsearch*, we establish a transformation $T : \mathbb{R}^M \to \widehat{\mathcal{U}}_{\rm ad}^M$ and search for the maximum of the composite function $\widehat{\Psi} \circ T : \mathbb{R}^M \to \mathbb{R}$. We used the transformation T given in the following way:

1. Let the parameters of the admissible set $\widehat{\mathcal{U}}_{\rm ad}^M$ be: M, $a_{\min}$, $a_{\max}$, C_L, x_C.
2. Let $x = (x_1, \ldots, x_M) \in \mathbb{R}^M$ be arbitrary. We obtain the corresponding value $T(x) = \alpha = (\alpha_1, \ldots, \alpha_M) \in \widehat{\mathcal{U}}_{\rm ad}^M$ as follows:

   For the first component of α we define
   $$\alpha_1 = a_{\min} + \frac{(a_{\max} - a_{\min})(\frac{\pi}{2} + \arctan x_1)}{\pi},$$
   for $\alpha_i,\ i = 2, \ldots, M$, we define
   $$\alpha_i = \alpha_{i-1} + \frac{K(\frac{\pi}{2} + \arctan x_i)}{\pi},$$
   where $K = \min\{\frac{C_L x_C}{M-1}, a_{\max} - \alpha_{i-1}\}$.

We observe that, to be allowed to use the *fminsearch* algorithm to solving of our maximization problem, we apply this algorithm to function $-\widehat{\Psi} \circ T$.

Now, we will present concrete numerical examples. We consider problem (3.15), respectively its finite-dimensional parallel (4.1). Let the parameters of admissible set $\mathcal{U}_{\rm ad}$ be: $a_{\min} = 1$, $a_{\max} = 6$, $C_L = 0.3$, and $x_C = 10$. Let the dimension of $\mathcal{U}_{\rm ad}^M$ be $M = 11$ and the dimension of the finite element space V_h be $N = 50$. We will solve the state problem (4.1) with two different right-hand sides f_1 and f_2. In the concrete, $f_1(x) = 300x(1-x)$, and

$$f_2(x) = \begin{cases} 100 & \text{for} \quad 0 \le x \le \frac{2}{3} \\ -100 & \text{for} \quad \frac{2}{3} < x \le 1. \end{cases}$$

The worst scenario problem (3.3) is solved with the following criterion functional:

$$\Phi(a, u(a)) = -10^6 \int_0^1 [u(a) - u_h(a^{M0})]^2 \, \mathrm{d}x, \qquad (4.2)$$

where $u_h(a^{M0}) \in V_h$ is the state solution of problem (4.1) computed for the selected (and afterwards searched) parameter a^{M0}. This parameter is determined by the vector of nodal values $\alpha^0 = (3.00, 3.10, 3.30, 3.40, 3.45, 3.50, 3.70, 3.80, 3.95, 4.00, 4.20) \in \widehat{\mathcal{U}}_{\rm ad}^M$. It is not difficult to show that the functional Φ defined by (4.2) satisfies the assumption (vi) from Section 3.1.

Figures 4.1 – 4.4 present the obtained results. The approximation $a_{\rm appr}^{M0}$ of the searched parameter a^{M0} is calculated for the right-hand sides f_1 and f_2 with using two different initial points. Also, the parameter $a_{\rm in}$ corresponding to selected initial point $\alpha_{\rm in} \in \widehat{\mathcal{U}}_{\rm ad}^M$ and, for comparison, the parameter a^{M0} are presented. In addition, the value $\widehat{\Psi}(\alpha_{\rm in})$ of the function $\widehat{\Psi}$ at the initial point, the final value $\widehat{\Psi}(\alpha_{\rm appr}^0)$ at the point $\alpha_{\rm appr}^0 \in \widehat{\mathcal{U}}_{\rm ad}^M$ corresponding to the parameter $a_{\rm appr}^{M0}$, and the number of iterations k of the Nelder-Mead simplex method are presented.

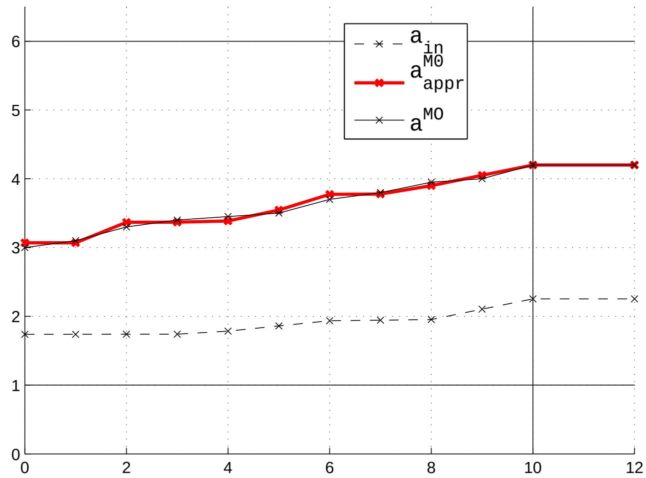

Figure 4.1 The approximation $a^{M0}_{\rm appr}$ of the searched parameter a^{M0} for the right-hand side f_1 and the given initial point $\alpha_{\rm in} \in \widehat{\mathcal{U}}^M_{\rm ad}$ corresponding to the parameter $a_{\rm in}$ ($\widehat{\Psi}(\alpha_{\rm in}) = -1.2828.10^6$, $\widehat{\Psi}(\alpha^{M0}_{\rm appr}) = -0.86.10^{-2}$, $k = 3000$)

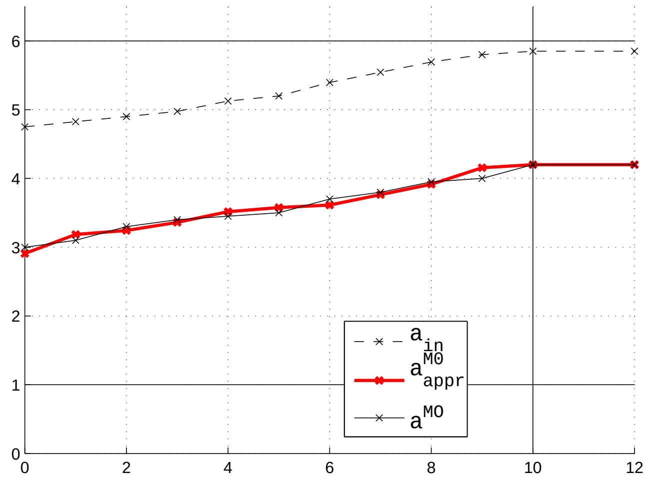

Figure 4.2 The approximation $a^{M0}_{\rm appr}$ of the searched parameter a^{M0} for the right-hand side f_1 and the given initial point $\alpha_{\rm in} \in \widehat{\mathcal{U}}^M_{\rm ad}$ corresponding to the parameter $a_{\rm in}$ ($\widehat{\Psi}(\alpha_{\rm in}) = -1.3589.10^5$, $\widehat{\Psi}(\alpha^{M0}_{\rm appr}) = -0.399.10^{-1}$, $k = 2500$)

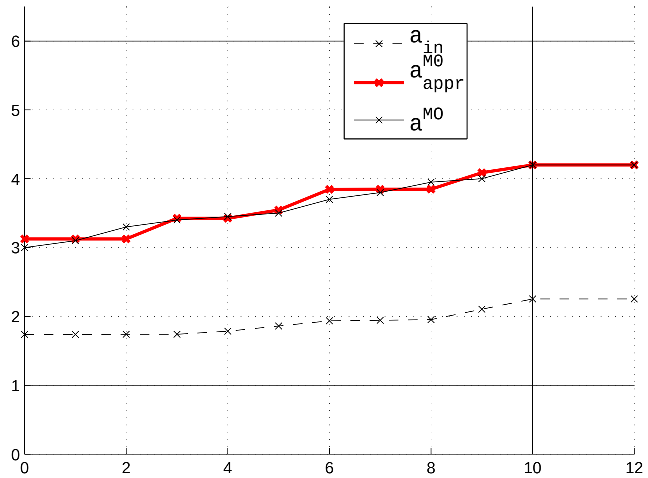

Figure 4.3 The approximation a^{M0}_{appr} of the searched parameter a^{M0} for the right-hand side f_2 and the given initial point $\alpha_{\text{in}} \in \widehat{\mathcal{U}}^M_{\text{ad}}$ corresponding to the parameter a_{in} ($\widehat{\Psi}(\alpha_{\text{in}}) = -9.1308.10^5,\ \widehat{\Psi}(\alpha^{M0}_{\text{appr}}) = -0.1144,\ k = 6700$)

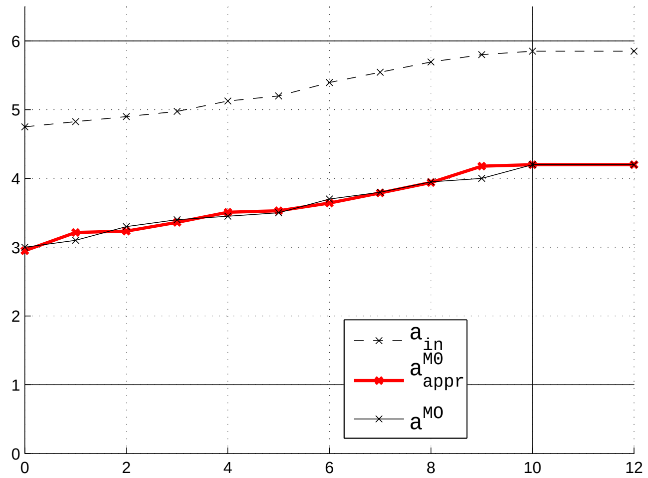

Figure 4.4 The approximation a^{M0}_{appr} of the searched parameter a^{M0} for the right-hand side f_2 and the given initial point $\alpha_{\text{in}} \in \widehat{\mathcal{U}}^M_{\text{ad}}$ corresponding to the parameter a_{in} ($\widehat{\Psi}(\alpha_{\text{in}}) = -9.7035.10^4,\ \widehat{\Psi}(\alpha^{M0}_{\text{appr}}) = -0.126.10^{-1},\ k = 4600$)

5 Publications

1. *P. Harasim*: On the worst scenario method: A modified convergence theorem and its application to an uncertain differential equation. Preprint Series in Mathematical Analysis, Mathematical Institute in Opava, Silesian University in Opava, MA 61/2007.

2. *P. Harasim*: On the worst scenario method: A modified convergence theorem and its application to an uncertain differential equation. Appl. Math. *53* (2008), 583- 598.

3. *P. Harasim*: On the worst scenario method: A modified convergence theorem and its application to an uncertain differential equation. Proceedings of SNA'09, Institute of Geonics AS CR, Ostrava, February 2009, 34-38

4. *P. Harasim*: Worst scenario method and other approaches to uncertainty. Proceedings of Ph.D. Workshop 2009 (CD), Institute of Geonics AS CR, Ostrava, November 2009

5. *P. Harasim*: On the worst scenario method: Application to a quasilinear elliptic 2D-problem with uncertain coefficients. Appl. Math. Accepted

6 Presentations

1. Seminar on Numerical Analysis 2009 (SNA'09), Institute of Geonics AS CR, Ostrava, February 2009. Lecture: On the worst scenario method: A modified convergence theorem and its application to an uncertain differential equation.

2. Modelling 2009, Institute of Geonics AS CR, VŠB-Technical University of Ostrava and International Association for Mathematics and Computers in Simulations (IMACS), Rožnov pod Radhoštěm, Czech Republic, June 2009. Lecture: On the worst scenario method: Application to an uncertain differential equation and numerical examples.

3. Ph.D. Workshop 2009, Institute of Geonics AS CR, Ostrava, November 2009. Lecture: Worst scenario method and other approaches to uncertainty.

7 Long-term visits

1. Universidad de Murcia, October-December 2005, Murcia, Spain. Socrates-Erasmus

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[9] *P. Harasim*: On the worst scenario method: Application to a quasilinear elliptic 2D-problem with uncertain coefficiens. Appl. Math. Accepted

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