Proceedings of the Seminar on Differential Geometry
Mathematical Publications Volume 2
Silesian University in Opava, Opava, 2000, 175–204

From semisprays to connections, from geometry of regular O.D.E. in mechanics to geometry of horizontal Pfaffian P.D.E. on fibered manifolds (and vice versa)[1]

A. Vondra

Abstract. The paper sumarizes motivations and interim investigations which have let to a recently established formalism related to the geometry of higher-order equations represented by connections on prolongations of a fibered manifold. Then the crucial ideas and results of the theory are presented.

Keywords and phrases. Connection, semispray, Pfaffian system, fibered manifold.

MS classification. 34A26, 35A30, 53C05, 58A17, 70H35.

1. Motivations

The classical results from the Riemannian and Finslerian geometries characterizing the extremals of some specific (arc length and (kinetic) energy) lagrangians as geodesics of canonical (*Levi-Civita*, *Cartan*) *connections* and the role of vector fiels called *sprays* as generators of corresponding second-order ordinary differential equations are well-known for a long time; we can refer to [1], [31], [35], [59], [68], [74] in particular.

The classical underlying structure is here made of a differentiable m-dimensional manifold M with local coordinates (q^σ). A *linear connection* on M is defined as a covariant derivative $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ (or equivalently as the corresponding

[1] Research supported by the Grant VS 96003 “Global Analysis” of the Czech Ministry of Education, Youth and Sports, and by the Grant 201/98/0853 of the Czech Grant Agency.
This paper is in final form and no part of it will be published elsewhere.

parallel lift) on vector fields on M. A *geodesics* is a curve $c : J \subset \mathbf{R} \to M$, whose tangent vector field $dc/dt : J \to TM$ is parallel with respect to the connection (covariant derivative vanishes). The corresponding condition is nothing but a system of m linear second-order ordinary differential equations for the components $c^\sigma = q^\sigma \circ c$ of c:

$$\frac{d^2c^\sigma}{dt^2} + \Gamma^\sigma_{ij}\frac{dc^i}{dt}\frac{dc^j}{dt} = 0, \quad \sigma = 1, \dots, m, \tag{1}$$

where the functions $\Gamma^\sigma_{ij} \in \mathcal{F}(M)$ are the well-known *Christoffels*. A *spray* is a vector field on the tangent bundle TM, horizontal with respect to the projection $TTM \to TM$ and compatible with homotheties on TM. A *geodesic spray* ζ for a linear connection ∇ is defined through the connection mapping and found out to be the only spray defining just the linear second order differential equations for geodesics of the connection. In coordinates,

$$\zeta = q^\sigma_{(1)}\frac{\partial}{\partial q^\sigma} + \Gamma^\sigma_{ij} q^i_{(1)} q^j_{(1)}\frac{\partial}{\partial q^\sigma_{(1)}}. \tag{2}$$

The classical approach, defining very often geometric objects as transformation rules of local coordinate expressions, survived in investigations of corresponding generalizations within some mathematical groups; cf. [3], [5], [32], [60], [75].

Another approach allowing more global point of view to the topic was introduced and then worked out in [7], [8], [13], [19], [33], [34], [69], [76]. The corresponding results were still intrinsically related to the underlying structures of *tangent bundle* TM of a manifold M. The studied connection is no more necessarily linear, being defined as a horizontal vector distribution in TM or equivalently as $(1,1)$-tensor field Γ on TM compatible with the so-called *almost tangent structure* J on TM (inspired by [30]): $\Gamma J = C$, $J\Gamma = J$, with C being the *Liouville vector field* on TM. The notion of the spray is generalized to a vector field called *semispray* and the relationship between connections and semisprays is studied in details. The relation to an autonomous Lagrangian L and Hamiltonian formalism is then studied through the *Poincare–Cartan* 2-form ω_L related to the energy $E = \partial_L L - L$ of the lagrangian L in the *Euler–Lagrange equation*

$$i_\zeta \omega_L = dE. \tag{3}$$

If L is regular and homogeneous, then there is a unique solution ζ_L of (3), which is a spray and the canonical projections of whose integral sections are just the extremals of L. Consequently, $\Gamma = -\partial_{\zeta_L} J$ is the unique (linear) connection without torsion whose paths are precisely the extremals of L.

Afterwards, analogous results were presented also on T^kM e.g. in [10], [11], [12], [14], [22].

These autonomous ideas has been then naturally extended to the time dependent situation: [9], [20], [21], [22]. First, the underlying structure is here $\mathbf{R} \times TM$, to where all the structures from TM are naturally extended. The crucial tool is again a canonical vertical endomorphism, now defined by $S = J - C \otimes dt$. A *semispray* is then a vector field ζ on $\mathbf{R} \times TM$ described by $S\zeta = 0$ and $J\zeta = C$. A *path* of ζ is a curve c in M, such that $\dot{c} := (t, c, dc/dt)$ is an integral curve of ζ. In addition to autonomous situation, the so-called *dynamical connections* are appearing, defining further decompositions of associated tangent bundles to strong and weak horizontal distributions. Just through these distributions, the *associated semisprays* are defined. Namely, a *dynamical connection* is an endomorphism on $T(\mathbf{R} \times TM)$, such that $J\Gamma = S\Gamma = S, \Gamma S = -S, \Gamma J = -J$. It can be identified with an $f(3, -1)$ *structure*, which means that $\Gamma^3 - \Gamma = 0$. Its path is the so-called *weak horizontal* curve in M and it is shown that there is a dynamical connection $\Gamma = -\partial_\zeta S$ with the same paths for any semispray ζ. In a usual way it is possible to associate uniquely the so-called *Lagrange vector field* to any regular lagrangian L on $\mathbf{R} \times TM$. Thus a dynamical connection whose paths are just the extremals of L can be found.

Our approach, introduced in [78–81], was based on a generalization of the notion of *higher-order semisprays* to a general fibered manifold $\pi : Y \to X$ with one-dimensional base X, which had to be reflected in the invariancy of all the concepts with respect to the changes of fibered coordinates. Thus we defined the *semispray distribution* as an horizontal subbundle with respect to the corresponding fibration, and we applied the properties of (generalized) *higher-order connections* in order to relate them with semispray distributions. We also described the conditions for connections on $\pi_{r-1,r}$ to be associated to a given connection of order $(r+1)$ on π in terms of relations of the corresponding horizontal distributions and consequently the equations. Then we discussed the whole class of *natural dynamical connections* on $J^r\pi$ canonically associated to a given connection of order $(r+1)$ on π as a generalization of the corresponding objects on $\mathbf{R} \times T^rM$. All the structures were intrinsically related to the geometry of underlying jet bundles. On the other hand, the one-dimensional base allowed to consider a special class of *natural affinors* (according to [23] for $\mathbf{R} \times T^rM$ and [72] for $J^r\pi$), being in particular generated by volume forms on the base X of the fibered manifold. As the main sources of the formalism and for the motivations we used [42],[44],[45] and [72]. The crucial definitions and properties of various connections on fibered manifolds were due to [56],[57]. The obtained results were then applied to a description of the geometry of *regular dynamics*, using again [42], [44–45] together with [46], through which the corresponding approach to the Lagrange and Hamilton formalism in time-dependent higher-order dynamics by means of the regular lagrangian and its Lepagean equivalent was applied. Moreover, we used the results of the papers [48], [49], which developed the Hamilton theory directly from locally *variational equations*.

2. Interim investigations

All the motivations have lead to an essential requirements: to investigate connections as equations and to do this in the most general situation, i.e., on a general fibered manifold with an arbitrary dimensional base. In particular, it meant the study of special “horizontal” kind of Pfaffian partial differential equations represented by the connections. The investigations went in two parallel and closely related directions.

First, a theory of *natural operators (differential invariants)* in sense of [40] and [47] has been found of particular usefulness when studying certain natural operations between various connections on prolongations of a fibered manifold in [24], [25]. A more detailed analysis of the formalism used amounted to the conclusion that there is a general framework the problems could be studied within. In [26], a new approach to the study of connections in 2-*fibered manifolds* was introduced and the role of naturality for this situation was discussed.

Following [36], a 2-*fibered manifold* is a quintuple $Z \xrightarrow{\rho} Y \xrightarrow{\pi} X$, where $\pi : Y \to X$ and $\rho : Z \to Y$ and thus also $\pi \circ \rho : Z \to X$ are fibered manifolds. Our contribution rests upon the study of the role of an arbitrary fibered morphism $\Phi : Z \to J^1\pi$. The point is that one of the most interesting particular cases of such a morphism is represented by $\Phi = \Gamma \circ \rho$ with $\Gamma : Y \to J^1\pi$ being a connection on π. The adopted approach was used in two particular situations: first, we studied natural relations between connections in $J^1\pi \xrightarrow{\pi_{1,0}} Y \xrightarrow{\pi} X$ with $\pi : Y \to X$ being a general fibered manifold and $\pi_{1,0} : J^1\pi = J^1Y \to Y$ the canonical affine bundle generated by π. The situation can be described diagamatically by

(4)
$$\begin{tikzcd}
X \arrow[d, "\mathrm{id_X}"] & J^1\pi \arrow[l, "\pi_1"'] \arrow[d, "\pi_{1,0}"'] & & J^1\pi_1 \arrow[ll, "J^1(\pi_{1,0},\mathrm{id_X})"'] \arrow[d, "(\pi_1)_{1,0}"'] & \\
X \arrow[d, "\mathrm{id_X}"] & Y \arrow[l, "\pi"'] \arrow[d, "\pi"'] & & J^1\pi \arrow[ll, "\pi_{1,0}"'] \arrow[d, "\pi_1"'] & J^1\pi_{1,0} \arrow[l, "(\pi_{1,0})_{1,0}"'] \\
X & X \arrow[l, "\mathrm{id_X}"'] & & X. \arrow[ll, "\mathrm{id_X}"'] &
\end{tikzcd}$$

Within this scheme, an alternative definition of the well-known *semiholonomic jets* has been given and the *formal curvature map* $R : J^1\pi_{1,0} \to \pi_{1,0}^*(V_\pi Y \otimes \pi^*(\Lambda^2 T^*X))$ was introduced. Among the results, we have shown that all natural operators transforming a connection Γ on π and a connection Ψ on $\pi_{1,0}$ into a connection Ξ on π_1 being of the zero order in Ψ form a 4-parameter family

$$(\Gamma, \Psi) \mapsto \mathbf{k}_\Gamma^{a,b} \circ \Psi + c\,(R \circ j^1\Gamma \circ \pi_{1,0}) + d\kappa_\Gamma \circ \Psi$$

for all $a, b, c, d \in \mathbf{R}$, where R is the formal curvature map, $\mathbf{k}_\Gamma^{a,b} = \mathbf{k}_{\Gamma_a} + bR$, $\Gamma_a = \mathrm{id}_{\mathrm{J}^1\pi} + \mathrm{a}(\Gamma \circ \pi_{1,0} - \mathrm{id}_{\mathrm{J}^1\pi})$, $\mathbf{k}_\Phi : J^1\pi_{1,0} \to J^1\pi$ is an affine bundle morphism defined for any fibered morphism $\Phi : J^1\pi \to J^1\pi$ (and especially for Γ_a) as the composition

$$J^1\pi_{1,0} \xrightarrow{(\pi_{1,0})_{1,0}\times \mathrm{id}} J^1\pi \times_Y J^1\pi_{1,0} \xrightarrow{\Phi\times\mathrm{id}} J^1\pi \times_Y J^1\pi_{1,0} \xrightarrow{\mathbf{k}} J^1\pi_1,$$

$\mathbf{k}$ is a canonical fibered morphism realizing the 'derivative of composed sections'; and κ_Γ is (analogously to formal curvature map) the *formal mixed curvature map* $\kappa_\Gamma : J^1\pi_{1,0} \to \pi_{1,0}^*(V_\pi Y \otimes \pi^*(\Lambda^2 T^*X))$.

Secondly, we worked with a 2-fibered manifold $V_\pi Y \xrightarrow{\rho} Y \xrightarrow{\pi} X$, where π is again a fibered manifold and $\rho = \tau_Y|_{V_\pi Y} : V_\pi Y \to Y$ its vertical bundle. Here, the situation is the following

(5)
$$\begin{tikzcd}
X \arrow[d, "\mathrm{id_X}"] & J^1Y \arrow[l, "\pi_1"'] \arrow[d, "\pi_{1,0}"] & J^1(\pi\circ\tau_Y) \arrow[l, "J^1(\tau_Y,\mathrm{id_X})"'] \arrow[r, "\nu_1"] \arrow[d, "(\pi\circ\tau_Y)_{1,0}"'] & V_{\pi_1}J^1\pi \\
X \arrow[d, "\mathrm{id_X}"] & Y \arrow[l, "\pi"'] \arrow[d, "\pi"'] & V_\pi Y \arrow[l, "\tau_Y"'] \arrow[d, "(\pi\circ\tau_Y)"'] & J^1\tau_Y \arrow[l, "(\tau_Y)_{1,0}"'] \\
X & X \arrow[l, "\mathrm{id_X}"'] & X\,, \arrow[l, "\mathrm{id_X}"'] &
\end{tikzcd}$$

where by ν_1 we denote the canonical isomorphism between $J^1(\pi \circ \tau_Y|_{V_\pi Y})$ and the subbundle $V_{\pi_1} J^1\pi$ of π_1-vertical vectors on $J^1\pi$. Here, all natural operators transforming a connection Γ on π and a connection Ψ on $\tau_Y : V_\pi Y \to Y$ into the connection Ξ on $\pi \circ \tau_Y : V_\pi Y \to X$ being of the zero order in Ψ form a 2–parameter family

$$(\Gamma, \Psi) \mapsto \mathbf{k}_{\Phi^a_{\Gamma,\Psi}} \circ \Psi + b\, D(k_\Gamma \circ \Psi, \mathcal{V}\Gamma)$$

for all $a, b \in \mathbf{R}$, where we refer to [26] for the details. Notice here only that $\mathcal{V}\Gamma$ is the *vertical prolongation* of the connection Γ, defined (following [37]) by $V\Gamma = \nu_1 \circ \mathcal{V}\Gamma$, being thus a connection on $(\pi \circ \tau_Y)$. This has been effectively used for finding a linear connection on $\tau_Y : V_\pi Y \to Y$ whose integral sections are just the symmetries of Γ.

The usefulness of such considerations for a description of the geometry of *first and second-order differential equations systems* represented by these connections became apparent in [54] and [82].

In [54], two dual indirect integration methods were discussed, both transferring the given integration problem to that of solving related connections. In case of the first-order system represented by a connection Ξ on $\pi_{1,0}$, the *method of characteristics* means that the uniquely determined 2-connection $\Gamma^{(2)} : J^1\pi \to J^2\pi$, called *characteristic* to Ξ, was solved. More specifically, a connection Ξ on $\pi_{1,0}$ is called *characterizable* if $R \circ \Xi = 0$, where R is the formal curvature map. If Ξ is characterizable and H_Ξ its horizontal distribution, then a 2-connection $\Gamma^{(2)} : J^1\pi \to J^2\pi$ (within the framework of (4) as a special type of a connection on π_1) is called the *characteristic connection* of Ξ, if its horizontal distribution $H_{\Gamma^{(2)}}$ is related to H_Ξ and the canonical *Cartan distribution* $C_{\pi_{1,0}}$ by $H_{\Gamma^{(2)}} = H_\Xi \cap C_{\pi_{1,0}}$. The distribution $H_{\Gamma^{(2)}}$ of the characteristic connection $\Gamma^{(2)}$ of Ξ is called the *characteristic distribution* of Ξ and the integral manifolds of $H_{\Gamma^{(2)}}$ of maximal dimension are called the *characteristics* of the connection Ξ. Clearly, the maximal integral manifolds of H_Ξ (integral sections of Ξ) are foliated by the maximal integral manifolds of the characteristic distribution (characteristics, 1-jet prolongations of integral sections of the characteristic connection). Accordingly, integral sections of Ξ are "glued together" from the characteristics.

Conversely, the *method of fields of paths* for the second-order system represented by $\Gamma^{(2)}$ was introduced. First, we have shown that if Ξ is an integrable characterizable connection on $\pi_{1,0}$ and $\Gamma^{(2)}$ its characteristic 2-connection on π, then $\Gamma^{(2)}$ is integrable, and each integral section of $\Gamma^{(2)}$ is locally embedded in a field of paths Γ, which is an integral section of Ξ. A connection $\Gamma : Y \supset V \to J^1\pi$ is a *field of paths* of $\Gamma^{(2)}$ if and only if $\Gamma(V)$ is foliated by first jets of integral sections of $\Gamma^{(2)}$, i.e., if $H_{\Gamma^{(2)}}|_{\Gamma(V)} \equiv C^\Gamma_{\pi_{1,0}}$. Then a local connection Ξ on $\pi_{1,0}$ is called an *integral* of $\Gamma^{(2)}$ if Ξ is integrable and $\Gamma^{(2)}$ is its characteristic connection. We have shown that the existence of integrals of an integrable 2-connection is a direct consequence of the integrability property, and that one can construct an integral of $\Gamma^{(2)}$ by means of a set of independent first integrals of $H_{\Gamma^{(2)}}$. This procedure generalized the well-known *Hamilton–Jacobi theory* of calculus of variations in the sense of [49], [50] (and afterwards [53]), to non-variational and partial differential equations.

The complementary constructions (in the sense of decompositions generated by a connection in question) were introduced in [82], having to do with symmetries of corresponding equations. Here, the *vertical prolongations* of first and second-order connections in the sense of [36], [37], [40], [61], and [88] appeared to be of importance, and certain related "strong horizontal" concepts (*reduced connections*) were established, following the ideas of [88].

The application of the above formalism for the first and second order ordinary differential equations was presented in [83].

3. Higher-order equations represented by connections

A natural requirement was to generalize the whole theory to the higher-order situation, i.e., to higher order connections and equations. This has been done in [84], the material of which has been prepared for publication in [85] (for O.D.E.) and [86]. In what follows, we thus work with standard framework and notation of jet prolongations of a fibered manifold $\pi : Y \to X$, according to [72]. Following the aim of this paper and space limitations, we do not give precise description of all notions we work with and we refer to the above mentioned papers for full details.

First, the equations represented by higher-order connections are described, following and combining the ideas and formalism of [2], [6], [29], [62], [63], [67], [72], [77].

By a *k-th order differential equation* on a fibered manifold $\pi : Y \to X$ is meant a fibered submanifold $\mathcal{E}^{(k)}$ of $\pi_k : J^k\pi \to X$ such that

$$\pi_{k,k-1}^{-1} \circ \pi_{k,k-1}\left(\mathcal{E}^{(k)}\right) \neq \mathcal{E}^{(k)}.$$

A *solution* of $\mathcal{E}^{(k)}$ is a section $\gamma \in \mathcal{S}_U(\pi)$ such that $j^k\gamma \subset \mathcal{E}^{(k)}$. Equations are frequently defined by fibered morphisms. Thus if $\Phi : J^k\pi \to Y'$ is a fibered morphism of constant rank between π_k and π' over X, the corresponding *differential operator* is the mapping $\mathcal{D}_\Phi : \mathcal{S}_{\mathrm{loc}}(\pi) \to \mathcal{S}_{\mathrm{loc}}(\pi')$ defined by $\mathcal{D}_\Phi(\gamma)(x) = (\Phi \circ j^k\gamma)(x)$, and for any $\psi \in \mathcal{S}_U(\pi')$ satisfying $\psi(U) \subset \mathrm{Im}\Phi$, the k-th order differential equation determined by Φ and ψ is

$$\mathcal{E}^{(k)}_{\Phi,\psi} = \ker_\psi \Phi = \left\{ j^k_x\gamma;\, \Phi(j^k_x\gamma) = \psi(x) \right\} \subset J^k\pi.$$

Accordingly, a solution of $\mathcal{E}^{(k)}_{\Phi,\psi}$ is $\gamma \in \mathcal{S}_V(\pi)$ such that $\mathcal{D}_\Phi(\gamma) = \psi|_V$, which in coordinates means a system of P.D.E.

$$\Phi^\sigma\left(x^i, \gamma^\lambda(x^i), \frac{\partial\gamma^\lambda}{\partial x^j}(x^i), \dots, \frac{\partial^k\gamma^\lambda}{\partial x^{j_1}\cdots\partial x^{j_k}}(x^i)\right) = \psi^\sigma(x^i),$$

where $\sigma = 1, \dots, \dim\pi'$. The *Cartan distribution* of the k-th order equation $\mathcal{E}^{(k)} \subset J^k\pi$ is the intersection

$$C^{\mathcal{E}^{(k)}} = C_{\pi_{k,k-1}} \cap T\mathcal{E}^{(k)},$$

carrying the most important information on the equation.

The *equation of order* $(k+1)$ *represented by a* $(k+1)$*-connection* $\Gamma^{(k+1)} : J^k\pi \to J^{k+1}\pi$ *on* π is the submanifold

$$\mathcal{E}^{\Gamma^{(k+1)}} = \Gamma^{(k+1)}(J^k\pi) \subset J^{k+1}\pi,$$

realizing (generally nonlinear) system of P.D.E. *in normal form*, i.e., explicitly solved with respect to the highest derivatives:

$$\frac{\partial^{k+1}\gamma^\sigma}{\partial x^{j_1}\cdots\partial x^{j_{k+1}}} = \Gamma^\sigma_{j_1\cdots j_{k+1}}\left(x^i, \gamma^\lambda, \dots, \frac{\partial^k\gamma^\lambda}{\partial x^{j_1}\cdots\partial x^{j_k}}\right). \tag{6}$$

A section $\gamma \in \mathcal{S}_{\mathrm{loc}}(\pi)$ is called the *integral section* (*path*) of $\Gamma^{(k+1)}$ if it is the solution of $\mathcal{E}^{\Gamma^{(k+1)}}$; i.e., if $j^{k+1}\gamma = \Gamma^{(k+1)} \circ j^k\gamma$. Evidently,

$$\mathcal{E}^{\Gamma^{(k+1)}} \equiv \mathcal{E}^{(k+1)}_{\nabla_{\Gamma^{(k+1)}},0},$$

which corresponds to the characterization of integral sections as those $\gamma \in \mathcal{S}_{\mathrm{loc}}(\pi)$ whose *covariant derivative* $\nabla_{\Gamma^{(k+1)}}(\gamma) := \nabla_{\Gamma^{(k+1)}} \circ j^{k+1}\gamma$ vanishes. On the other hand, a $(k+1)$-connection $\Gamma^{(k+1)}$ represents a *Pfaffian system*

$$\left.\begin{aligned} \omega^\sigma &= 0 \\ &\vdots \\ \omega^\sigma_{j_1\cdots j_{k-1}} &= 0 \\ \omega^{\Gamma^{(k+1)}\sigma}_{j_1\cdots j_k} &= 0 \end{aligned}\right\} \equiv \left\{\begin{aligned} dy^\sigma &= y^\sigma_i\, dx^i \\ &\vdots \\ dy^\sigma_{j_1\cdots j_{k-1}} &= y^\sigma_{j_1\cdots j_{k-1}i}\, dx^i \\ dy^\sigma_{j_1\cdots j_k} &= \Gamma^\sigma_{j_1\cdots j_k i}\, dx^i, \end{aligned}\right. \tag{7}$$

hence $\gamma \in \mathcal{S}_U(\pi)$ is an integral section of $\Gamma^{(k+1)}$ if, and only if, $j^k\gamma(U)$ is an integral manifold of $H_{\Gamma^{(k+1)}}$, i.e., for each $x \in U$ it holds $T_x j^k\gamma(T_xU) \subset H_{\Gamma^{(k+1)}}(j^k_x\gamma)$. In terms of $h_{\Gamma^{(k+1)}}$ it means

$$h_{\Gamma^{(k+1)}}|_{j^k\gamma} \equiv Tj^k\gamma \circ T\pi_k : T_{j^k_x\gamma}J^k\pi \to T_{j^k_x\gamma}J^k\pi.$$

A $(k+1)$-connection $\Gamma^{(k+1)}$ on π is *integrable* if, and only if, one of the following equivalent conditions holds:

– For an arbitrary $y \in Y$, there is a unique integral section of $\Gamma^{(k+1)}$ passing through it.

– The horizontal distribution $H_{\Gamma^{(k+1)}}$ is completely integrable.

– $[D_{\Gamma^{(k+1)}i}, D_{\Gamma^{(k+1)}p}] = 0$ for all i, p.

– The connection $\Gamma^{(k+1)}$ is *flat*, i.e., $R_{\Gamma^{(k+1)}} = 0$.

– $J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)} \in \mathrm{J}^{k+2}\pi$.

– The components of $\Gamma^{(k+1)}$ satisfy $D_{\Gamma^{(k+1)}i}(\Gamma^\sigma_{j_1\ldots j_k p}) = D_{\Gamma^{(k+1)}p}(\Gamma^\sigma_{j_1\cdots j_k i})$ for arbitrary $i, p = 1, \ldots, n$.

Denote by

$$C^{\Gamma^{(k+1)}} := C_{\pi_{k+1,k}} \cap T\Gamma^{(k+1)}(J^k\pi)$$

the Cartan distribution of the equation represented by $\Gamma^{(k+1)}$. Clearly, it is a regular n-dimensional distribution on the submanifold $\Gamma^{(k+1)}(J^k\pi) \subset J^{k+1}\pi$, annihilated by the forms $\omega^\sigma_{j_1\cdots j_\ell}$ ($\ell = 0, \ldots, k-1$) together with $dy^\sigma_{j_1\cdots j_k} - \Gamma^\sigma_{j_1\cdots j_k i}\, dx^i$ and $dy^\sigma_{j_1\cdots j_{k+1}} - d\Gamma^\sigma_{j_1\cdots j_{k+1}}$, or equivalently spanned by the vector fields

$$\begin{aligned} &T\Gamma^{(k+1)}(D_{\Gamma^{(k+1)}i}) \\ &\quad = \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k-1} y^\sigma_{j_1\ldots j_\ell i}\frac{\partial}{\partial y^\sigma_{j_1\cdots j_\ell}} + \Gamma^\sigma_{j_1\cdots j_k i}\frac{\partial}{\partial y^\sigma_{j_1\cdots j_k}} \\ &\quad + D_{\Gamma^{(k+1)}p}(\Gamma^\sigma_{j_1\cdots j_k i})\frac{\partial}{\partial y^\sigma_{j_1\ldots j_k p}}. \end{aligned}$$

Then it is easy to prove that a $(k+1)$-connection $\Gamma^{(k+1)}$ on π is integrable if, and only if, the distribution $C^{\Gamma^{(k+1)}}$ is completely integrable, and a section γ is an integral section of $\Gamma^{(k+1)}$ if, and only if, $j^{k+1}\gamma$ is the integral mapping of $C^{\Gamma^{(k+1)}}$.

In accordance with the above general situation, a $(k+1)$-connection $\Gamma^{(k+1)}$ on $\pi : \mathbf{R} \times M \to \mathbf{R}$ is a section $\Gamma^{(k+1)} : \mathbf{R} \times T^k M \to \mathbf{R} \times T^{k+1} M$ of $\mathrm{id}_{\mathbf{R}} \times \tau_{\mathrm{M}}^{\mathrm{k+1,k}}$. Any $(k+1)$-connection is characterized by its horizontal form $h_{\Gamma^{(k+1)}} = D_{\Gamma^{(k+1)}} \otimes dt$, where the absolute derivative

$$(8) \qquad D_{\Gamma^{(k+1)}} = \frac{\partial}{\partial t} + \sum_{i=0}^{k-1} q^{\sigma}_{(i+1)} \frac{\partial}{\partial q^{\sigma}_{(i)}} + \Gamma^{\sigma}_{(k+1)} \frac{\partial}{\partial q^{\sigma}_{(k)}}$$

is the so-called *semispray* on $\mathbf{R} \times T^k M$, defining the one-dimensional π_k-horizontal *semispray distribution* $H_{\Gamma^{(k+1)}}$. Due to the product structure and analogously to the first-order case, $\Gamma^{(k+1)}$ can be represented by the vector field

$$w^{(k+1)} = \sum_{i=0}^{k-1} q^{\sigma}_{(i+1)} \frac{\partial}{\partial q^{\sigma}_{(i)}} + \Gamma^{\sigma}_{(k+1)} \frac{\partial}{\partial q^{\sigma}_{(k)}}$$

along $\mathrm{pr}_2 : \mathbf{R} \times \mathrm{T^k M} \to \mathrm{T^k M}$, which is nothing but a *time-dependent semispray* on $T^k M$; in the autonomous situation, a semispray on $T^k M$ is a section of $\tau_M^{k+1,k} : T^{k+1} M \to T^k M$.

The $(k+1)$*-th order* (generally nonlinear) *system of O.D.E. represented by a* $(k+1)$*-connection* $\Gamma^{(k+1)}$ on $\pi : \mathbf{R} \times M \to \mathbf{R}$ can be described both globally as the $((k+1)\, m + 1)$-dimensional submanifold

$$\Gamma^{(k+1)}(\mathbf{R} \times T^k M) \subset \mathbf{R} \times T^{k+1} M$$

of $\mathbf{R} \times T^{k+1} M$ and locally by

$$(9) \qquad \frac{d^{k+1} c^{\sigma}}{dt^{k+1}} = \Gamma^{\sigma}_{(k+1)} \left(t, c^{\lambda}, \ldots, \frac{d^k c^{\lambda}}{dt^k} \right);$$

the Pfaffian version of which is

$$dq^{\sigma} = q^{\sigma}_{(1)}\, dt, \quad \ldots \quad , \quad dq^{\sigma}_{(k-1)} = dq^{\sigma}_{(k)}\, dt, \quad dq^{\sigma}_{(k)} = \Gamma^{\sigma}_{(k+1)}\, dt.$$

The integral sections of $\Gamma^{(k+1)}$ are thus the ‘graphs’ of the geodesics of the above semispray $w^{(k+1)}$ in the sense that $w^{(k+1)} \circ j^{k+1}\gamma = c^{(k+1)}$.

4. Prolongations and fields of paths

The *r-th jet prolongation* of the equations is studied. In general, the prolongation of an equation carries the information on the equation together with a given number of “consequences”, obtained by differentiating the original equation. In case of connections, the construction of the prolongation in terms of the prolongations of corresponding morphisms results in a very transparent characterization, which follows the definition of a *field of paths* as a local lower-order connection representing an order-reduction of the initial equations.

Let $\mathcal{E}^{(k)} \subset J^k \pi$ be a k-th order equation on π. The *r-th prolongation* of $\mathcal{E}^{(k)}$ is the subset

$$\mathcal{E}^{(k)(r)} = J^r \mathcal{E}^{(k)} \cap J^{k+r} \pi$$

with $J^r\mathcal{E}^{(k)} \subset J^r\pi_k$. For the equation $\mathcal{E}^{(k)}_{\Phi,\psi}$ defined by a fibered morphism (Ψ, ψ),

$$\mathcal{E}^{(k)(r)}_{\Phi,\psi} = \left\{ j^{k+r}_x\gamma;\ J^r(\Phi, \mathrm{id_X}) \circ \iota_{\mathrm{r,k}}(\mathrm{j^{k+r}_x}\gamma) = \mathrm{j^r_x}\psi \right\} \subset J^{k+r}\pi$$

is again a differential equation, now of the $(k+r)$-th order. In fact, $\mathcal{E}^{(k)(r)}_{\Phi,\psi}$ represents the family of P.D.E. obtained by differentiating the original equations $0, 1, \dots, r$ times with respect to the independent variables.

Let $k \geq 0$ and $\Gamma^{(k+1)} : J^k\pi \to J^{k+1}\pi$ be an integrable $(k+1)$-connection on π. The *r-th prolongation* of the equation $\mathcal{E}^{\Gamma^{(k+1)}} \subset J^{k+1}\pi$ represented by $\Gamma^{(k+1)}$ is defined to be the submanifold

$$\mathcal{E}^{\Gamma^{(k+1)}(r)} = \mathrm{Im}\Gamma^{\mathrm{(k+1)(r)}} \subset \mathrm{J^{k+r+1}}\pi,$$

where $\Gamma^{(k+1)(r)}$ is the last term of the sequence of sections

$$\left(\Gamma^{(k+1)(0)}, \Gamma^{(k+1)(1)}, \dots, \Gamma^{(k+1)(r)}\right)$$

recurrently defined for each $\ell = 1, \dots, r$ by

$$\Gamma^{(k+1)(\ell)} := J^1\left(\Gamma^{(k+1)(\ell-1)}, \mathrm{id_X}\right) \circ \iota_{1,\mathrm{k}} \circ \Gamma^{\mathrm{(k+1)}} : \mathrm{J^k}\pi \to \mathrm{J^{k+\ell+1}}\pi$$

with $\Gamma^{(k+1)(0)} := \Gamma^{(k+1)}$.

Then it is easy to see that the equation $\mathcal{E}^{\Gamma^{(k+1)}(r)}$ consists of $(k+r+1)$-jets of integral sections of $\Gamma^{(k+1)}$; in fact $j^{k+r+1}\gamma = \Gamma^{(k+1)(r)} \circ j^k\gamma$.

By the *r-th order Cartan distribution* $C^{\Gamma^{(k+1)}(r)}$ of an integrable $(k+1)$-connection $\Gamma^{(k+1)}$ on π is meant the Cartan distribution of the r-th prolongation $\mathcal{E}^{\Gamma^{(k+1)}(r)}$, i.e.,

$$C^{\Gamma^{(k+1)}(r)} := C_{\pi_{k+r+1,k+r}} \cap T\Gamma^{(k+1)(r)}(J^k\pi).$$

By definition, $C^{\Gamma^{(k+1)}(0)} = C^{\Gamma^{(k+1)}}$, and $C^{\Gamma^{(k+1)}(r)}$ is a regular n-dimensional distribution on $\Gamma^{(k+1)(r)}(J^k\pi) \subset J^{k+r+1}\pi$ annihilated by the forms $\omega^\sigma, \dots, \omega^\sigma_{j_1\cdots j_{k+1}}$ restricted to $\Gamma^{(k+1)(r)}(J^k\pi)$ together with

$$dy^\sigma_{j_1\cdots j_{k+1}i_1\dots i_r} - d\left(D_{i_1\dots i_r}(\Gamma^\sigma_{j_1\cdots j_{k+1}})\right) \circ \Gamma^{(k+1)(r-1)},$$

or equivalently spanned by the vector fields $T\Gamma^{(k+1)(r)}(D_{\Gamma^{(k+1)}i})$. Let $r \geq 1$. Then

$$T\Gamma^{(k+1)(r-1)} \circ D_{\Gamma^{(k+1)}i} = D^{k+r+1,k+r}_i \circ \Gamma^{(k+1)(r)}.$$

Let $k \geq 0, r \geq 1$. A $(k+1)$-connection $\Gamma^{(k+1)}$ on π is integrable if, and only if, its r-th order Cartan distribution $C^{\Gamma^{(k+1)}(r)}$ is completely integrable, and a section γ is an integral section of $\Gamma^{(k+1)}$ if, and only if, $j^{k+r+1}\gamma$ is the integral mapping of $C^{\Gamma^{(k+1)}(r)}$.

A $(k+1)$-connection $\Gamma^{(k+1)} \in \mathcal{S}_V(\pi_{k+1,k})$ is called a *field of paths* of a $(k+r+1)$-connection $\Gamma^{(k+r+1)} : J^{k+r}\pi \to J^{k+r+1}\pi$ if on V holds

$$\Gamma^{(k+r+1)} \circ \Gamma^{(k+1)(r-1)} = \Gamma^{(k+1)(r)}.$$

By definition, each field of paths is integrable, and

$$H_{\Gamma^{(k+r+1)}}|_{\Gamma^{(k+1)(r-1)}(V)} \equiv C^{\Gamma^{(k+1)}(r-1)}.$$

Equivalently, if γ is an integral section of $\Gamma^{(k+1)}$, then if γ is an integral section of a field of paths $\Gamma^{(k+1)}$ of $\Gamma^{(k+r+1)}$, then it is the integral section (a path) of $\Gamma^{(k+r+1)}$. In other words, $H_{\Gamma^{(k+1)}}$ defines a foliation of V such that each leaf of this foliation is an integral section of $\Gamma^{(k+r+1)}$.

Globally speaking, each field of paths represents a (local) order-reduction of the given equation. In this respect, the problem of finding the integral sections of a given integrable higher-order connection can be transferred to the problem of looking for and then solving of its fields of paths; the transitivity of the relation 'to be a field of paths of a higher-order connection' is evident. In this respect, the *method of fields of paths* will be discussed later on.

5. Symmetries and vertical prolongations

Infinitesimal symmetries as the generators of invariant transformations in sense of [28] are studied in terms of the corresponding decompositions on tangent bundles. The use of the *vertical prolongation* $\mathcal{V}\Gamma^{(k+1)}$ finds its application within the 2-fibered manifold

$$V_{\pi_k} J^k\pi \xrightarrow{\tau_{J^k\pi}} J^k\pi \xrightarrow{\pi_k} X,$$

where a linear connection on $\tau_{J^k\pi}$ whose integral sections are the symmetries is found. Finally, the relations between symmetries for a connection and its field of paths are derived, again in terms of vertical prolongations.

Let $\Gamma^{(k+1)}$ be an integrable $(k+1)$-connection on π, and let $\zeta^{(k)} \in \mathcal{X}(J^k\pi)$. Then there is a direct sum decomposition of its r-th prolongation

$$\begin{aligned}\mathcal{J}^r\zeta^{(k)} \circ \Gamma^{(k+1)(r-1)} &= \mathcal{J}^r(h_{\Gamma^{(k+1)}} \circ \zeta^{(k)}) \circ \Gamma^{(k+1)(r-1)} \\ &\quad + V\Gamma^{(k+1)(r-1)} \circ v_{\Gamma^{(k+1)}} \circ \zeta^{(k)} + (\mathcal{J}^r\zeta^{(k)} \circ \Gamma^{(k+1)(r-1)})^{\pi_{k+r,k}},\end{aligned}$$

where

$$\begin{aligned}&\mathcal{J}^r(h_{\Gamma^{(k+1)}} \circ \zeta^{(k)}) \circ \Gamma^{(k+1)(r-1)} \in C^{\Gamma^{(k+1)}(r-1)},\\ &V\Gamma^{(k+1)(r-1)} \circ v_{\Gamma^{(k+1)}} \circ \zeta^{(k)} \in V\Gamma^{(k+1)(r-1)}(V_{\pi_k} J^k\pi),\\ &(\mathcal{J}^r\zeta^{(k)} \circ \Gamma^{(k+1)(r-1)})^{\pi_{k+r,k}} \in V_{\pi_{k+r,k}} J^{k+r}\pi.\end{aligned}$$

The decomposition represents a contribution to the *internal* geometry of equations under consideration, and as such it can be viewed as an internal version of results presented in terms of the so-called *characterizable connections*. The bridge between these points of view is created by fields of paths. For instance, for $\Gamma^{(k+1)}$ being a field of paths of $\Gamma^{(k+r+1)}$ it holds

$$\mathcal{J}^r D_{\Gamma^{(k+1)}i} \circ \Gamma^{(k+1)(r-1)} = D_{\Gamma^{(k+r+1)}i} \circ \Gamma^{(k+1)(r-1)}.$$

In what follows, $\Gamma^{(k+1)}$ is supposed to be an integrable $(k+1)$-connection on π. A vector field $\zeta^{(k)} \in \mathcal{X}(J^k\pi)$ is called a *k-th order symmetry* (briefly *k-symmetry*) of $\Gamma^{(k+1)}$ if $\zeta^{(k)}$ and $\mathcal{J}^1\zeta^{(k)}$ are $\Gamma^{(k+1)}$-related, i.e.,

$$\mathcal{J}^1\zeta^{(k)} \circ \Gamma^{(k+1)} = T\Gamma^{(k+1)} \circ \zeta^{(k)}.$$

The set of all k-symmetries of $\Gamma^{(k+1)}$ is denoted by $\mathrm{Sym}^{(k)}(\Gamma^{(k+1)})$.

It is evident that any $\Gamma^{(k+1)}$-horizontal vector field is a k-symmetry of $\Gamma^{(k+1)}$, which leads to the fact that a vector field $\zeta^{(k)} \in \mathcal{X}(J^k\pi)$ is a k-symmetry of $\Gamma^{(k+1)}$ if, and only if one of the following equivalent conditions holds:

$$\mathcal{J}^1(v_{\Gamma^{(k+1)}} \circ \zeta^{(k)}) \circ \Gamma^{(k+1)} = V\Gamma^{(k+1)} \circ v_{\Gamma^{(k+1)}} \circ \zeta^{(k)},$$
$$\mathcal{L}_{v_{\Gamma^{(k+1)}}(\zeta^{(k)})} h_{\Gamma^{(k+1)}} = 0.$$

In this arrangement, the k-symmetries of $\Gamma^{(k+1)}$ are just the symmetries of the horizontal distribution $H_{\Gamma^{(k+1)}}$.

A π_k-projectable vector field $\zeta^{(k)}$ on $J^k\pi$ is a k-symmetry of $\Gamma^{(k+1)}$ if, and only if, equivalently $\mathcal{L}_{\zeta^{(k)}} h_{\Gamma^{(k+1)}} = 0$ ($\mathcal{L}$ is here the Lie derivative) or the flow of $\zeta^{(k)}$ permutes the k-jets of integral sections of $\Gamma^{(k+1)}$.

Denote by $\mathrm{Sym}^{(k)}_{v}(\Gamma^{(k+1)}) \subset \mathrm{Sym}^{(k)}(\Gamma^{(k+1)})$ the submodule of π_k-vertical k-symmetries of $\Gamma^{(k+1)}$, by $\mathrm{Char}(H_{\Gamma^{(k+1)}})$ the ideal of *characteristic symmetries* of $H_{\Gamma^{(k+1)}}$ (e.i. those lying within $H_{\Gamma^{(k+1)}}$) and by $\mathrm{Shuf}(H_{\Gamma^{(k+1)}})$ the quotient algebra

$$\mathrm{Shuf}(H_{\Gamma^{(k+1)}}) = \mathrm{Sym}(H_{\Gamma^{(k+1)}})/\mathrm{Char}(H_{\Gamma^{(k+1)}})$$

of the so-called *shuffling symmetries*. Recall that while the flow of a characteristic symmetry moves integral manifolds along themselves, any shuffling symmetry represents the whole class of symmetries whose flow rearranges the integral manifolds in the same way. Then it holds $H_{\Gamma^{(k+1)}} \cong \mathrm{Char}(H_{\Gamma^{(k+1)}})$ and $\mathrm{Sym}^{(k)}_{v}(\Gamma^{(k+1)}) \cong \mathrm{Shuf}(\Gamma^{(k+1)})$.

The structure of higher-order jet prolongations and corresponding projections allowed us to define some other types of symmetries. A vector field $\zeta^{(r)} \in \mathcal{X}(J^r\pi)$, $0 \le r \le k-1$, is called the *r-symmetry* of $\Gamma^{(k+1)}$ if $\mathcal{J}^{k-r}\zeta^{(r)} \in \mathrm{Sym}^{(k)}(\Gamma^{(k+1)})$. The set of all r-symmetries of $\Gamma^{(k+1)}$ is denoted by $\mathrm{Sym}^{(r)}(\Gamma^{(k+1)})$. Then a π_r-projectable vector field $\zeta^{(r)}$ on $J^r\pi$ is the r-symmetry of $\Gamma^{(k+1)}$ if, and only if, its flow permutes the r-jets of integral sections of $\Gamma^{(k+1)}$.

Of course, our main concern is with vector fields on Y as generators of invariant transformations on sections; in this respect, zero-symmetries are referred to briefly as *symmetries*. In this case, $\zeta \in \mathcal{X}(Y)$ is a symmetry of an integrable $\Gamma^{(k+1)}$ if, and only if, one of the following equivalent conditions holds:

$$\mathcal{J}^{k+1}\zeta \circ \Gamma^{(k+1)} = T\Gamma^{(k+1)} \circ \mathcal{J}^k\zeta,$$
$$\mathcal{J}^1(v_{\Gamma^{(k+1)}} \circ \mathcal{J}^k\zeta) \circ \Gamma^{(k+1)} = V\Gamma^{(k+1)} \circ v_{\Gamma^{(k+1)}} \circ \mathcal{J}^k\zeta,$$
$$\mathcal{L}_{v_{\Gamma^{(k+1)}}(\mathcal{J}^k\zeta)} h_{\Gamma^{(k+1)}} = 0,$$
$$[D_{\Gamma^{(k+1)}i}, \mathcal{J}^k\zeta] = D_{\Gamma^{(k+1)}i}(\zeta^j)\, D_{\Gamma^{(k+1)}j},$$

where $D_{\Gamma^{(k+1)}i}(\zeta^j)$ denotes briefly just $D_{\Gamma^{(k+1)}i}(\pi^*_{k,0}(\zeta^j)) \equiv \pi^*_{k,1}(D^{1,0}(\zeta^j))$. If in addition $\zeta \in \mathcal{X}_X(Y)$, then it is a symmetry of $\Gamma^{(k+1)}$ if, and only if, its flow permutes the integral sections of $\Gamma^{(k+1)}$.

The symmetries of the Cartan distribution $C_{\pi_{k,k-1}}$ on $J^k\pi$ are called *contact* vector fields. By the well-known *Bäcklund's theorem*, in the case of $m = \dim\pi = 1$ and if $\zeta^{(k)}$ is contact, then it is the $(k-1)$-th prolongation of a contact vector field on $J^1\pi$. If $m > 1$, then $\zeta^{(k)}$ is the k-th prolongation of a vector field on Y. In this respect, the *external symmetry* of an equation $\mathcal{E}^{(k)} \subset J^k\pi$ is a contact vector field on $J^k\pi$ tangent to $\mathcal{E}^{(k)}$. In other words, its flow preserves both the Cartan distribution and the equation. The restriction of an external symmetry to $\mathcal{E}^{(k)}$ defines a symmetry of $C^{\mathcal{E}^{(k)}}$ and just the symmetries of the distribution $C^{\mathcal{E}^{(k)}}$ are called the *internal symmetries* of the equation $\mathcal{E}^{(k)}$. It can be shown that $\zeta^{(r)} \in \mathcal{X}(J^r\pi)$ is the r-symmetry of an integrable $\Gamma^{(k+1)}$

if $\mathcal{J}^{k-r+1}\zeta^{(r)}|_{\Gamma^{(k+1)}(J^k\pi)} \in \mathcal{X}(\Gamma^{(k+1)}(J^k\pi))$ is an internal symmetry of $\Gamma^{(k+1)}(J^k\pi)$. In particular, if $\zeta \in \mathcal{X}_X(Y)$ is such that $\mathcal{J}^{k+1}\zeta \circ \Gamma^{(k+1)} \in C^{\Gamma^{(k+1)}}$, then its flow acts on the integral sections of $\Gamma^{(k+1)}$ trivially – moves them along themselves. On the other hand, a π-vertical symmetry can be viewed as representing the whole class of symmetries rearranging the integral sections in the same way.

The *vertical prolongation* $\mathcal{V}\Gamma^{(k+1)}$ of $\Gamma^{(k+1)}$ is a $(k+1)$-connection on $(\pi \circ \tau_Y|_{V_\pi Y})$ defined by

$$V\Gamma^{(k+1)} \circ \nu_k = \nu_{k+1} \circ \mathcal{V}\Gamma^{(k+1)},$$

which is projectable over $\Gamma^{(k+1)}$ within the 2-fibered manifold

$$J^k(\pi \circ \tau_Y|_{V_\pi Y}) \xrightarrow{J^k(\tau_Y|_{V_\pi Y}, \mathrm{id_X})} J^k\pi \xrightarrow{\pi_k} X.$$

In fact, to eliminate the formalism including the ν's, we work with a slight inaccuracy directly with the izomorphic

$$V_{\pi_k}J^k\pi \xrightarrow{\tau_{J^k\pi}} J^k\pi \xrightarrow{\pi_k} X.$$

Then the following assertion can be easily verified by means of the results obtained in [26].

Let $\Gamma^{(k+1)}$ be an integrable $(k+1)$-connection on π and Ψ a connection on $\tau_{J^k\pi} : V_{\pi_k}J^k\pi \to J^k\pi$, satisfying $\mathbf{k}_{\Gamma^{(k+1)}} \circ \Psi = \mathcal{V}\Gamma^{(k+1)}$, (where $\mathbf{k}_{\Gamma^{(k+1)}}$ is defined analogously to Section 2 in Section 6). Then if $\zeta^{(k)} \in \mathcal{X}_X(J^k\pi)$ is an integral section of Ψ, then $\zeta^{(k)} \in \mathrm{Sym}_{\mathrm{v}}^{(\mathrm{k})}(\Gamma^{(\mathrm{k+1})})$.

Let finally $\Gamma^{(k+1)} \in \mathcal{S}_V(\pi_{k+1,k})$ be a field of paths of $\Gamma^{(k+r+1)} : J^{k+r}\pi \to J^{k+r+1}\pi$. Then one might ask on the relationship between the vertical (zeroth-order) symmetries of the above connections. First, since each integral section of $\Gamma^{(k+1)}$ is the integral section of $\Gamma^{(k+r+1)}$, then if $\zeta \in \mathcal{X}_X^v(Y)$ is a symmetry of $\Gamma^{(k+r+1)}$, then $\zeta|_{\pi_{k,0}(V)}$ is a symmetry of $\Gamma^{(k+1)}$. To obtain the well-known result affirming that each vertical symmetry of an equation is the symmetry of its prolongation, the relation between the corresponding vertical prolongations must be clarified. In fact, one can prove that a $(k+1)$-connection $\Gamma^{(k+1)}$ is a field of paths of a $(k+r+1)$-connection $\Gamma^{(k+r+1)}$ if, and only if, $\mathcal{V}\Gamma^{(k+1)}$ is a field of paths of $\mathcal{V}\Gamma^{(k+r+1)}$. As a corollary, one gets: if ζ is a symmetry of $\Gamma^{(k+1)}$, then it is a symmetry of $\Gamma^{(k+1)(r)}$.

6. Characterizable connections

The most interesting part of the theory is that having to do with the interrelations between equations represented by connections on various fibrations.

In this part, the 2-fibered manifold $J^{k+1}\pi \xrightarrow{\pi_{k+1,k}} J^k\pi \xrightarrow{\pi_k} X$ finds the application. The generalization of the diagramm (4) is thus

(10)
$$\begin{array}{ccccccc}
X & \xleftarrow{(\pi_k)_1} & J^1\pi_k & \xleftarrow{J^1(\pi_{k+1,k},\mathrm{id_X})} & J^1\pi_{k+1} & & \\
\Big\downarrow{\scriptstyle \mathrm{id_X}} & & \Big\downarrow{\scriptstyle (\pi_k)_{1,0}} & & \Big\downarrow{\scriptstyle (\pi_{k+1})_{1,0}} & & \\
X & \xleftarrow{\pi_k} & J^k\pi & \xleftarrow{\pi_{k+1,k}} & J^{k+1}\pi & \xleftarrow{(\pi_{k+1,k})_{1,0}} & J^1\pi_{k+1,k} \\
\Big\downarrow{\scriptstyle \mathrm{id_X}} & & \Big\downarrow{\scriptstyle \pi_k} & & \Big\downarrow{\scriptstyle \pi_{k+1}} & & \\
X & \xleftarrow{\mathrm{id_X}} & X & \xleftarrow{\mathrm{id_X}} & X. & &
\end{array}$$

The canonical map $\mathbf{k} : J^1\pi_k \times_{J^k\pi} J^1\pi_{k+1,k} \to J^1\pi_{k+1}$ does not effect the coordinates up to $y^\sigma_{j_1\cdots j_k;\,i}$ and its equations are

$$y^\sigma_{j_1\cdots j_{k+1};\,i} = z^\sigma_{j_1\cdots j_{k+1}i} + \sum_{\ell=0}^{k} z^\sigma r_1 \cdots r_\ell j_1 \cdots j_{k+1}\lambda \, y^\lambda_{r_1\cdots r_\ell;\,i},$$

where by

$$z^\sigma_{j_1\cdots j_{k+1}i}, z^\sigma_{j_1\cdots j_{k+1}\lambda}, \ \ldots, \ z^\sigma i_1 \cdots i_{k\,j_1\cdots j_{k+1}\lambda}$$

we denote the induced derivative coordinates on $J^1\pi_{k+1,k}$. The first order of business is to mention the role of the canonical embedding $\iota_{1,k} : J^{k+1}\pi \hookrightarrow J^1\pi_k$, which is in coordinates expressed by

$$y^\sigma_{;i} = y^\sigma_i, \ldots, y^\sigma_{j_1\cdots j_k;\,i} = y^\sigma_{j_1\cdots j_k i}. \tag{11}$$

If $\Phi : J^{k+1}\pi \to J^1\pi_k$ is an arbitrary fibered morphism over $J^k\pi$, then since the vertical bundle associated to $(\pi_k)_{1,0}$ is $V_{\pi_k}J^k\pi \otimes \pi_k^*(T^*X)$, the difference $\Phi - \iota_{1,k}$ is a fibered morphism $J^k\pi \to V_{\pi_k}J^k\pi \otimes \pi_k^*(T^*X)$ and thus

$$\Phi_a := \iota_{1,k} + a(\Phi - \iota_{1,k})$$

is a fibered morphism $J^{k+1}\pi \to J^1\pi_k$ over $J^k\pi$ for any $a \in \mathbf{R}$. The *formal curvature map* is then the map

$$R : J^1\pi_{k+1,k} \to \pi^*_{k+1,k}\big(V_{\pi_k}J^k\pi \otimes \pi_k^*(\Lambda^2T^*X)\big)$$

defined for each $j^1_{j^k_x\gamma}\chi \in J^1\pi_{k+1,k}$ by

$$R(j^1_{j^k_x\gamma}\chi) = r_{k+1} \circ J^1(\chi, \mathrm{id_X}) \circ \iota_{1,\mathrm{k}} \circ \chi(\mathrm{j^k_x}\gamma).$$

Then

$$R : J^1\pi_{k+1,k} \to V_{\pi_{k+1}}J^{k+1}\pi \otimes \pi^*_{k+1}(T^*X).$$

Consequently, one can define (for $a, b \in \mathbf{R}$) the affine morphism

$$\mathbf{k}^{a,b}_\Phi : J^1\pi_{k+1,k} \to J^1\pi_{k+1}$$

between $(\pi_{k+1,k})_{1,0}$ and $(\pi_{k+1})_{1,0}$ over $J^{k+1}\pi$ by $\mathbf{k}^{a,b}_\Phi = \mathbf{k}_{\Phi_a} + bR$.

It is easy to see that regarding a curvature of the connections in question, one gets that

$$\begin{aligned} R_{\Gamma^{(k+1)}} &= -\mathrm{pr}_2 \circ \mathrm{R} \circ \mathrm{j}^1\Gamma^{(\mathrm{k+1})} \\ &= -\mathrm{pr}_2 \circ \mathrm{r_{k+1}} \circ \mathrm{J}^1(\Gamma^{(\mathrm{k+1})}, \mathrm{id_X}) \circ \iota_{1,\mathrm{k}} \\ &\circ \Gamma^{(k+1)} : J^k\pi \to V_{\pi_{k,k-1}}J^k\pi \otimes \pi_k^*(\Lambda^2T^*X). \end{aligned}$$

As to be expected, the same characterization can be presented for a (first-order) connection Γ on π, i.e.,

$$R_\Gamma = -\mathrm{pr}_2 \circ \mathrm{r}_1 \circ \mathrm{J}^1(\Gamma, \mathrm{id_X}) \circ \Gamma : \mathrm{Y} \to \mathrm{V}_\pi\mathrm{Y} \otimes \pi^*(\Lambda^2\mathrm{T}^*\mathrm{X}),$$

hence $k = 0$ is allowed when speaking on the curvature of a $(k+1)$-connection on π.

Recall that $\mathbf{k}_{\iota_{1,k}} : J^1\pi_{k+1,k} \to \widehat{J}^{k+2}\pi$ is by definition

$$j^1_z\Gamma^{(k+1)} \mapsto \mathbf{k}(\Gamma^{(k+1)}(z), j^1_z\Gamma^{(k+1)}) = J^1(\Gamma^{(k+1)}, \mathrm{id_X}) \circ \iota_{1,\mathrm{k}} \circ \Gamma^{(\mathrm{k+1})}(\mathrm{z}).$$

Therefore, if $\gamma \in \mathcal{S}_U(\pi)$ is an arbitrary section of the $(k+1)$-connection $\Gamma^{(k+1)}$, then

$$\begin{aligned}\mathbf{k}_{\iota_{1,k}}(j^1_{j^k_x\gamma}\Gamma^{(k+1)}) &= J^1(\Gamma^{(k+1)}, \mathrm{id_X}) \circ \iota_{1,\mathrm{k}} \circ \Gamma^{(\mathrm{k+1})}(\mathrm{j^k_x}\gamma) \\ &= \iota_{1,k+1}(j^{k+2}_x\gamma) \in \iota_{1,k+1}(J^{k+2}\pi).\end{aligned}$$

Secondly, for any $\Gamma^{(k+1)}$, $\mathbf{k}_{\Gamma^{(k+1)}} := \mathbf{k}_{\iota_{1,k}\circ\Gamma^{(k+1)}\circ\pi_{k+1,k}} : J^1\pi_{k+1,k} \to J^1\pi_{k+1}$ reads

$$j^1_z\chi \xrightarrow{\mathbf{k}_{\Gamma^{(k+1)}}} J^1(\chi, \mathrm{id_X}) \circ \iota_{1,\mathrm{k}} \circ \Gamma^{(\mathrm{k+1})}(\mathrm{z}),$$

which means that for an integrable $\Gamma^{(k+1)}$ holds

$$\begin{aligned}\mathbf{k}_{\Gamma^{(k+1)}} \circ j^1\Gamma^{(k+1)} &= \mathbf{k}_{\iota_{1,k}} \circ j^1\Gamma^{(k+1)} J^1(\Gamma^{(k+1)}, \mathrm{id_X}) \circ \iota_{1,\mathrm{k}} \circ \Gamma^{(\mathrm{k+1})} \\ &= \Gamma^{(k+1)(1)}.\end{aligned}$$

On the other hand, let $\Xi : J^{k+1}\pi \to J^1\pi_{k+1,k}$ be a connection on $\pi_{k+1,k}$, and $\Phi : J^{k+1}\pi \to J^1\pi_k$ be a fibered morphism over $J^k\pi$. Then

$$\Sigma^{a,b}_{\Phi,\Xi} := \mathbf{k}^{a,b}_{\Phi} \circ \Xi : J^{k+1}\pi \to J^1\pi_{k+1}$$

is a connection on π_{k+1} for an arbitrary $a, b \in \mathbf{R}$. In particular, a (local) connection $\Gamma^{(k+1)}$ can be considered representing both the morphism $\Phi = \iota_{1,k} \circ \Gamma^{(k+1)} \circ \pi_{k+1,k}$ and the section of $\pi_{k+1,k}$. Then denoting by $\Sigma_{\Gamma^{(k+1)},\Xi} = \mathbf{k}_{\Gamma^{(k+1)}} \circ \Xi$, the following assertion can be presented.

Let $\Gamma^{(k+1)}$ be an integral section of a connection Ξ on $\pi_{k+1,k}$. Then γ is an integral section of $\Gamma^{(k+1)}$ if, and only if, $\Gamma^{(k+1)} \circ j^k\gamma$ is the integral section of $\Sigma_{\Gamma^{(k+1)},\Xi}$.

For an arbitrary connection Ξ on $\pi_{k+1,k}$ and $b \in \mathbf{R}$,

$$\widehat{\Gamma}^{(k+2)}_{\Xi,b} := \Sigma^{0,b}_{\iota_{1,k},\Xi} = \mathbf{k}^{0,b}_{\iota_{1,k}} \circ \Xi : J^{k+1}\pi \to \widehat{J}^{k+2}\pi$$

is a semiholonomic connection on π_{k+1}, which can be decomposed to the $(k+2)$-connection

$$\Gamma^{(k+2)}_{\Xi} := s_{k+1} \circ \widehat{\Gamma}^{(k+2)}_{\Xi,b},$$

and to a certain multiple of the composition $R \circ \Xi$ of the formal curvature R with Ξ.

Then a connection Ξ on $\pi_{k+1,k}$ is called *characterizable*, if $R \circ \Xi = 0$. The $(k+2)$-connection $\Gamma^{(k+2)}_{\Xi} = \mathbf{k}_{\iota_{1,k}} \circ \Xi$ is then called the *characteristic connection* of Ξ. Accordingly, the horizontal distribution $H_{\Gamma^{(k+2)}_{\Xi}}$ is called the *characteristic distribution* of Ξ and the maximal-dimensional integral manifolds of the characteristic distribution (i.e. $(k+1)$-jets of integral sections of $\Gamma^{(k+2)}_{\Xi}$) are the *characteristics* of Ξ.

A $(k+2)$-connection $\Gamma^{(k+2)}$ on π is the characteristic connection of a connection Ξ on $\pi_{k+1,k}$ if, and only if, one of the following equivalent conditions holds:

$$\Gamma^{\sigma}_{j_1\cdots j_{k+1}i} = \Xi^{\sigma}_{j_1\cdots j_{k+1}i} + \sum_{\ell=0}^{k} \Xi^{\sigma}r_1\cdots r_{\ell\, j_1\cdots j_{k+1}\lambda} y^{\lambda}_{r_1\cdots r_\ell i};$$

$$D_{\Gamma^{(k+2)}i} = D_{\Xi i} + \sum_{\ell=0}^{k} D^{j_1\cdots j_\ell}_{\Xi\lambda} y^{\lambda}_{j_1\cdots j_\ell i};$$

$$h_{\Xi} - h_{\Gamma^{(k+2)}} = \sum_{\ell=0}^{k} D_{\Xi\lambda}^{j_1\cdots j_\ell} \otimes \omega_{j_1\cdots j_\ell}^{\lambda};$$

$$H_{\Gamma^{(k+2)}} = H_{\Xi} \cap C_{\pi_{k+1,k}}.$$

A class of characterizable connections on $\pi_{k+1,k}$ with the same characteristic $(k+2)$-connection on π is generated by the class of $\iota_{1,k}$*-admissible deformations* on $\pi_{k+1,k}$. More precisely, if we call any such Ξ *associated* to $\Gamma^{(k+2)}$, then for each soldering form

$$\varphi : J^{k+1}\pi \to V_{\pi_{k+1,k}} J^{k+1}\pi \otimes \pi_{k+1,k}^{*}(T^{*}J^{k}\pi)$$

satisfying locally

$$\varphi_{j_1\cdots j_{k+1}i}^{\sigma} + \sum_{\ell=0}^{k} \varphi_{j_1\cdots j_{k+1}\lambda}^{\sigma r_1\cdots r_\ell} y_{r_1\cdots r_\ell i}^{\lambda} = 0,$$

$h_{\Xi} + \varphi$ is the horizontal form of another connection on $\pi_{k+1,k}$ associated to $\Gamma^{(k+2)}$.

Let us again consider $\pi : \mathbf{R} \times M \to \mathbf{R}$. There is an interesting submanifold of $J^1\pi_{k+1,k}$ having to do with the relations between the autonomous and the time-dependent situations. Namely, there is a canonical inclusion

$$\mathbf{R} \times J^1\tau_M^{k+1,k} \hookrightarrow J^1\pi_{k+1,k},$$

defined by

$$(x, j_y^1 w^{(k+1)}) \longmapsto j_{(x,y)}^1 \Gamma^{(k+1)}$$

for $\Gamma^{(k+1)}$ being defined by $w^{(k+1)}$ (see Section 4). In this respect, the restriction of the morphism

$$\mathbf{k}_{\iota_{1,k}} : J^1\pi_{k+1,k} \to \mathbf{R} \times T^{k+2}M$$

to the above submanifold generates the morphism

$$\mathbf{k}_M^{(k+1)} : J^1\tau_M^{k+1,k} \to T^{k+2}M$$

over $T^{k+1}M$.

Accordingly, a connection Λ on $\tau_M^{k+1,k}$ can be considered as a connection on $\pi_{k+1,k}$ of the particular type

$$\Xi = \mathrm{id}_{\mathbf{R}} \times \Lambda : \mathbf{R} \times \mathrm{T}^{\mathrm{k+1}}\mathrm{M} \to \mathbf{R} \times \mathrm{J}^1\tau_{\mathrm{M}}^{\mathrm{k+1,k}}$$

with the components $\Xi_{(k+1)}^{\sigma} = 0$ and $\Xi_{(k+1)\lambda}^{\sigma(i)} \in \mathcal{F}(T^{k+1}M)$. The corresponding horizontal distribution is

$$h_{\Xi} = \frac{\partial}{\partial t} \otimes dt + \sum_{i=0}^{k} D_{\Xi\lambda}^{(i)} \otimes dq_{(i)}^{\lambda} = \mathrm{id}_{\mathrm{T}\mathbf{R}} + \mathrm{h}_{\Lambda},$$

and the integral sections can be identified with the semisprays on T^kM (for $k \geq 1$) or the vector fields on M (for $k = 0$). Just the case of $k = 0$ might be of particular importance due to the fact that Λ represents a (generally *non-linear*) *connection* on $\tau_M : TM \to M$ with integral sections being the vector fields on M whose covariant derivative with respect to Λ vanishes, i.e., those *parallel* with respect to Λ.

The deformations of connections on $\pi_{k+1,k}$ are the soldering forms on $\pi_{k+1,k}$; a local expression of any such a $\pi_{k+1,k}$-vertical endomorphism on $\mathbf{R} \times T^{k+1}M$ is

$$\varphi = \frac{\partial}{\partial q^{\sigma}_{(k+1)}} \otimes \left(\varphi^{\sigma}_{(k+1)}\, dt + \sum_{i=0}^{k} \varphi^{\sigma(i)}_{(k+1)\lambda}\, dq^{\lambda}_{(i)} \right).$$

Nevertheless, there is a distinguished subfamily of the above soldering forms created by the *natural soldering forms* on $\pi_{k+1,k}$.

Recall first the family of *natural vector-valued one-forms* on $\mathbf{R} \times T^kM$, which is expressed by

$$\sum_{i=1}^{k} c_i J_i^{(k)} + \sum_{i=k+1}^{2k} c_i C_{i-k}^{(k)} \otimes dt + c_{2k+1} I_{T^kM} + c_{2k+2} I_{\mathbf{R}},$$

where $c_i \in \mathcal{F}(\mathbf{R})$, I_{T^kM} and

$$J_i^{(k)} = \sum_{j=1}^{k-i+1} j \frac{\partial}{\partial q^{\sigma}_{(i+j-1)}} \otimes dq^{\sigma}_{(j-1)}$$

(for $i = 1, \ldots, k$) are the unique natural $(1,1)$-tensor fields on T^kM,

$$I_{\mathbf{R}} = \frac{\partial}{\partial t} \otimes dt,$$

and

$$C_i^{(k)} = \sum_{j=1}^{k-i+1} \frac{(i+j-1)!}{(j-1)!}\, q^{\sigma}_{(j)} \frac{\partial}{\partial q^{\sigma}_{(i+j-1)}}$$

(for $i = 1, \ldots, k$) are the *absolute (generalized Liouville) vector fields* on T^kM. Consequently, any such natural soldering form is expressed by

$$\varphi = f_1 J_{k+1}^{(k+1)} + f_2 C_{k+1}^{(k+1)} \otimes dt$$

for $f_1,\ f_2 \in \mathcal{F}(\mathbf{R})$, i.e.,

$$\varphi^{\sigma}_{(k+1)} = (k+1)!\ f_2 q^{\sigma}_{(1)}, \qquad \varphi^{\sigma}_{(k+1)\lambda} = f_1 \delta^{\sigma}_{\lambda}$$

and the rest of the components vanishes identically. As a consequence we get that all natural $\iota_{1,k}$-admissible deformations on $\pi_{k+1,k}$ are of the form

$$\varphi = f S_{k+1}^{(k+1)}$$

with

$$S_{k+1}^{(k+1)} = J_{k+1}^{(k+1)} - \frac{1}{(k+1)!} C_{k+1}^{(k+1)} \otimes dt$$

and $f \in \mathcal{F}(\mathbf{R})$. In coordinates,

$$S_{k+1}^{(k+1)} = \frac{\partial}{\partial q^{\sigma}_{(k+1)}} \otimes \left(dq^{\sigma} - q^{\sigma}_{(1)}\, dt \right).$$

7. The method of characteristics

In fact, the construction generalizes that of the *associated semispray* to a given *dynamical connection* and it results in the *method of characteristics* for Ξ. As regards both the name and the meaning, the approach is quite near to the ideas dealing with Pfaffian systems in [67] and particularly [73]. Reaping the benefit of the fact that each integral section of Ξ is the field of paths of $\Gamma^{(k+2)}$, the integral 'surfaces' of Ξ are foliated by $(k+1)$-jets of integral "curves" of $\Gamma^{(k+2)}$ (= *characteristics*). The relation between the equations studied can be roughly (and non-geometrically) expressed as follows (suppose $k=0$): if the equations for Ξ are given by

$$dy^\sigma_i = \Xi^\sigma_{ij}\, dx^j + \Xi^\sigma_{i\lambda}\, dy^\lambda,$$

then those for its characteristic $\Gamma^{(2)}$ are

$$y^\sigma_{ij} = \frac{dy^\sigma_i}{dx^j} = \Xi^\sigma_{ij}\frac{dx^j}{dx^j} + \Xi^\sigma_{i\lambda}\frac{dy^\lambda}{dx^j} = \Xi^\sigma_{ij} + \Xi^\sigma_{i\lambda} y^\lambda_j.$$

Let Ξ be a characterizable connection on $\pi_{k+1,k}$, and $\Gamma^{(k+2)}_\Xi$ be its characteristic $(k+2)$-connection on π. Then each integral section $\Gamma^{(k+1)}$ of Ξ is a field of paths of $\Gamma^{(k+2)}_\Xi$. Since $H_{\Gamma^{(k+2)}}(z) \equiv C^{\Gamma^{(k+1)}}(z) \subset T_z\Gamma^{(k+1)}(V)$ for each $z \in \Gamma^{(k+1)}(V)$, one can say that $\Gamma^{(k+1)}$ is an 'integral including manifold' of $H_{\Gamma^{(k+2)}}$.

Since each field of paths is integrable, if Ξ is characterizable and integrable, then its characteristic connection $\Gamma^{(k+2)}_\Xi$ is integrable, as well. In fact, the maximal integral manifolds of H_Ξ (integral sections of Ξ) are foliated by the characteristics, whose equations are

$$\begin{aligned}\frac{\partial^{k+2}\gamma^\sigma}{\partial x^{j_1}\cdots\partial x^{j_{k+1}}\partial x^i} &= \Xi^\sigma_{j_1\cdots j_{k+1}i}\left(x^r, \gamma^\nu, \dots, \frac{\partial^{k+1}\gamma^\nu}{\partial x^{r_1}\cdots\partial x^{r_{k+1}}}\right)\\ &+ \sum_{\ell=0}^{k} \Xi^{\sigma r_1\cdots r_\ell}_{j_1\cdots j_{k+1}\lambda}\left(x^r, \gamma^\nu, \dots, \frac{\partial^{k+1}\gamma^\nu}{\partial x^{r_1}\cdots\partial x^{r_{k+1}}}\right)\frac{\partial^{\ell+1}\gamma^\lambda}{\partial x^{r_1}\cdots\partial x^{r_\ell}\partial x^i}.\end{aligned}$$

In other words, under the integrability conditions, the looking for solutions of the first-order system represented by Ξ can be transferred to the looking for the solutions of the above $(k+2)$-th order system – the integral sections of Ξ are "pieced together" by characteristics.

Moreover, knowing an r-dimensional integral submanifold M_r of H_Ξ, the characteristics can be applied when constructing an integral submanifold $M_{\geq r}$ of dimension $\geq r$ containing M_r – this task is the well-known *Cauchy initial problem*. Clearly, the case when M_r in itself is foliated by characteristics must be eliminated, in such a case $M_{\geq r} \equiv M_r$. In this respect, a point $z \in M_r$ can be called *characteristic* (with respect to Ξ) if $T_zM_r \supset H_{\Gamma^{(k+2)}_\Xi}(z)$, and the Cauchy problem is solvable just around the non-characteristic points of M_r. It is evident that the integrability of Ξ is not necessary for the integrability of $\Gamma^{(k+2)}_\Xi$. Nevertheless, the above method of characteristics can be applied, as well.

The relation between the characterizability of connections on $\pi_{k+1,k}$ and the integrability of $(k+1)$-connections on π is hidden within the following construction.

Let $\Gamma^{(k+1)}$ be a $(k+1)$-connection on π. The *formal mixed* $\Gamma^{(k+1)}$*-curvature map* is the map

$$\kappa_{\Gamma^{(k+1)}} : J^1\pi_{k+1,k} \to \pi^*_{k+1,k}(V_{\pi_{k,k-1}} J^k\pi \otimes \pi^*_k(\Lambda^2 T^*X))$$

defined for each $j^1_{j^k_x\gamma}\chi \in J^1\pi_{k+1,k}$ by means of the F-N bracket as

$$\kappa_{\Gamma^{(k+1)}}(j^1_{j^k_x\gamma}\chi) = (\chi(j^k_x\gamma), [h_{\Gamma^{(k+1)}} - h_\chi, h_\chi](j^k_x\gamma)).$$

The motivation of the definition is similar to that of formal curvature map; namely, if $\widetilde{\Gamma}^{(k+1)}$ is another $(k+1)$-connection on π, then

$$\begin{aligned}\kappa(\widetilde{\Gamma}^{(k+1)}, \Gamma^{(k+1)}) &:= \mathrm{pr}_2 \circ \kappa_{\Gamma^{(k+1)}} \circ \mathrm{j}^1\widetilde{\Gamma}^{(k+1)} \\ &= [h_{\Gamma^{(k+1)}} - h_{\widetilde{\Gamma}^{(k+1)}}, h_{\widetilde{\Gamma}^{(k+1)}}] : J^k\pi \to V_{\pi_{k,k-1}} J^k\pi \otimes \pi^*_k(\Lambda^2 T^*X)\end{aligned}$$

is the so-called *mixed curvature* of the pair $\Gamma^{(k+1)}$ and $\widetilde{\Gamma}^{(k+1)}$. Since

$$\varphi = h_{\Gamma^{(k+1)}} - h_{\widetilde{\Gamma}^{(k+1)}}$$

is a soldering form on π_k, the mixed curvature $\kappa(\widetilde{\Gamma}^{(k+1)}, \Gamma^{(k+1)})$ is nothing but the φ-torsion τ_φ of $\widetilde{\Gamma}^{(k+1)}$. Moreover, we have

$$\kappa(\widetilde{\Gamma}^{(k+1)}, \Gamma^{(k+1)}) = R_{\Gamma^{(k+1)}} - R_{\widetilde{\Gamma}^{(k+1)}} - \tfrac{1}{2}[\varphi, \varphi]$$

and thus e.g. also

$$\kappa(\widetilde{\Gamma}^{(k+1)}, \Gamma^{(k+1)}) - \kappa(\Gamma^{(k+1)}, \widetilde{\Gamma}^{(k+1)}) = 2(R_{\Gamma^{(k+1)}} - R_{\widetilde{\Gamma}^{(k+1)}}).$$

Clearly,

$$\kappa_{\Gamma^{(k+1)}} = \widehat{R}_{\Gamma^{(k+1)}} - R - \tfrac{1}{2}\widehat{\kappa}_{\Gamma^{(k+1)}}$$

with $\widehat{R}_{\Gamma^{(k+1)}} := R \circ j^1\Gamma^{(k+1)} \circ (\pi_{k+1,k})_1$ and $\widehat{\kappa}_{\Gamma^{(k+1)}}$ being defined analogously to $\kappa_{\Gamma^{(k+1)}}$ by

$$\widehat{\kappa}_{\Gamma^{(k+1)}}(j^1_{j^k_x\gamma}\chi) = \big(\chi(j^k_x\gamma), [h_{\Gamma^{(k+1)}} - h_\chi, h_{\Gamma^{(k+1)}} - h_\chi](j^k_x\gamma)\big).$$

Let now Ξ be a connection on $\pi_{k+1,k}$. Then

$$\kappa_{\Gamma^{(k+1)},\Xi} = \kappa_{\Gamma^{(k+1)}} \circ \Xi : J^{k+1}\pi \to \pi^*_{k+1,k}(V_{\pi_{k+1,k}} J^k\pi \otimes \pi^*_k(\Lambda^2 T^*X))$$

represents a "curvature-like" term generated by $\Gamma^{(k+1)}$ and Ξ, where

$$\widehat{R}_{\Gamma^{(k+1)}} \circ \Xi = R \circ j^1\Gamma^{(k+1)} \circ \pi_{k+1,k}$$

does not depend on Ξ and it vanishes if, and only if, $\Gamma^{(k+1)}$ is integrable, $R \circ \Xi$ does not depend on $\Gamma^{(k+1)}$ and it vanishes if, and only if, Ξ is characterizable, and finally $\widehat{\kappa}_{\Gamma^{(k+1)}} \circ \Xi$ integrates $\Gamma^{(k+1)}$ and Ξ together: if $\Gamma^{(k+1)}$ is an integral section of Ξ, then $\widehat{\kappa}_{\Gamma^{(k+1)}} \circ \Xi \circ \Gamma^{(k+1)} = \widehat{\kappa}_{\Gamma^{(k+1)}} \circ j^1\Gamma^{(k+1)} = 0$. In particular, if Ξ is characterizable with the integral section $\Gamma^{(k+1)}$, then $\kappa_{\Gamma^{(k+1)},\Xi} = 0$.

Owing to the dimension of the base, each connection Ξ on $\pi_{k+1,k} : \mathbf{R} \times T^{k+1}M \to \mathbf{R} \times T^kM$ is characterizable and a semispray connection $\Gamma^{(k+2)} : \mathbf{R} \times T^{k+1}M \to \mathbf{R} \times$

$T^{k+2}M$ is the characteristic $(k+2)$-connection of Ξ if, and only if, for the corresponding semispray $D_{\Gamma^{(k+2)}}$ on $\mathbf{R}\times T^{k+1}M$ holds

$$D_{\Gamma^{(k+2)}} = D_{\Xi 0} + \sum_{i=0}^{k} D^{(i)}_{\Xi\lambda} q^{\lambda}_{(i+1)},$$

which means

$$\Gamma^{\sigma}_{(k+2)} = \Xi^{\sigma}_{(k+1)} + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+1)\lambda} q^{\lambda}_{(i+1)}.$$

The above semispray $D_{\Gamma^{(k+2)}}$ can be called *characteristic* to Ξ, as well. Thus we have the diagram

$$\begin{array}{ccccccc}
\mathbf{R}\times T^{k+1}M & \xrightarrow{J^1(\Gamma^{(k+1)},\mathrm{id}_{\mathbf{R}})} & \mathbf{R}\times T^{k+2}M & = & \mathbf{R}\times T^{k+2}M & = & \mathbf{R}\times T^{k+2}M \\
\uparrow{\scriptstyle \Gamma^{(k+1)}} & & \uparrow{\scriptstyle \Gamma^{(k+2)}} & & {\scriptstyle \mathbf{k}_{\iota_{\mathbf{1},\mathbf{k}}}}\uparrow & & {\scriptstyle \mathrm{id}_{\mathbf{R}}\times\mathbf{k}^{(k+1)}_{\mathrm{M}}}\uparrow \\
\mathbf{R}\times T^{k}M & \xrightarrow{\Gamma^{(k+1)}} & \mathbf{R}\times T^{k+1}M & \xrightarrow{\Xi} & J^1\pi_{k+1,k} & \hookleftarrow & \mathbf{R}\times J^1\tau^{k+1,k}_{M},
\end{array}$$

which in particular defines the *characteristic semispray* on $T^{k+1}M$ for a connection Λ on $\tau^{k+1,k}_{M}$ in the autonomous case.

The *equations for characteristics* are then

$$\frac{d^{k+2}c^{\sigma}}{dt^{k+2}} = \Xi^{\sigma}_{(k+1)}\left(t, c^{\nu}, \ldots, \frac{d^{k+1}c^{\nu}}{dt^{k+1}}\right) + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+1)\lambda}\left(t, c^{\nu}, \ldots, \frac{d^{k+1}c^{\nu}}{dt^{k+1}}\right)\frac{d^{i+1}c^{\lambda}}{dt^{i+1}}$$

and with respect to the above general ideas, the looking for the solutions of the first-order P.D.E. system can be transferred to the looking for the solutions of the $(k+2)$-th order O.D.E. system.

8. The method of fields of paths: Part I

A dual *method of fields of paths* can be introduced, as well. Here the *integral* of an integrable $\Gamma^{(k+2)}$ is an integrable Ξ on $\pi_{k+1,k}$ whose characteristic connection is just $\Gamma^{(k+2)}$. The existence of such an integral allows the order-reduction of $\Gamma^{(k+2)}$ to (local) integral sections of Ξ. In this respect, the existence of both local and global integrals is discussed.

Actually, if Ξ is an integrable characterizable connection on $\pi_{k+1,k}$ associated to the $(k+2)$-connection $\Gamma^{(k+2)}$ on π, then each integral section of $\Gamma^{(k+2)}$ is locally embedded in a field of paths $\Gamma^{(k+1)}$ which is the integral section of Ξ. Accordingly, the problem of the looking for the solutions of the $(k+2)$-th order system represented by $\Gamma^{(k+2)}$ can be transferred to the looking for an integrable and characterizable connection Ξ on $\pi_{k+1,k}$ associated to $\Gamma^{(k+2)}$, and after this to the solving of the corresponding $(k+1)$-th order fields of paths. As already mentioned, if $\Gamma^{(k)}$ is a field of paths of $\Gamma^{(k+1)}$ which is the field of paths of $\Gamma^{(k+2)}$, then $\Gamma^{(k)}$ is a field of paths of $\Gamma^{(k+2)}$, and the procedure can be repeated.

Let $\Gamma^{(k+2)}$ be an integrable $(k+2)$-connection on π. A (generally local) integrable connection Ξ on $\pi_{k+1,k}$ associated to $\Gamma^{(k+2)}$ is called an *integral* of $\Gamma^{(k+2)}$.

Denoting here by $\Xi_{(\ell+1)}$ the integral of $\Gamma^{(\ell+2)}$, the following diagram can be presented:

$$
\begin{array}{ccccccc}
X & \xrightarrow{j^{k+1}\gamma} & J^{k+1}\pi & \xrightarrow{\Gamma^{(k+2)}} & J^{k+2}\pi & & \\
{\scriptstyle \mathrm{id_X}}\downarrow & & {\scriptstyle \pi_{k+1,k}}\downarrow & & {\scriptstyle \Gamma^{(k+2)}}\uparrow & & \\
X & \xrightarrow{j^{k}\gamma} & J^{k}\pi & \xrightarrow{\Gamma^{(k+1)}} & J^{k+1}\pi & \xrightarrow{\Xi_{(k+1)}} & J^1\pi_{k+1,k} \\
{\scriptstyle \mathrm{id_X}}\downarrow & & {\scriptstyle \pi_{k,k-1}}\downarrow & & {\scriptstyle \Gamma^{(k+1)}}\uparrow & & \\
X & \xrightarrow{j^{k-1}\gamma} & J^{k-1}\pi & \xrightarrow{\Gamma^{(k)}} & J^{k}\pi & \xrightarrow{\Xi_{(k)}} & J^1\pi_{k,k-1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\vdots & & \vdots & & \vdots & & \vdots \\
X & \xrightarrow{j^{1}\gamma} & J^{1}\pi & \xrightarrow{\Gamma^{(2)}} & J^{2}\pi & \xrightarrow{\Xi_{(2)}} & J^1\pi_{2,1} \\
{\scriptstyle \mathrm{id_X}}\downarrow & & {\scriptstyle \pi_{1,0}}\downarrow & & {\scriptstyle \Gamma^{(2)}}\uparrow & & \\
X & \xrightarrow{\gamma} & Y & \xrightarrow{\Gamma} & J^{1}\pi & \xrightarrow{\Xi} & J^1\pi_{1,0} \\
{\scriptstyle \mathrm{id_X}}\downarrow & & {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi_1}\downarrow & & \\
X & \xrightarrow{\mathrm{id_X}} & X & \xrightarrow{\mathrm{id_X}} & X. & &
\end{array}
$$

Natural question on the existence of integrals for a given $(k+2)$-connection may be considered both locally and globally. The former case can be answered in terms of first integrals.

Notice first that each first integral of a characterizable Ξ is the first integral of its characteristic $\Gamma^{(k+2)}$. The converse is not true in general, nevertheless, the following assertion holds.

Let $\Gamma^{(k+2)}$ be an integrable $(k+2)$-connection on π and $\{a^1, \ldots, a^K\}$, where $K = \dim \pi_{k+1,k}$, be a set of independent first integrals of $\Gamma^{(k+2)}$, defined on some open subset $W \subset J^{k+1}\pi$. If the matrix

$$A = \left(\frac{\partial a^L}{\partial y^\sigma_{j_1 \cdots j_{k+1}}} \right)$$

is regular on W, then there is an integral Ξ of $\Gamma^{(k+2)}$ on W, defined by

$$H_\Xi = \mathrm{anih}\big\{\mathrm{da}^1, \ldots, \mathrm{da}^{\mathrm{K}}\big\}.$$

It should be noticed that if $\Gamma^{(k+2)}$ is integrable, then the existence of a set of independent first integrals satisfying the above condition is due to the horizontality of $H_{\Gamma^{(k+2)}}$.

The problem of global integrals is much more complicated. In fact, two questions appear in terms of the above considerations. First, whether there exist transformations, allowing a global assignment $\Gamma^{(k+2)} \mapsto \Xi$, and secondly, what conditions force Ξ to be the integral of $\Gamma^{(k+2)}$? Especially the first question represents an *open problem* for $\dim X > 1$ and $k \geq 1$. For $k = 0$, the following assertion can be presented, reformulating the corresponding result of [25]. It should be stressed that all concepts involved are *global*.

Let $\Gamma^{(2)}$ be a 2-connection on π and Λ a linear connection on X. Then there is a connection $\Xi^\Lambda_a = g^\Lambda_a \circ j^1\Gamma^{(2)}$ on $\pi_{1,0}$ associated to $\Gamma^{(2)}$, being determined in virtue of a natural fibered morphism $g^\Lambda_a : J^1\pi_{2,1} \to J^1\pi_{1,0}$ over $J^1\pi$ which is locally expressed by

$$z^\sigma_{i\lambda} = \tfrac{1}{2}(z^{\sigma k}_{ik\lambda} + \delta^\sigma_\lambda \Lambda^k_{ki}) + a\delta^\sigma_\lambda(\Lambda^k_{ik} - \Lambda^k_{ki}),$$
$$z^\sigma_{ij} = y^\sigma_{ij} - z^\sigma_{i\lambda} y^\lambda_j$$

for an arbitrary $a \in \mathbf{R}$. It appears that the presence of a linear connection on the base X is essential, it cannot be omitted. If $\mathcal{T} : TX \to V_{\tau_X}TX \otimes \Lambda^2 T^*X$,

$$\mathcal{T} = \Lambda^k_{ij}\frac{\partial}{\partial x^k} \otimes dx^i \wedge dx^j,$$

is the classical *torsion* of Λ, then its *contraction* is a one-form $\widehat{\mathcal{T}} = \mathcal{T}_i\, dx^i$ on X with $\mathcal{T}_i = \Lambda^k_{ik} - \Lambda^k_{ki}$. It can be shown that the linear connection Λ on X canonically generates the *soldering form of type* $\widehat{\mathcal{T}}$ on $\pi_{1,0}$, which locally reads

$$S_\Lambda = \mathcal{T}_i \frac{\partial}{\partial y^\sigma_i} \otimes (dy^\sigma - y^\sigma_j\, dx^j),$$

hence, it is trivial if, and only if, Λ is symmetric (torsion free).

As a consequence, the connection Ξ^Λ_a can be written in the form

$$\Xi^\Lambda_a = \Xi^\Lambda_0 + a\, S_\Lambda,$$

where the components of Ξ^Λ_0 are

$$\Xi^\sigma_{i\lambda} = \frac{1}{2}\left(\frac{\partial \Gamma^\sigma_{ik}}{\partial y^\lambda_k} + \delta^\sigma_\lambda \Lambda^k_{ki}\right),$$
$$\Xi^\sigma_{ij} = \Gamma^\sigma_{ij} - \Xi^\sigma_{i\lambda} y^\lambda_j,$$

with Γ^σ_{ij} being the components of $\Gamma^{(2)}$.

Recall finally that the family of connections on $\pi_{1,0}$ associated to $\Gamma^{(2)}$ can be obtained by means of $\iota_{1,0}$-admissible deformations on $\pi_{1,0}$, where $\iota_{1,0} \equiv \mathrm{id}_{J^1\pi}$.

On the other hand, there is a construction of global associated connections for $k \geq 0$, but with $\dim X = 1$, established in [81]. In this situation, the role of a linear connection is played by a *volume form* on X, as we present at the very and of this section.

Let us now again consider $\pi : \mathbf{R} \times M \to \mathbf{R}$. Here, the role of another natural $(1, 1)$-tensor field appears; namely,

$$h_{\Xi_0} = \frac{1}{2}\left[h_{\Gamma^{(k+2)}} + I + \frac{1}{k+2}\left(kv_{\Gamma^{(k+2)}} - 2\mathcal{L}_{D_{\Gamma^{(k+2)}}} S^{(k+1)}_1\right)\right]$$

is the horizontal form of a connection Ξ_0 on $\pi_{k+1,k}$ associated to $\Gamma^{(k+2)}$, where

$$S^{(k+1)}_1 = J^{(k+1)}_1 - C^{(k+1)}_1 \otimes dt,$$

i.e.,

$$S_1^{(k+1)} = \sum_{i=1}^{k+1} i \frac{\partial}{\partial q^\sigma_{(i)}} \otimes (dq^\sigma_{(i-1)} - q^\sigma_{(i)}\, dt).$$

The components of Ξ_0 are then

$$\Xi^{\sigma(i)}_{(k+1)\lambda} = \frac{i+1}{k+2} \frac{\partial \Gamma^\sigma_{(k+2)}}{\partial q^\lambda_{(i+1)}}, \quad i = 0, \dots k,$$

$$\Xi^\sigma_{(k+1)} = \Gamma^\sigma_{(k+2)} - \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+1)\lambda}\, q^\lambda_{(i+1)}.$$

Accordingly, the family of all connections on $\pi_{k+1,k}$ *naturally associated* to a $(k+2)$-connection $\Gamma^{(k+2)}$ on $\pi : \mathbf{R} \times M \to \mathbf{R}$ is defined by

$$h_\Xi = h_{\Xi_0} + f S^{(k+1)}_{k+1}$$

with $f \in \mathcal{F}(\mathbf{R})$.

Let us finally recall the result of [81]. Le $\pi : Y \to X$ be an arbitrary fibered manifold over one-dimensional base X endowed by a volume form $\Omega = \omega\, dt$. By [72], there is a naturally defined vector-valued one-form

$$S^{(k+1)}_\Omega = \sum_{j+i=1}^{k} \binom{j+i+1}{i} \frac{d^j\omega}{dt^j} \frac{\partial}{\partial q^\sigma_{(j+i+1)}} \otimes \left(dq^\sigma_{(i)} - q^\sigma_{(i+1)}\, dt\right)$$

on $J^{k+1}\pi$, where i, j are non-negative integers and $d^0\omega/dt^0 \equiv \omega$. Then

$$h_{\Xi_\Omega} = \frac{1}{2}\left[h_{\Gamma^{(k+2)}} + I + \frac{1}{k+2}\left(k v_{\Gamma^{(k+2)}} - \frac{2}{\omega} \mathcal{L}_{D_{\Gamma^{(k+2)}}} S^{(k+1)}_\Omega\right)\right]$$

is the horizontal form of a (global) connection Ξ_Ω on $\pi_{k+1,k}$, associated to $\Gamma^{(k+2)}$. Clearly, the above h_{Ξ_0} corresponds to $\Omega = dt$ and $S^{(k+1)}_{dt} \equiv S^{(k+1)}_1$.

On the other hand, the result can be related to the most general situation of π, for $k = 0$ and $\dim X = 1$. The "strong horizontal components" of Ξ_Ω are then

$$\Xi^\sigma_\lambda = \frac{1}{2}\left(\frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} - \frac{d\omega}{dt} \frac{1}{\omega} \delta^\sigma_\lambda\right)$$

with the quantity $\Lambda(t) = -(d\omega/dt)(1/\omega)$ being transformed in the same way like the component of a linear connection on X. Consequently, there is a geometric interpretation of Ξ^Λ_0 in this situation; namely, it is just Ξ_Ω for an arbitrary volume form Ω on X which is the integral section (i.e. $\Lambda^* \circ \Omega = j^1\Omega$) of the dual connection Λ^* on τ^*_X.

9. The method of fields of paths: part II

The generalization of methods of fields of paths was completely motivated by [50]. The background is the 2-fibered manifold $J^{k+r}\pi \xrightarrow{\pi_{k+r,k}} J^k\pi \xrightarrow{\pi_k} X$. If $\Gamma^{(k+r+1)}$ is a $(k+r+1)$-connection on π, then the method gives a $(k+1)$-connection $\Gamma^{(k+1)}$ on π

representing the order-reduction of the equations represented by $\Gamma^{(k+r+1)}$, all for $r \geq 2$. In fact, this is obtained by means of looking for the prolongation of $\Gamma^{(k+1)}$, which is a section of $\pi_{k+r,k}$ (a *jet field*). In this respect, the connections on $\pi_{k+r,k}$ are studied, as well, which results in the definition of the $\pi_{k+r,k}$-*integral* of $\Gamma^{(k+r+1)}$.

The corresponding diagram generalizing (10), is now

$$\begin{tikzcd}
X & J^1\pi_k \arrow[l, "(\pi_k)_1"'] \arrow[d, "(\pi_k)_{1,0}"'] & J^1\pi_{k+r} \arrow[l, "J^1(\pi_{k+r,k},\mathrm{id}_X)"'] \arrow[d, "(\pi_{k+r})_{1,0}"'] & \\
X \arrow[u, leftarrow, "\mathrm{id}_X"'] \arrow[d, "\mathrm{id}_X"'] & J^k\pi \arrow[l, "\pi_k"'] \arrow[d, "\pi_k"'] & J^{k+r}\pi \arrow[l, "\pi_{k+r,k}"'] \arrow[d, "\pi_{k+r}"'] & J^1\pi_{k+r,k} \arrow[l, "(\pi_{k+r,k})_{1,0}"'] \\
X & X \arrow[l, "\mathrm{id}_X"'] & X. \arrow[l, "\mathrm{id}_X"'] &
\end{tikzcd} \tag{12}$$

The map $\mathbf{k} : J^1\pi_k \times_{J^k\pi} J^1\pi_{k+r,k} \to J^1\pi_{k+r}$, defined for $\psi \in \mathcal{S}_{\mathrm{loc}}(\pi_k)$ and $\varphi \in \mathcal{S}_{\mathrm{loc}}(\pi_{k+r,k})$, $\mathrm{Im}\,\psi \subset \mathrm{Dom}\,\varphi$, by

$$\mathbf{k}(j^1_x\psi, j^1_{\psi(x)}\varphi) = j^1_x(\varphi \circ \psi),$$

locally does not effect the coordinates

$$x^i, y^\sigma, \ldots, y^\sigma_{j_1\cdots j_{k+1}}, y^\sigma_{;i}, \ldots, y^\sigma_{j_1\ldots j_k;i},$$

and

$$\mathbf{k} : \begin{cases} y^\sigma_{j_1\cdots j_{k+1};\,i} = z^\sigma_{j_1\cdots j_{k+1}i} + \displaystyle\sum_{\ell=0}^{k} z^{\sigma r_1\cdots r_\ell}_{j_1\ldots j_{k+1}\lambda} y^\lambda_{r_1\ldots r_\ell;\,i} \\ \qquad\vdots \\ y^\sigma_{j_1\ldots j_{k+r};\,i} = z^\sigma_{j_1\ldots j_{k+r}i} + \displaystyle\sum_{\ell=0}^{k} z^{\sigma r_1\cdots r_\ell}_{j_1\ldots j_{k+r}\lambda} y^\lambda_{r_1\ldots r_\ell;\,i} \end{cases}$$

with z's being the induced coordinates on $J^1\pi_{k+r,k}$. Clearly, there is a natural candidate for a morphism between $\pi_{k+r,k}$ and $(\pi_k)_{1,0}$ over $J^k\pi$; namely, denote by

$$\Phi_0 = \iota_{1,k} \circ \pi_{k+r,k+1} : J^{k+r}\pi \to J^1\pi_k$$

the composition, whose coordinate expression coincides with (11). Then the affine morphism $\mathbf{k}_{\Phi_0} : J^1\pi_{k+r,k} \to J^1\pi_{k+r}$ defines an affine subbundle $A_{\pi_{k+r,k}} \subset J^1\pi_{k+r}$ consisting of the points $z \in J^1\pi_{k+r}$ satisfying

$$\iota_{1,k} \circ \pi_{k+r,k+1} \circ (\pi_{k+r})_{1,0}(\mathrm{z}) = J^1(\pi_{k+r,k}, \mathrm{id}_X)(\mathrm{z}).$$

Such elements are called $\pi_{k+r,k}$-*semiholonomic jets*; the local expression is again just (11). Thus there is a canonical inclusion

$$J^{k+r+1}\pi \subset \widehat{J}^{k+r+1}\pi \subset A_{\pi_{k+r,k}},$$

which corresponds to the associated vector bundle

$$\overline{A}_{\pi_{k+r,k}} = V_{\pi_{k+r,k}} J^{k+r}\pi \otimes \pi^*_{k+r}(T^*X) \subset V_{\pi_{k+r}} J^{k+r}\pi \otimes \pi^*_{k+r}(T^*X).$$

Remark here that the study of invariant subspaces of the above nature has been presented in [27], obtained by means of the methods of natural operators.

Notice now some properties of the sections of $\pi_{k+r,k}$, called *jet fields;* again, we work with global sections for the simplicity only, the same applies (under appropriate restrictions) for the local ones.

A section $\gamma \in \mathcal{S}_U(\pi)$ is called an *integral section* (or a *path*) of a jet field $\varphi \in \mathcal{S}(\pi_{k+r,k})$ if it is a solution of the equation $\mathcal{E}^\varphi = \varphi(J^k\pi) \subset J^{k+r}\pi$, i.e., if $\varphi \circ j^k\gamma = j^{k+r}\gamma$ on U. In this respect, φ is called *integrable* if there is an integral section of φ through each point of Y. In coordinates, the equations of φ are

$$y^\sigma_{j_1\cdots j_{k+1}} = \varphi^\sigma_{j_1\cdots j_{k+1}}, \ldots, y^\sigma_{j_1\cdots j_{k+r}} = \varphi^\sigma_{j_1\cdots j_{k+r}}$$

with the *components* of φ being functions on $J^k\pi$.

For an arbitrary jet field $\varphi \in \mathcal{S}(\pi_{k+r,k})$, there is a distinguished associated projection; namely, by $\Gamma^{(k+1)}_\varphi = \pi_{k+r,k+1} \circ \varphi$ we get a $(k+1)$-connection $\Gamma^{(k+1)}_\varphi$ on π; in coordinates,

$$\Gamma^\sigma_{j_1\cdots j_{k+1}} = \varphi^\sigma_{j_1\cdots j_{k+1}}.$$

Then one can show that a jet field $\varphi \in \mathcal{S}(\pi_{k+r,k})$ is integrable if, and only if, $\Gamma^{(k+1)}_\varphi$ is integrable and $\varphi = \Gamma^{(k+1)(r-1)}_\varphi$.

As for higher-order connections, there is an n-dimensional π_{k+r-1}-horizontal distribution H_φ on $J^{k+r-1}\pi$ naturally associated with φ. In fact,

$$H_\varphi = \mathrm{span}\{\mathrm{D}_{\varphi\mathrm{i}},\ \mathrm{i} = 1, \ldots, \mathrm{n}\},$$

where the generators $D_{\varphi i}$ are defined by $D_{\varphi i} = D_i^{k+r,k+r-1} \circ \varphi \circ \pi_{k+r-1,k}$, i.e., locally

$$D_{\varphi i} = \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k-1} y^\sigma_{j_1\ldots j_\ell i} \frac{\partial}{\partial y^\sigma_{j_1\cdots j_\ell}} + \sum_{\ell=k}^{k+r-1} \varphi^\sigma_{j_1\ldots j_\ell\, i} \frac{\partial}{\partial y^\sigma_{j_1\cdots j_\ell}}. \tag{13}$$

As to be expected, a section $\gamma \in \mathcal{S}_U(\pi)$ is an integral section of φ if, and only if, $j^{k+r-1}\gamma(U)$ is an integral manifold of H_φ.

It must be remarked that due to the horizontality, H_φ is involutive (= completely integrable) if, and only if, $[D_{\varphi i}, D_{\varphi p}] = 0$ for all i, p. It should be stressed that this condition is *not* equivalent with the integrability of φ in the above presented sense. Nevertheless, the integral section of φ could be defined to be $\psi \in \mathcal{S}_U(\pi_{k+r-1})$ such that $\psi(U)$ is an integral manifold of H_φ. Of course, now the equations must be considered on $J^1\pi_{k+r-1}$.

Adding sections and connections, the diagram (12) turns out to be of the form

$$\begin{array}{ccccccc}
X & \xrightarrow{j^1\psi} & J^1\pi_k & \xrightarrow{J^1(\varphi,\mathrm{id_X})} & J^1\pi_{k+r} & = & J^1\pi_{k+r} \\
{\scriptstyle \mathrm{id_X}}\downarrow & & {\scriptstyle \Sigma_{(k)}}\uparrow & & {\scriptstyle \Sigma_{(k+r)}}\uparrow & & {\scriptstyle \mathbf{k}_{\Phi_0}}\uparrow \\
X & \xrightarrow{\psi} & J^k\pi & \xrightarrow{\varphi} & J^{k+r}\pi & \xrightarrow{\Xi} & J^1\pi_{k+r,k} \\
{\scriptstyle \mathrm{id_X}}\downarrow & & {\scriptstyle \pi_k}\downarrow & & {\scriptstyle \pi_{k+r}}\downarrow & & \\
X & \xrightarrow{\mathrm{id_X}} & X & \xrightarrow{\mathrm{id_X}} & X, & &
\end{array} \tag{14}$$

where by $\Sigma_{(\ell)}$ we denote a connection on π_ℓ. As regards $\Sigma_{(k+r)}$, it can be called $\pi_{k+r,k}$-*semiholonomic*, if

$$\Sigma_{(k+r)} : J^{k+r}\pi \to A_{\pi_{k+r,k}},$$

which means just

$$D_{\Sigma i} = \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k} y^\sigma_{j_1\ldots j_\ell i}\frac{\partial}{\partial y^\sigma_{j_1\cdots j_\ell}} + \sum_{\ell=k+1}^{k+r} \Sigma^\sigma_{j_1\ldots j_\ell;\, i}\frac{\partial}{\partial y^\sigma_{j_1\cdots j_\ell}}.$$

In this respect, if $\varphi \in \mathcal{S}(\pi_{k+r,k})$ is a jet field, then φ can be identified with a (special type of) $\pi_{k+r-1,k-1}$-semiholonomic connection on π_{k+r-1}.

Our main concern is with connections on $\pi_{k+r,k}$, i.e. $\Xi : J^{k+r}\pi \to J^1\pi_{k+r,k}$. The point is that the integral sections (if any) of a connection Ξ on $\pi_{k+r,k}$ are (local) jet fields from $\mathcal{S}(\pi_{k+r,k})$ satisfying $j^1\varphi = \Xi \circ \varphi$.

A connection Ξ on $\pi_{k+r,k}$ is called *characterizable*, if the connection $\mathbf{k}_{\Phi_0} \circ \Xi$ is holonomic. The connection $\Gamma^{(k+r+1)}_\Xi = \mathbf{k}_{\Phi_0} \circ \Xi$ is called *characteristic* to Ξ.

One can see that $\mathbf{k}_{\Phi_0} \circ \Xi$ is $\pi_{k+r,k}$-semiholonomic for an arbitrary Ξ; it is semiholonomic if, and only if,

$$y^\sigma_{j_1\cdots j_{k+1}i} = \Xi^\sigma_{j_1\cdots j_{k+1}i} + \sum_{\ell=0}^{k} \Xi^{\sigma r_1\cdots r_\ell}_{j_1\cdots j_{k+1}\lambda} y^\lambda_{r_1\ldots r_\ell\, i}$$

$$\vdots$$

$$y^\sigma_{j_1\ldots j_{k+r-1}i} = \Xi^\sigma_{j_1\cdots j_{k+r-1}i} + \sum_{\ell=0}^{k} \Xi^{\sigma r_1\cdots r_\ell}_{j_1\cdots j_{k+r-1}\lambda} y^\lambda_{r_1\ldots r_\ell\, i},$$

and it is holonomic if, moreover, the functions

$$\Gamma^\sigma_{j_1\ldots j_{k+r}i} = \Xi^\sigma_{j_1\ldots j_{k+r}i} + \sum_{\ell=0}^{k} \Xi^{\sigma r_1\cdots r_\ell}_{j_1\ldots j_{k+r}\lambda} y^\lambda_{r_1\ldots r_\ell\, i}$$

are totally symmetric. Then we can see that a $(k+r+1)$-connection $\Gamma^{(k+r+1)}$ on π is the characteristic connection of a connection Ξ on $\pi_{k+r,k}$ if, and only if, $H_{\Gamma^{(k+r+1)}} \subset H_\Xi$ or, equivalently,

$$D_{\Gamma^{(k+r+1)}i} = D_{\Xi i} + \sum_{\ell=0}^{k} D^{j_1\cdots j_\ell}_{\Xi\lambda} y^\lambda_{j_1\ldots j_\ell i}.$$

The motivation of the above constructions is the following. Let Ξ be a characterizable connection on $\pi_{k+r,k}$, and $\Gamma^{(k+r+1)}_\Xi$ its characteristic connection. Let $\varphi \in \mathcal{S}_{\mathrm{loc}}(\pi_{k+r,k})$ be an integral section of Ξ and $\Gamma^{(k+1)}_\varphi$ the $(k+1)$-connection on π, defined by $\Gamma^{(k+1)}_\varphi = \pi_{k+r,k+1} \circ \varphi$. Then $\Gamma^{(k+1)}_\varphi$ is a field of paths of $\Gamma^{(k+r+1)}_\Xi$ and $\varphi = \Gamma^{(k+1)(r-1)}_\varphi$.

As usually, the situation may be described diagrammatically:

$$\begin{tikzcd}
J^{k+1}\pi \arrow[r, "J^1(\varphi,\mathrm{id}_X)"] & J^{k+r+1}\pi \arrow[r, equal] & J^{k+r+1}\pi \\
J^k\pi \arrow[u, "\Gamma^{(k+1)}_\varphi"] \arrow[r, "\varphi"] & J^{k+r}\pi \arrow[u, "\Gamma^{(k+r+1)}_\Xi"] \arrow[r, "\Xi"] & J^1\pi_{k+r,k}. \arrow[u, "\mathbf{k}_{\Phi_0}"]
\end{tikzcd}$$

In this arrangement, the following definition appears very naturally; again, any connection Ξ on $\pi_{k+r,k}$ whose characteristic connection is the given $\Gamma^{(k+r+1)}$ is called *associated* to it.

Let $\Gamma^{(k+r+1)}$ be an integrable $(k+r+1)$-connection on π. A (generally local) integrable connection Ξ on $\pi_{k+r,k}$ associated to $\Gamma^{(k+r+1)}$ is called the $\pi_{k+r,k}$-*integral* of $\Gamma^{(k+r+1)}$.

In other words, a second version of the method of fields of paths was presented. In contradiction to Section 8, now we are not looking for fields of paths directly, but through their prolongations. It is evident that the crucial problem is again that of the existence of $\pi_{k+r,k}$-integrals. In this respect, the following assertion can be proved.

Let $\Gamma^{(k+r+1)}$ be an integrable $(k+r+1)$-connection on π and $\{a^1, \dots, a^K\}$, where $K = \dim \pi_{k+r,k}$, be a set of independent first integrals of $\Gamma^{(k+r+1)}$, defined on some open $W \subset J^{k+r}\pi$. If the matrix

$$A = \left(\frac{\partial a^L}{\partial y^\sigma_{j_1 \dots j_\ell}} \right),$$

where $\ell = k+1, \dots, k+r$, is regular on W, then $H_\Xi = \operatorname{anih}\{\mathrm{d}a^1, \dots, \mathrm{d}a^K\}$ defines an $\pi_{k+r,k}$-integral of $\Gamma^{(k+r+1)}$ on W.

For an application (and in fact the motivation) of the above considerations, we refer to [85], dealing with the particular case of one-dimensional base X (and thus O.D.E.) and generalizing the *Hamilton–Jacobi* method from variational analysis studied in [50] and [53].

References

[1] R. Abraham, J.E. Marsden, *Foundations of Mechanics*, Benjamin/Cummings Publ. Comp., Reading, 1978.

[2] D.V. Alekseevskiĭ, A.M. Vinogradov, V.V. Lychagin, *Fundamental Ideas and Concepts of Differential Geometry*, Itogi nauki i tekhniki, Sovremennye problemy matematiki 28, VINITI, 1988, Moscow, (Russian).

[3] M. Anastasiei, H. Kawaguchi, A geometrical theory of the time dependent Lagrangians, I. Non-linear connections, Tensor, N.S. 48 (1989) 273–282; II. M-connections, Tensor, N.S. 48 (1989) 283–293; III. Applications, Tensor, N.S. 49 (1990) 296–304.

[4] V.I. Arnol'd, *Additional Parts of the Theory of Ordinary Differential Equations*, Nauka, 1978, Moscow, (Russian).

[5] V.I. Bliznikas, R.V. Vosylius, Non-holonomic connections, Liet. matem. rink. 27 (1) (1987) 15–27 (Russian).

[6] H. Cartan, *Calcul différentiel formes différentielles*, Mir, 1971, Moscow, transl. to Russian.

[7] M. Crampin, On the differential geometry of the Euler–Lagrange equations and the inverse problem of Lagrangian dynamics, J. Phys. A: Math. Gen. 14 (1981) 2567–2575.

[8] M. Crampin, Alternative Lagrangians in particle dynamics, Proc. Conf. Diff. Geom. and Its Appl., Brno, 1986, 1–12.

[9] M. Crampin, G.E. Prince, G. Thompson, A geometrical version of the Helmholtz conditions in time dependent Lagrangian dynamics, J. Phys. A. Math. Gen. 17 (1984) 1437–1447.

[10] M. Crampin, W. Sarlet, F. Cantrijn, Higher-order differential equations and higher-order lagrangian mechanics, Math. Proc. Cambridge Philos. Soc. 99 (1986) 565–587.

[11] L.C. de Andres, M. de León, P.R. Rodrigues, Connections on tangent bundles of higher order, Demonstratio Math. 22 (3) (1989) 607–632.

[12] L.C. de Andres, M. de León, P.R. Rodrigues, Connections on tangent bundles of higher order associated to regular Lagrangians, Geom. Dedicata 39 (1991) 12–18.

[13] A. Dekrét, Vector fields and connections on TM, Časopis Pěst. Mat. 115 (1990) 360–367.

[14] A. Dekrét, Ordinary differential equations and connections, Proc. Conf. Diff. Geom. and Its Appl., Brno,1989, 27–32.

[15] M. de León, J. C. Marrero, Time-dependent linear Lagrangians: the inverse problem, symmetries and constants of motion, International Symposium on Hamiltonian Systems and Celestian mechanics, Guanajnato, Mexico, 1991, Advanced Series in Nonlinear Dynamics, 4, World Scientific, Singapore 1993, 55–83.

[16] M. de León, D. Martín de Diego, Non-autonomous submersive second order differential equations and Lie symmetries, Internacional Journal of Theoretical Physisc 33 (8) (1994) 1759–1781.

[17] M. de León, D. Martín de Diego, Symmetries and constants of the motion for higher order Lagrangian systems, J. Math. Phys. 36 (8) (1995) 4138–4161.

[18] M. de León, D. Martín de Diego, Symmetries and constants of the motion for higher order Lagrangian systems. II: the non-autonomous case, Extracta Mathematicae 9 (2) (1994) 111–134.

[19] M. de León, P.R. Rodrigues, *Generalized Classical Mechanics and Field Theory*, North-Holland Mathematical Studies 112, North-Holland, 1985, Amsterdam.

[20] M. de León, P.R. Rodrigues, Higher order almost tangent geometry and non-autonomous Lagrangian dynamics, Supp. Rend. Circolo Mat. Palermo 16 (1987) 157–171.

[21] M. de León, P.R. Rodrigues, Dynamical connections and non-autonomous Lagrangian systems, Ann. Fac. Sci. Toulouse IX (1988) 171–181.

[22] M. de León, P.R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, North-Holland Mathematics Studies 158, North-Holland, 1989, Amsterdam.

[23] M. Doupovec, I. Kolář, Natural affinors on time-dependent Weil bundles, Arch Math. (Brno) 27 (1991) 205–209.

[24] M. Doupovec, A. Vondra, On certain natural transformations between connections, Proc. Conf. Diff. Geom. and Its Appl., Opava, 1992 (Silesian University, Opava, 1993) 273–279.

[25] M. Doupovec, A. Vondra, Some natural operations between connections on fibered manifolds, Rendiconti del Circolo Matematico di Palermo, Serie II - Suppl. 39 (1996) 73–84.

[26] M. Doupovec, A. Vondra, Natural relations between connections in 2-fibred manifolds, New Developments in Differential Geometry, edited by L. Tammasi and J. Szenthe, Kluwer Academic Publishers, Dordrecht (1996) 113–130.

[27] M. Doupovec, A. Vondra, Invariant subspaces in higher-order jet prolongations of a fibered manifold, Czechoslovak Math. J. 50 (125) (2000) 209–220.

[28] S.V. Duzhin, V.V. Lychagin, Symmetries of distributions and quadrature of ordinary differential equations, Acta Appl. Math. 24 (1991) 29–57.

[29] H. Goldschmidt, Integrability criteria for systems of non-linear differential equations, J. Differential Geom. 1 (1969) 269–307.

[30] J. Grifone, Structure presque-tangente et connexions, I (II), Ann. Inst. Fourier, Grenoble 22 (3,4) (1972) 287–334 (291–338).

[31] D. Gromoll, W. Klingenberg, W. Meyer, *Riemanische Geometrie im Grossen*, Springer, Berlin, 1968.

[32] H. Kawaguchi, The d-connections in Lagrange geometry, Proc. Conf. Diff. Geom. and Its Appl., Brno, 1989, 230–235.

[33] J. Klein, Geometry of sprays. Lagrangian case. Principle of least curvature Proc. IUTAM–ISIMM Symposium on Modern Developments in Analytical Mechanics, Torino, Volume I–Geometrical Dynamics (1982) 177–196.

[34] J. Klein, Almost symplectic structures in dynamics, Proc. Conf. Diff. Geom. and Its Appl., Brno, 1986, 79–91.

[35] S. Kobayashi, K. Nomidzu, *Foundations of Differential Geometry*, I, Nauka, Moscow, 1981, Russian.

[36] I. Kolář, Connections in 2-fibred manifolds, Arch. Math. (Brno) 17 (1981) 23–30.

[37] I. Kolář, Some natural operations with connections, J. Nat. Acad. Math. 5 (1987) 127–141.

[38] I. Kolář, M. Modugno, On the algebraic structure on the jet prolongation of fibred manifolds, Czechoslovak Math. J. 40 (115) (1990) 601–611.

[39] I. Kolář, M. Modugno, Torsions of connections on some natural bundles, Diff. Geom. Appl. 2 (1992) 1–16.

[40] I. Kolář, P.W. Michor, J. Slovák, *Natural operations in differential geometry*, Springer, 1993.

[41] I. Kolář, J. Slovák, Prolongations of vector fields to jet bundles, Proc. Winter School on Geometry and Physics, Srní, 1989; Suppl. Rendiconti Circolo Mat. Palermo, Serie II 21 (1989) 103–111.

[42] D. Krupka, Some geometrical aspects of variational problems on fibred manifolds, Folia Fac. Sci. Natur. Univ. Purk. Brun. Phys. XIV (1973) 1–65

[43] D. Krupka, On the higher-order Hamilton theory in fibered spaces, Geometrical Methods in Physics, Proc. Conf. Diff. Geom. and Its Appl., Nové Město na Moravě, 1983, J. E. Purkyně Univ., Brno, 1984, 167–183

[44] D. Krupka, Geometry of Lagrangian structures 1, Arch. Math. (Brno) 22 (1986) 159–174; Geometry of Lagrangian structures 2, Arch. Math. (Brno) 22 (1986) 211–228.

[45] D. Krupka, Regular lagrangians and Lepagean forms, Proc. Conf. Diff. Geom. and Its Appl., Brno, 1986, 111–148.

[46] D. Krupka, J. Musilová, Hamilton extremals in higher order mechanics, Arch. Math. (Brno) 22 (1984) 21–31.

[47] D. Krupka, J. Janyška, *Lectures on differential invariants*, Folia Fac. Sci. Natur. Univ. Purk. Brun. Phys., 1990, Brno.

[48] O. Krupková, Lepagean 2-forms in higher order Hamiltonian mechanics. I. Regularity, Arch. Math. (Brno) 22 (1986) 97–120; Lepagean 2-forms in higher order Hamiltonian mechanics. II. Inverse problem, Arch. Math. (Brno) 23 (1987) 155–170.

[49] O. Krupková, Hamilton–Jacobi distributions, preprint, Brno, 1990.

[50] O. Krupková, Variational analysis on fibered manifolds over one-dimensional bases, Ph.D. Thesis, Dept. of Mathematics, Silesian Univ. at Opava, Czechoslovakia, 1992, 1–67.

[51] O. Krupková, Variational metrics on $R \times TM$ and the geometry of nonconservative mechanics, Math. Slovaca 44 (3) (1994) 315–335.

[52] O. Krupková, Symmetries and first integrals of time-dependent higher-order constrained systems, J. Geom. Phys. 18 (1996) 38–58.

[53] O. Krupková, *The Geometry of Ordinary Variational Equations*, Lecture Notes in Mathematics 1678, Springer-Verlag, 1997, 251 pp.

[54] O. Krupková, A. Vondra, On some integration methods for connections on fibered manifolds, Proc. Conf. Diff. Geom. and Its Appl., Opava, 1992, Silesian University, Opava (1993) 89–101.

[55] V. Lychagin, *Lectures on Geometry of Differential Equations*, Università “La Sapienza”, 1992, Roma.

[56] L. Mangiarotti, M. Modugno, Fibred Spaces, Jet Spaces and Connections for Field Theories, Proceedings of International Meeting “Geometry and Physics”, Florence, 1982 (1983) Pitagora Editrice, Bologna, 135–165.

[57] L. Mangiarotti, M. Modugno, Connections and differential calculus on fibred manifolds. Aplications to field theory, Istituto di Matematica Applicata “G. Sansone”, Firense (1989) 1–147.

[58] E. Massa, E. Pagani, Jet bundle geometry, dynamical connections, and the inverse problem of Lagrangian mechanics, Ann. Inst. Henri Poincaré: Phys. Theor. 61 (1994) 17–62.

[59] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Shikagen, 1986.
[60] R. Miron, Metrical Finsler structures and metrical Finsler connections, J. Math. Kyoto Univ. 23 (2) (1983) 219–224.
[61] M. Modugno, Jet involution and prolongation of connections, Časopis Pěst. Mat. 114 (4) (1989) 356–365.
[62] P.J. Olver, *Aplications of Lie Groups to Differential Equations*, Springer, 1986.
[63] J.F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudogroups*, Mir, 1983, Moscow, transl. to Russian.
[64] M.M. Postnikov, *Lectures on Geometry IV. Differencial Geometry*, Nauka, 1988, Moscow, Russian.
[65] G.E. Prince, Toward a classification of dynamical symmetries in classical mechanics, Bull. Austral. Math. Soc. 27 (1983) 53–71.
[66] G.E. Prince, A complete classification of dynamical symmetries in classical mechanics, Bull. Austral. Math. Soc. 32 (1985) 299–308.
[67] P.K. Rashevskiĭ, *Geometrical Theory of Partial Differential Equations*, OGIZ, 1947, Moscow, Russian.
[68] H. Rund, *The differential geometry of Finsler spaces*, Springer, Berlin, 1959.
[69] W. Sarlet, F. Cantrijn, M. Crampin, A new look at second–order equations and Lagrangian machanics, J. Phys. A: Math. Gen. 17 (1984) 1999–2009.
[70] W. Sarlet, Symmetries and alternative Lagrangians in higher-order mechanics, Physics Lett. A 108 (1) (1985) 14–18.
[71] W. Sarlet, A. Vandecasteele, F. Cantrijn, E. Martínez, Derivations of forms along a map: the framework for time-dependent second-order equations, Diff. Geom. Appl. 5 (1995) 171–203.
[72] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, 1989, London Mathematical Society Lecture Note Series 142, Cambridge.
[73] V.V. Stepanov, *Course on Differential Equations*, Moscow, 1950, Russian.
[74] S. Sternberg, *Lectures on Differential Geometry*, Prentice–Hall, 1965, New York.
[75] J. Szenthe. Lagrangians and sprays, Annales Univ. Sci. Budapest 35 (1992) 1–5.
[76] G. Thompson, Second–order equation fields and the inverse problem of Lagrangian dynamics, J. Math. Phys. 28 (112) (1987) 2851–2857.
[77] A.M. Vinogradov, I.S. Krasilʹshchik, V.V. Lychagin, *Introduction to the Geometry of Nonlinear Differential Equations*, Nauka, 1986, Moscow, Russian.
[78] A. Vondra, Semisprays, connections and regular equations in higher order mechanics, In Proc. Conf. Diff. Geom. and Its Appl., Brno 1989, (1990) Word Scientific 276–287 Singapore.
[79] A. Vondra, On some connections related to the geometry of regular higher-order dynamics, Sborník VA, řada B 2 (1992) 7–18.
[80] A. Vondra, Sprays and homogeneous connections on $R \times TM$, Arch. Math. (Brno) 28 (1992) 163–173.
[81] A. Vondra, Natural dynamical connections, Czechoslovak Math. J. 41 (1991) 724–730.
[82] A. Vondra, Symmetries of connections on fibered manifolds, Arch. Math. (Brno) 30 (2) (1994) 97–115.
[83] A. Vondra, Geometry of second-order connections and ordinary differential equations, Mathematica Bohemica 120 (2) (1995) 145–167.
[84] A. Vondra, Towards a geometry of higher-order differential equations represented by connections on fibered manifolds, Thesis, Military Academy, Brno, 199, pp. 114.
[85] A. Vondra, Prolongations and fields of paths for higher-order O.D.E. represented by connections on a fibered manifold, Extracta Mathematicae 11 (1) (1996) 229–242.
[86] A. Vondra, Higher-order differential equations represented by connections on prolongations of a fibered manifold, to appear in Extracta Mathematical, 2000.
[87] C. von Westenholz, *Differential Forms in Mathematical Physics*, North-Holland, 1981, Amsterdam, revised edition.

[88] R.V. Vosylius, The contravariant theory of the differential prolongation on a space with a connection, VINITI, Itogi nauki i tekhniki, Problemy geometrii (14), Moscow (1983), 101–176 Russian.

Alexandr Vondra
Mathematical Institute
Silesian University in Opava
Bezručovo nám. 13, 746 01 Opava
Czech Republic
E-mail: Alexandr.Vondra@math.slu.cz

Received 10 December 1999