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Contents

0. Introduction

The Thesis is based on four independent papers connected by one common subject — they all study the theory of chaotic discrete dynamical systems generated by continuous maps of a compact metric space into itself. (For the first two papers see [Ba1] and [Ba2], the third [Ba3] and fourth one [Ba4] will be published in 2001.)

The first part provides a counterexample which disproves a conjecture about orbit-enclosing ω-limit sets stated by Agronsky and Ceder in 1991. The second part shows by several examples that triangular maps of the unit square admit phenomena which cannot occur in the one-dimensional case. The third part proves that any bitransitive continuous map of the interval is conjugate to a map extremely chaotic in the sense of Li and Yorke almost everywhere. Finally, the fourth part shows that a similar assertion holds for distributional chaos, too: Any bitransitive continuous map of the interval is conjugate to a map distributionally chaotic almost everywhere.

1. Basic terminology and notation

Let A be a topological space, $f : A \to A$ a continuous map, $x \in A$ and n a nonnegative integer. By $f^n(x)$ we denote the n-th iteration of x under f. The sequence $\{f^n(x)\}_{n=0}^{\infty}$, where $f^0(x) = x$, is the *trajectory* of x under f, and the set $\omega_f(x)$ of all limit points of the trajectory is the *ω-limit set* of x. An ω-limit set is *maximal* if it is not propely contained in any other

ω-limit set, and an ω-limit set is *orbit-enclosing*, if it contains the trajectory.

If for any non-void subsets U and V of A there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$, then we say that f is *(topologically) transitive* on A; f is *bitransitive* if f^2 is transitive. By a *continuum* we mean any compact connected set which contains more than one point. A set $M \subset A$ is *arcwise connected* if each two points in M belong to some homeomorph of $[0,1]$ which lies in M. A map F from a subset of $A \times A$ into itself is called *triangular* if it is of the form $F(x,y) = (f(x), g(x,y))$.

Let $I = [0,1]$ be the unit interval. By $C(I,I)$ we denote the set of continuous maps $f : I \to I$. Function $f \in C(I,I)$ is *semiconjugate* to $g \in C(I,I)$ if there is a surjective map $h \in C(I,I)$ such that $h \circ f = g \circ h$. If h is bijective, then f and g are *conjugate*.

A map $f \in C(I,I)$ is called *chaotic in the sense of Li and Yorke*, briefly, *LY-chaotic* (resp. *extremely LY-chaotic*) if there is an $\varepsilon > 0$ and a set $S \subset I$ containing at least two points such that, for every $x, y \in S$ with $x \neq y$, $\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geq \varepsilon$ (resp. $\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1$) and $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$. The set S is called *LY-scrambled set* (resp. *extremely LY-scrambled set*) of f. We say that a function $f \in C(I,I)$ is *LY-chaotic almost everywhere* if there is a LY-scrambled set S of f with $\lambda(S) = 1$, where λ denotes the Lebesgue measure.

For $f \in C(I,I)$, $x, y \in I$, $t \in \mathbf{R}$, and a positive integer n, let

$$\xi(x,y,n,t) = \sharp\{i; 0 \leq i < n \text{ and } |f^i(x) - f^i(y)| < t\}.$$

Put $F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x,y,n,t)$, and $F_{xy}(t) = \liminf_{n\to\infty}$

$\frac{1}{n}\xi(x,y,n,t)$. Then both F_{xy} and F^*_{xy} are nondecreasing functions, with $0 \leq F_{xy} \leq F^*_{xy} \leq 1$, $F^*_{xy}(t) = 0$ for $t < 0$, and $F_{xy}(t) = 1$ for $t > 1$. We refer to F^*_{xy} and F_{xy} as the *upper* and *lower distribution function* of x and y, respectively. The map f is *distributionally chaotic* (briefly, *d-chaotic*) *in the wider sense* if there is a set $S \subset I$ containing at least two points such that, for any $x \neq y$ in S, $F_{xy} < F^*_{xy}$ (by this we mean that $F_{xy}(t) < F^*_{xy}(t)$ for all t in a non-degenerate interval). Such S is a *d-scrambled set* for f. If, in adition, for any $x \neq y$ in S, $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$, then we say that f is *d-chaotic in the narrow sense* (see [SS]). Obviously, d-chaos in the narrow sense implies LY-chaos, but for d-chaos in the wider sense it is not true (see Chapter 3). Moreover, a d-scrambled set S is *uniform* if there is a (probability) distribution function F such that, for any $x \neq y$ in S, $F_{xy} \leq F < F^*_{xy} \equiv 1$. The *principal measure of chaos* of f is the number

$$\mu_p(f) = \sup_{x,y\in S} \int_0^1 (F^*_{xy}(t) - F_{xy}(t))dt$$

.

A pair of points (x,y), $x,y \in I$, is called *isotectic* if, for every positive integer n, the ω-limit sets $\omega_{f^n}(x)$ and $\omega_{f^n}(y)$ are subsets of the same maximal ω-limit set of f^n. The *spectrum* of f, denoted by $\Sigma(f)$, is the set of minimal elements of the set $\{F_{xy}; (x,y) \text{ is isotectic}\}$.

2. Conjecture of Agronsky and Ceder

A number of examples have induced Agronsky and Ceder to formulate the following conjecture (see [AC1]).
A continuum $K \subset E^k$ is an orbit-enclosing ω-limit set if and only if it is arcwise connected.

The aim of this section is to give a counterexample disproving this assertion.

The crucial point of this section is the construction of a transitive map $\varphi : D \to D$, where $D = I^2 \cup ([1, \infty) \times \{1/2\})$, such that for any $x \in [0, \infty)$, $\lim_{x\to\infty} ||\varphi(x, 1/2) - (x, 1/2)|| = 0$. Then the map φ is transformed by a homeomorphism $h : D \to I^2 \cup W$, where W is the graph of the curve $y = \frac{1}{2} + \frac{1}{2} \sin \frac{\pi}{x-2}$, for $x \in [1, 2)$, into a map $F : S \to S$, where $S = I^2 \cup W \cup (\{2\} \times I)$, and with the help of a theorem proved by Agronsky and Ceder in [AC2] (in the Thesis Theorem 2) it can be easily seen that the above described set is an orbit-enclosing ω-limit set with regard to F. That means that the set S together with the map $F : S \to S$ is the required counterexample which disproves the Agronsky and Ceder conjecture. Moreover, in this example the set S has a non-empty interior and the map F is triangular.

3. Distributional chaos for triangular maps

The natural question arising during the study of one-dimensional dynamical systems is whether the results can be generalized to higher-dimensional dynamical systems.

It is known (see e.g. [FPS], [Ko]) that there are phenomena in higher-dimensional dynamical systems which are impossible in the one-dimensional case, and that such phenomena occur even for two-dimensional dynamical systems generated by triangular maps of the unit square, i.e. by the most simple non-trivial two-dimensional mappings.

The aim of this section is to prove that the below listed properties of one-dimensional dynamical systems disappear if we go over to dynamical systems generated by triangular maps of the unit square:

Property 1 (see [SS]). *For any $f \in C(I,I)$ the spectrum of f is non-empty and finite.*

Property 2 (see [J]). *For any $f \in C(I,I)$ the principal measure of chaos of f is generated by a pair of points.*

Property 3 (see [SS]). *Any distributionally chaotic map $f \in C(I,I)$ is chaotic in the sense of Li and Yorke.*

This aim has been achieved by construction of three triangular mappings of the unit square: the first one has an infinite spectrum, the second one is d-chaotic but its principal measure of chaos is not generated by any pair of points and its spectrum is empty, and the third one is d-chaotic in the wider sense but not chaotic in the sense of Li and Yorke.

4. LY-chaos and transitivity

In the early eighties Gy. Targoński formulated the question whether there are any functions $f \in C(I,I)$ with scrambled sets of positive Lebesgue measure. After several partial result (see e.g. in [S1] the Smítal's example of a function with LY-scrambled set with full outer Lebesgue measure). The question was finally answered in the affirmative in 1984 by Kan and Smítal who gave independently of each other examples of functions with LY-scrambled sets of positive Lebesgue measure (see [Ka] and [S2]). Moreover, in 1985 Misiurewicz (see [M]) and in 1987 Bruckner and Hu (see [BH]) presented examples of functions with LY-scrambled set with full Lebesgue measure (i.e. examples of functions chaotic almost everywhere).

As follows from this survey, up to now only individual examples of functions with LY-scrambled sets of positive Lebesgue measure have been published. The natural question is therefore to give a universal description of all such functions. This section provides the first step in this direction — it shows that the property of being LY-chaotic almost everywhere is universal for all bitransitive maps up to a homeomorphism. Actually, the main result of this section goes much deeper than that — it shows that (up to a homeomorphism) all bitransitive maps are not only LY-chaotic but even extremely LY-chaotic almost everywhere. More precisely, the following theorem is proved.

Theorem A. *Any bitransitive map $f \in C(I, I)$ is topologically conjugate to an almost everywhere extremely LY-chaotic map $g \in C(I, I)$.*

Using a result of A. M. Blokh (see [B]) we get as a consequence of Theorem A that for any map $f \in C(I, I)$ with positive topological entropy there is a positive integer k such that f^k is semiconjugate to a continuous map extremely LY-chaotic almost everywhere.

5. Distributional chaos and transitivity

The last part of the Thesis studies the effects of replacing chaos in the sense of Li and Yorke in Theorem A by distributional chaos. It turns out that the main idea underlying the proof of Theorem A can be used for proving its direct anologue for distributional chaos. Through this section d-chaos means d-chaos in the narrow sense.

Theorem B. *Any bitransitive map $f \in C(I, I)$ is topologically conjugate to an almost everywhere d-chaotic map $g \in C(I, I)$.*

Moreover, certain features of the construction used in the proof of Theorem B make it possible to prove in a short and simple way the following slightly stronger version of this theorem.

Theorem C. *Any bitransitive map $f \in C(I, I)$ is topologically conjugate to a map $g \in C(I, I)$ with uniform d-scrambled set S of the full Lebesgue measure.*

The result of A. M. Blokh (see [B]) can be used for distributional chaos in the same way as it was used for LY-chaos in the close of Section 4 — by applying Blokh's result to Theorem C we can show that for any map $f \in C(I, I)$ with positive topological entropy there is a positive integer k such that f^k is semiconjugate to a continuous map d-chaotic almost everywhere.

Finally, let us remark that the assertion "any bitransitive map $f \in C(I, I)$ is topologically conjugate to an almost everywhere LY-chaotic map $g \in C(I, I)$" (i.e. a weak version of Theorem A) can be easily deduced directly from Theorem C. However, there are good reasons for choosing the approach presented in the Thesis. First, Theorem C provides chaoticity but not extreme chaoticity of the map g, and second, the — rather technical and lengthy — proof of Theorem C is much more transparent on the background formed by techniques and ideas developed in the course of proving Theorem A. Last, but not least, the decisive motivation for studying the above described questions for distributional chaos comes from the successful proof of Theorem A for chaos in the sense of Li and Yorke.

6. Publications concerning the Thesis

[1] M. BABILONOVÁ, *On a conjecture of Agronsky and Ceder concerning orbit-enclosing ω-limit sets*. Real Analysis Exchange vol. 23 (1997/98), 773–777. MR 99i:26004, Zbl 939.37013.

[2] M. BABILONOVÁ, *Distributional chaos for triangular maps*. Annales Mathematicae Silesianae 13 (1999), 33–38. MR 2000k:37018, Zbl pre 991.30894.

[3] M. BABILONOVÁ, *The bitransitive continuous maps of the interval are conjugate to maps extremely chaotic a.e.* Preprint MA 11 A/1999, Mathematical Institute, Silesian University, Opava. Acta Math. Univ. Comen. — to appear.

[4] M. BABILONOVÁ, *Massive chaos*. Real Analysis Exchange vol. 25(1) (1999/2000), 43–44. Abstract of talk presented in [16].

[5] M. BABILONOVÁ, *Extreme chaos and transitivity*. Internat. J. Bifur. Chaos Appl. Sci. Engrg. — to appear.

7. Other publications

[6] M. BABILONOVÁ, *Solution of a problem of S. Marcus concerning J-convex functions*. Preprint MA 15/1999, Mathematical Institute, Silesian University, Opava. Accepted to Aequationes Mathematicae.

[7] M. BABILONOVÁ, *On stationary and determining sets for J-convex functions*. Preprint MA 17/1999, Mathematical Institute, Silesian University, Opava. Accepted to Real Analysis Exchange.

8. Quotations by other authors

[8] F. BALIBREA and V. JIMÉNEZ LÓPEZ, *The measure of scrambled sets: a survey*. Acta Univ. M. Belii, Math. no. 7 (1999), 3–11. (cf. [3])

[9] V. JIMÉNEZ LÓPEZ and J. SMÍTAL, *Two counterexamples to a conjecture by Agronsky and Ceder*. Acta Math. Hung. 88 (2000), No. 3, 193–204. (cf. [1])

[10] V. JIMÉNEZ LÓPEZ and J. SMÍTAL, *Omega limit sets for triangular mappings*. 13 pp. Accepted to Fundamenta Math. (cf. [1])

[11] K. JANKOVÁ, *Points generating the principal measure of chaos*. Accepted to Real Analysis Exchange. (cf. [2])

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9. Presentations

[12] 26th Winter School in Abstract Analysis, Křišt'anovice, January 1998. Paper on [1].

[13] European Conference on Iteration Theory — ECIT 98, Muszyna, Poland, August 30 – September 5, 1998. Invitation. Paper on [2].

[14] Seminars of prof. Ger and prof. Baron, Silesian University in Katowice, Poland, October 1998 – January 1999. Two talks on author's results.

[15] 27th Winter School in Abstract Analysis, Lhota nad Rohanovem, January 23 – 30, 1999. Paper on [2].

[16] 23th Summer Symposium on Real Analysis, Łódź, Poland, June 20 – 26, 1999. Paper on [4].

[17] 28th Frolík School in Abstract Analysis, Křišt'anovice, January 23 – 29, 2000. Paper on [6].

[18] Millenium Symposium on Real Analysis, Denton, Texas, USA, May 23 – 27, 2000. Invitation. Paper on [7].

[19] 4th Czech-Slovak Conference on Dynamical Systems, Praděd, June 22 – 28, 2000. Paper on [6].

[20] European Conference on Iteration Theory — ECIT 2000, La Manga, Spain, September 4 – 9, 2000. Invitation. Paper on [5].

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