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Obsah

1. Introduction i

2. Basic terminology i

3. Functions with connected G_δ graphs ii

4. Li-Yorke sensititvity iv

5. Publications concerning the Thesis v

6. Other publications v

7. Quotations by other authors vi

8. Conferences vi

9. Awards and other activities vii

Reference vii

1. Introduction

This Thesis is based on three independent papers [1]-[3]. Their common subject are discrete dynamical systems generated by maps of a compact metric space. In the first two papers, generally we consider the discontinuous functions of the interval.

In the first paper we define the topological entropy for discontinuous functions of a compact metric space with almost all of the standard properties. This part shows that any function of the interval with connected G_δ graph has positive topological entropy if and only if there is a periodic point of period different from 2^n, for any $n \in \mathbb{N}$. The second paper studies the properties of minimal sets of functions of the interval whose graphs are connected G_δ sets. In this part we introduce the notion ω-minimal set.

Finally, the third part gives counterexamples which disprove conjectures about Li-Yorke sensitivity stated by Ethan Akin and Sergii Kolyada in 2003.

2. Basic terminology

Let X be a nonvoid compact metric space with metric ρ and let $f : X \to X$ be a map. The pair (X, f) is called the *dynamical system.* Let $\mathcal{F}$ be the space of all maps $X \to X$ (including discontinuous functions). For a nonnegative integer i and $f \in \mathcal{F}$, we denote i-th iteration of x under f by $f^i(x)$. The *trajectory* of x under f is the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where $f^0(x) = x$. The set of all limit points of the trajectory of x is the *ω-limit set* of x and is denoted by $\omega_f(x)$. A point $x \in X$ is *periodic* of period $k \in \mathbb{N}$ if $f^k(x) = x$ and $f^i(x) \neq x$ for $i = 1, 2, \ldots, k-1$. The trajectory of periodic point is the *periodic orbit.* A point $x \in X$ is *recurrent* if $x \in \omega_f(x)$ and we denote the set of all recurrent points by $\mathrm{Rec}(f)$. A point $x \in X$ is *uniformly reccurent* if, for each open set $U \subset X$ containing x, there exists a positive integer N such that if $f^m(x) \in U$ with $m \geq 0$, then $f^{m+k}(x) \in U$ for some k with $0 < k \leq N$.

Let $f \in \mathcal{F}, n \in \mathbb{N}$ and $\varepsilon > 0$. A set $G \subset X$ is (n, ε)*-separeted set* if for every $x, y \in G$, $x \neq y$ there is $0 \leq i < n$ such that $\rho(f^i(x), f^i(y)) > \varepsilon$. A set $E \subset X$ is an (n, ε)*-span* if for every $x \in X$, there is a point $y \in E$ such that $\rho(f^i(x), f^i(y)) \leq \varepsilon$ for every $i \in \{0, 1, \ldots, n\}$. We denote an (n, ε)-separated set with maximal possible number of points by

$S(f,n,\varepsilon)$, and its cardinality by $s_n(\varepsilon)$. Analogously, $r_n(\varepsilon) = \min\{\#F : F$ is an $(n,\varepsilon) -$ span$\}$. The *topological entropy* of an $f \in \mathcal{F}$ is the number

$$h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon).$$

Denote by $\mathcal{J}$ the class of functions of the interval $I = [0,1]$ whose graph is a connected G_δ set. Let $\varphi \in \mathcal{J}$. A non-empty set $M \subset I$ is *minimal* if it is closed, $\varphi(M) = M$ and no proper subset of M has these properties. A nonvoid set $M_\omega \subset I$ is *ω-minimal* if $\omega_\varphi(x) = M_\omega$ for every $x \in M_\omega$.

Let $T : X \to X$ be continuous and surjective. A pair $(x,y) \in X \times X$ is *proximal* if $\liminf_{n\to\infty} \rho(T^n(x), T^n(y)) = 0$. The system (X,T) is called *Li-Yorke sensitive*, if there is an $\varepsilon > 0$ such that for every $x \in X$ there is a sequence $\{y_n\}_{n=1}^\infty$ of proximal points to x converging to x with

$$\limsup_{i\to\infty} \rho(T^i(x), T^i(y_n)) > \varepsilon, \quad \text{for any } n \in \mathbb{N}.$$

The system (X,T) is *spatio-temporally chaotic* if for any neighbourhood of any point $x \in X$ contains a point y proximal to x such that

$$\limsup_{i\to\infty} \rho(T^i(x), T^i(y)) > 0.$$

The map T is *topological transitive* if for every pair of open, nonempty subsets $U, V \subset X$ there is a positive integer n such that $U \cap T^{-n}(V) \neq \emptyset$, and is *weakly mixing* when the product map $T \times T$ of $X \times X$ is transitive. If Y is a compact metric space and $S : Y \to Y$ is a surjective, continuous map, the system (Y,S) is a *factor* of (X,T) if there is a semiconjugacy map $\pi : X \to Y$, i.e., π is surjective continuous map such that $T \circ \pi(x) = \pi \circ S(x)$, for all $x \in X$.

3. Functions with connected G_δ graphs

In this section we show that some classical results concerning dynamical properties of continuous functions of the interval are true for more general mappings of the interval whose graph is a connected G_δ set, thus for functions in the class $\mathcal{J}$.

For maps in $\mathcal{J}$, there are true some classical results, e.g., the Sharkovsky's theorem (cf. [Szu1]) and the Itinerary Lemma as nontrivial consequence of result in [Szu2]. The topological entropy of these functions has the following properties. In general, the following assertion is true for any map of a compact metric space, thus not necessarily continuous map.

Theorem A. (Cf. [1].) *Let $f \in \mathcal{F}$. Then*

(i) $h(f^k) = k \cdot h(f)$, *for every positive integer* k,

(ii) $h(f|_{A \cup B}) = \max\{h(f|_A), h(f|_B)\}$, *where* $A, B \subset X$,

(iii) $h(f) = h(f|_{\mathrm{Rec}(f)})$.

One of the main aims is to show that for maps in $\mathcal{J}$ another classical result is true – Misiurewicz's characterization of continuous maps of the interval.

Theorem B. (Cf. [1].) *Let $f \in \mathcal{J}$. Then f has positive topological entropy if and only if f has a periodic point whose period is not a power of* 2.

The key result is the fact, that topological entropy is supported by the set of recurrent points of the map. Some useful ideas were found in [Sz] and [AKLS].

The natural question is whether the results concerning minimality of continuous mappings of the interval can be generalized to the class $\mathcal{J}$. For continuous functions the notion ω-minimal set is the same as the notion minimal set. In general, the same is not true for functions from $\mathcal{J}$, but only for maps with zero topological entropy. For functions with positive topological entropy it may happen that, e.g., an ω-minimal set M_ω is disjoint from the image of M_ω. In fact, we have the following theorem.

Theorem C. (Cf. [2].) *Let $f \in \mathcal{J}$ have zero topological entropy, and let $\emptyset \neq M \subset I$. Then the following two conditions are equivalent.*

(i) M is a minimal set of f,

(ii) M is closed, $f(M) \subseteq M$, every point in M is recurrent, and no proper subset of M has these properties.

Moreover, if M is minimal then $f(M) = M$ and any point in M is uniformly recurrent.

4. Li-Yorke sensititvity

In [AK] Akin and Kolyada formulated the following conjectures:

(1) For minimal systems, spatio-temporal chaos is equivalent to Li-Yorke sensitivity.
(2) Every Li-Yorke sensitive, minimal system has a nontrivial, weak mixing factor.

The main result of this section is to give counterexamples disproving these conjectures. These examples are based on ideas from [FPS] and [BSS]. We describe constructions of triangular maps F_1 and F_2 of $Q \times S$, where S is the circle with radius $R > 0$ and Q is the Cantor set in $[0, 1]$.

The set Q is homeomorphic to the set $\{0, 1\}^{\mathbb{N}}$ of sequences of zeros and ones. Define the Adding Machine $\tau : Q \to Q$ by $\tau(\alpha) = \alpha + 10000\ldots$ where addition is modulo 2 from left to right. The map τ is continuous on Q and it is known that (Q, τ) is minimal system.

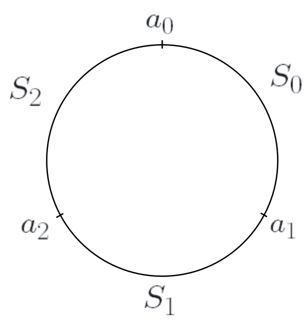
Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers. Define the maps $F_i : Q \times S \to Q \times S$, for $i = 1, 2$, by

$$F_i(\alpha, y) = \begin{cases} (\tau(\alpha), y) & \text{if} \quad \alpha = 1111\ldots, \\ (\tau(\alpha), \varphi_k^j(y)) & \text{otherwise,} \end{cases}$$

where $k \in \mathbb{N}$, $0 \le j \le 2^{n_k} - 2$ and φ_k^j is a homeomorphism of S.

If φ_{2k-1}^j is the rotation of S with suitable angle in suitable direction, the systems $(Q \times S, F_i)$ are minimal and there are only trivial weakly mixing factor.

Let S_i, for $i = 0, 1, 2$, be the parts of the circle S with the endpoints a_i, a_{i+1} and the same length (as in the Figure 1).



Obrázek 1. The circle S.

Without loss of generality we may consider any φ_{2k}^{j} on S_i as a map of the unit interval $[0,1]$. If we define this map by $\varphi_{2k}^{j}(y) = y^{t_k}$, for suitable sequences $\{t_k\}$ we get the required systems.

Thus our main result is the following.

Theorem D. (Cf. [3].) *There are continuous triangular maps $F_i : X \to X$, $F_i : (x,y) \mapsto (\tau(x), g_i(x,y))$, $i = 1,2$, with the following properties:*

(i) Both (X, F_1) and (X, F_2) are minimal systems, without weak mixing factors (i.e., neither of them is semiconjugate to a weak mixing system).
(ii) (X, F_1) is spatio-temporally chaotic but not Li-Yorke sensitive.
(iii) (X, F_2) is Li-Yorke sensitive.

5. Publications concerning the Thesis

[1] M. Čiklová, *Dynamical systems generated by functions with connected G_δ graphs.* Real Analysis Exchange **30** (2004-2005), no. 2, pp. 617–737. (MR 2006m:26005l, Zbl 1108.37011)

[2] M. Čiklová, *Minimal and ω-minimal sets of functions with connected G_δ graphs.* Real Analysis Exchange **32** (2006-2007), no. 2, pp. 397–408. (Zbl 1127.37034)

[3] M. Čiklová, *Li-Yorke sensitive minimal maps.* Nonlinearity **19** (2006), pp. 517–529. (MR 2007b:37015)

6. Other publications

[4] M. Čiklová, *Dynamical systems generated by functions with connected G_δ graphs.* Real Anal. Exchange **2005**, 29th Summer Symposium Conference, pp. 19–20. (Zbl 1096.37008)

[5] M. Čiklová, *Minimal sets of functions with connected G_δ graph.* Real Anal. Exchange **2006**, 30th Summer Symposium Conference, pp. 67–68. (Zbl 1116.37004)

[6] M. Čiklová, *On open problem concerning Li-Yorke sensitivity.* Real Anal. Exchange **2007**, 31st Summer Symposium Conference – to appear.

7. Quotations by other authors

[7] E. Akin and S. Kolyada, *For minimal systems does Li-Yorke sensitivity occur exactly for extensioins of nontrivial weak mixing systems?* http://www.math.iupui.edu/ mmisiure/open/ (updated 2007) (cf. [3]).

[8] J. Smítal, *Why is important to understand dynamics of triangular maps?*, J. of Difference Equations and Applications **14** (2008), pp. 597 – 606 (cf. [3]).

[9] T. K. Subrahmonian Moothathu, *Orbits of Darboux-like real functions.* – to appear (cf. [1]).

[10] H. Pawlak and R. J. Pawlak, *The stable points and the attractors of Darboux functions.* – to appear (cf. [1]).

[11] R. J. Pawlak, *On the entropy of Darboux functions.* – to appear (cf. [1]).

8. Conferences

[12] 8th Czech-Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, September 18-25, 2004. Talk on: "Dynamical systems generated by functions with G_δ connected graphs."

[13] Summer Symposium in Real Analysis XXIX, Walla Walla, Washington, June 21 - 25, 2005. Talk on: "Dynamical systems generated by functions with G_δ connected graphs."

[14] Summer Symposium in Real Analysis XXX, Asheville, North Carolina, June 13 - 17, 2006. Talk on: "Minimal sets of functions with connected G_δ graphs."

[15] 10th Czech-Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, June 25 - July 1, 2006. Talk on: "Minimal sets of functions with connected G_δ graphs."

[16] European Conference on Iteration Theory, Gargnano, Italy, September 10-16, 2006. Talk on: "Li-Yorke sensitive minimal maps."

[17] Visegrad Conference on Dynamical Systems, High Tatras, June 17 - 23, 2007. Talk on: "On open problems concernning Li-Yorke sensitivity."

[18] Summer Symposium in Real Analysis XXXI , Oxford, England, August 12 - 16, 2007. Talk on: "On open problems concernning Li-Yorke sensitivity."

[19] Conference in Honor of David Preiss, University of Warwick, England, August 17 - 19, 2007.

9. Awards and other activities

[20] The third prize in the Mathematical competition of students of Czech universities SVOČ, May 2004.

[21] The first prize in the Mathematical competition of students of Czech universities SVOČ, May 2005.

[22] Reviewer for Mathematical Reviews - 9 reviewed papers.

[23] Referee for international journals - 3 refereed papers (Discrete Contin. Dyn. Syst., Grazer Math. Ber., Real Analysis Exchange).

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[Szu2] P. Szuca, *Loops of intervals and Darboux Baire-1 fixed point problem.* Real Anal. Exchange **29** (2003/04), 205–209.