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# On recursion operators and nonlocal symmetries of evolution equations[1]

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**Abstract.** We consider the recursion operators with nonlocal terms of special form for evolution systems in $(1+1)$ dimensions, and extend them to well-defined operators on the space of nonlocal symmetries associated with the so-called universal Abelian coverings over these systems. The extended recursion operators are shown to leave this space invariant. These results apply, in particular, to the recursion operators of the majority of known today $(1+1)$-dimensional integrable evolution systems. We also present some related results and describe the extension of them and of the above results to $(1+1)$-dimensional systems of PDEs transformable into the evolutionary form. Some examples and applications are given.

**Keywords and phrases.** Nonlocal symmetries, evolution equations, universal Abelian covering, recursion operators.

**MS classification.** 35Q53, 35Q55, 35Q58, 37K05, 37K10, 37K45, 37K30.

## 1. Introduction

The scalar $(1+1)$-dimensional evolution equation possessing an infinite-dimensional commutative Lie algebra of *time-independent* local generalized (Lie–Bäcklund) symmetries is usually either linearizable or integrable via inverse scattering transform, see e.g. [1] – [5], [8, 9, 12, 14, 18] for the survey of known results and [10] for the generalization to (2+1) dimensions. The existence of such algebra is usually proved by exhibiting the recursion operator [12] or master symmetry [2]. But in order to possess the latter

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the equation in question must have higher order *time-dependent* symmetries, which usually turn out to be nonlocal. This fact is one of the main reasons for growing interest in the study of the whole algebra of time-dependent symmetries of evolution equations [1]. Moreover, for the evolution equations with time-dependent coefficients it is natural to consider their time-dependent symmetries from the very beginning [3, 7, 15].

Nowadays there seem to exist two basic approaches to the definition of nonlocal symmetries of PDEs. They are presented in the papers of Fuchssteiner [2, 3] and the book of Błaszak [1], and references therein, and in the works of Vinogradov et al., see e.g. [6, 16, 17] and references therein. Our approach is a combination of both.

In the present paper we shall consider the objects called by Vinogradov et al. the *shadows* of nonlocal symmetries associated with the so-called universal Abelian covering (UAC) over the evolution system (1). It was shown by Khor'kova [6] that in the case of UAC any shadow can be lifted to a nonlocal symmetry over UAC in the sense of [6, 16]. Moreover, it is easy to see that the nonlocal symmetries in the sense of [6, 16] with zero shadows are in a sense trivial and hardly represent any practical interest. Hence, we lose no essential information about the nonlocal symmetries of (1), when we restrict ourselves to considering just the shadows.

Since most of authors, see e.g. [1, 2, 3, 15] and references therein, call similar objects just "nonlocal symmetries", in this paper we shall essentially keep this tradition, calling the objects of our study "nonlocal UAC symmetries". This does not lead to any confusion, because nonlocal symmetries in the sense of [6, 16] will not appear in this paper. The precise definitions and further comments are given in Section 2 below. Note that our interest in nonlocal UAC symmetries was inspired by the results from [6, 16], where it was shown that in certain cases it is possible to extend the action of the recursion operator so that its application to these symmetries produces symmetries of the same kind.

In Section 3 we prove two main results of the present paper. The first one states that a large class of recursion operators for systems (1) can be extended to well-defined operators, namely to recursion operators in the sense of Guthrie [4], acting on the space of nonlocal UAC symmetries and leaving this space invariant. The second result states that under certain conditions, which are satisfied for the majority of known examples, the repeated application of (extended) recursion operator to weakly nonlocal UAC symmetries (i.e., to the symmetries that depend only on the nonlocal variables of the first level, that is, on the integrals of local conserved densities, and are linear in these variables) again yields weakly nonlocal UAC symmetries. An important consequence of the latter result is that the hereditary algebras (see e.g. [1] for definition) of time-dependent symmetries for evolution systems (1) are usually contained in the set of weakly nonlocal UAC symmetries. Note that our results are in a sense complementary to Wang's [18] sufficient conditions of *locality* of time-independent symmetries, obtained with usage of recursion operator.

In Section 4 we explain how the results of Section 3 can be transferred from (1+1)-dimensional evolution systems to arbitrary $(1+1)$-dimensional systems of PDEs transformable into the evolutionary form by the appropriate change of variables. This is illustrated by the example of sine-Gordon equation. Finally, in Section 5 we discuss further applications of our results and consider the example of Harry Dym equation.

In Appendices A and B we present two useful technical results: the characterization of kernel of the total $x$-derivative in the space of nonlocal UAC functions and the proof of the fact that the set of nonlocal UAC symmetries for (1) is a Lie algebra under the so-called Lie bracket—a natural commutator for nonlocal symmetries (cf. e.g. [1]). Note that if the hereditary algebra of time-dependent symmetries for (1) is generated by (a finite or infinite number of) nonlocal UAC symmetries, then by virtue of the latter result *all* elements of this algebra are nonlocal UAC symmetries.

## 2. Basic definitions and structures

Let us consider a $(1+1)$-dimensional system of evolution equations

(1) $$\partial \mathbf{u}/\partial t = \mathbf{F}(x, t, \mathbf{u}, \ldots, \mathbf{u}_n)$$

for the $s$-component vector function $\mathbf{u} = (u^1, \ldots, u^s)^T$. Here $\mathbf{u}_j = \partial^j \mathbf{u}/\partial x^j$, $\mathbf{u}_0 \equiv \mathbf{u}$; $\mathbf{F} = (F^1, \ldots, F^s)^T$; $^T$ denotes the matrix transposition. As in [11, 14], we make a *blanket assumption* that all functions below (**F**, symmetries **G**, etc.) are locally analytic functions of their arguments. This allows, in particular, to avoid the pathologies caused by the existence of divisors of zero in the ring of $C^\infty$-smooth functions.

**2.1. Universal Abelian covering.** Following [6, 16], describe the construction of universal Abelian covering over a system of PDEs for the particular case of system (1).

A function $f(x, t, \mathbf{u}, \mathbf{u}_1, \ldots)$ is called *local* (cf. [8, 9, 10]), if a) $f$ depends only on a finite number of variables $\mathbf{u}_k$ and b) $f$ is a locally analytic function of its arguments.

The operators of total $x$- and $t$-derivatives on the space of local functions are defined as (cf. e.g. [12, Ch. V])

$$D_x^{(0)} = \partial/\partial x + \sum_{I=1}^{s}\sum_{i=0}^{\infty} u_{i+1}^I \partial/\partial u_i^I,$$
$$D_t^{(0)} = \partial/\partial t + \sum_{I=1}^{s}\sum_{i=0}^{\infty} (D_x^{(0)})^i (F^I) \partial/\partial u_i^I.$$

Let $\{D_t^{(0)}(\rho_\alpha^{(0)}) = D_x^{(0)}(\sigma_\alpha^{(0)}) \mid \alpha \in \mathcal{I}_1\}$ be a basis for the space $\mathrm{CL}_F^{(0)}$ of nontrivial local conservation laws for (1) considered modulo trivial ones. Recall that locality means that $\rho_\alpha^{(0)}$ and $\sigma_\alpha^{(0)}$ are local functions and nontriviality means that $\rho_\alpha^{(0)} \notin \operatorname{Im} D_x^{(0)}$ (i.e., $\rho_\alpha^{(0)}$ cannot be represented as a total $x$-derivative of a local function).

We introduce [6, 16] nonlocal variables $\omega_\alpha^{(1)}$ of the first level as “integrals” of $\rho_\alpha^{(0)}$. Namely, we define them for all $\alpha \in \mathcal{I}_1$ as a solution of the system of PDEs

(2) $$\partial \omega_\alpha^{(1)}/\partial x = \rho_\alpha^{(0)}, \qquad \partial \omega_\alpha^{(1)}/\partial t = \sigma_\alpha^{(0)}.$$

It is clear that $\omega_\alpha^{(1)}$ are nothing but the *potentials* for the conserved currents $(\rho_\alpha^{(0)}, \sigma_\alpha^{(0)})$.

Now let us extend the action of operators $D_x^{(0)}$ and $D_t^{(0)}$ to the functions that depend on $\omega_\alpha^{(1)}$ by means of the formulae

$$D_x^{(1)} = D_x^{(0)} + \sum_{\alpha\in\mathcal{I}_1} \rho_\alpha^{(0)} \partial/\partial\omega_\alpha^{(1)}, \quad D_t^{(1)} = D_t^{(0)} + \sum_{\alpha\in\mathcal{I}_1} \sigma_\alpha^{(0)} \partial/\partial\omega_\alpha^{(1)}$$

and consider a basis $\{D_t^{(1)}(\rho_\alpha^{(1)}) = D_x^{(1)}(\sigma_\alpha^{(1)}) \mid \alpha \in \mathcal{I}_2\}$ in the set $\mathrm{CL}_F^{(1)}$ of all nontrivial (nontriviality now means that $\rho_\alpha^{(1)} \notin \operatorname{Im} D_x^{(1)}$) local conservation laws for the system of equations (1), (2). The densities $\rho_\alpha^{(1)}$ and fluxes $\sigma_\alpha^{(1)}$ may depend not only on $x, t, \mathbf{u}$, $\mathbf{u}_1, \mathbf{u}_2, \ldots$, but on $\vec{\omega}^{(1)}$ as well, and any given $\rho_\alpha^{(1)}$ or $\sigma_\alpha^{(1)}$ depends only on a finite number of $\mathbf{u}_r$ and of nonlocal variables $\omega_\alpha^{(1)}$. Here $\vec{\omega}^{(1)}$ denotes the totality of variables $\omega_\alpha^{(1)}$ for $\alpha \in \mathcal{I}_1$.

We further define the nonlocal variables of second level by means of the relations

$$\partial\omega_\alpha^{(2)}/\partial x = \rho_\alpha^{(1)}, \alpha \in \mathcal{I}_2,$$
$$\partial\omega_\alpha^{(2)}/\partial t = \sigma_\alpha^{(1)}, \alpha \in \mathcal{I}_2,$$

extend the action of $D_x^{(1)}$ and $D_t^{(1)}$ to the functions that may depend on $\omega_\alpha^{(2)}$, and so on.

Iterating this procedure infinite number of times, we obtain an infinite-dimensional covering $\mathcal{U}$ over (1), which is called *universal Abelian covering* (UAC) [6, 16]. More precisely, the covering constructed in this way is just a representative of the class of equivalent coverings, and the authors of [6, 16] identify UAC with this class.

Thus, $\mathcal{U}$ involves the infinite set of nonlocal variables $\omega_\alpha^{(j)}$ defined by the relations

$$\partial\omega_\alpha^{(j)}/\partial x = \rho_\alpha^{(j-1)}, \alpha \in \mathcal{I}_j,\ j \in \mathbb{N}, \tag{3}$$

$$\partial\omega_\alpha^{(j)}/\partial t = \sigma_\alpha^{(j-1)}, \alpha \in \mathcal{I}_j,\ j \in \mathbb{N}. \tag{4}$$

Here $\mathcal{I}_{k+1}$, $k \geq 1$, is a set of indices such that the conservation laws $D_t^{(k)}(\rho_\alpha^{(k)}) = D_x^{(k)}(\sigma_\alpha^{(k)})$ for $\alpha \in \mathcal{I}_{k+1}$ form a basis in the set $\mathrm{CL}_F^{(k)}$ of all nontrivial local conservation laws of the form $D_t^{(k)}(\rho) = D_x^{(k)}(\sigma)$ for (1) and (3), (4) with $j \leq k$. The locality means that the densities $\rho$ and fluxes $\sigma$ of conservation laws from $\mathrm{CL}_F^{(k)}$ depend only on $x, t, \mathbf{u}, \mathbf{u}_1, \ldots$ and $\vec{\omega}^{(1)}, \ldots, \vec{\omega}^{(k)}$, but not on $\vec{\omega}^{(m)}$ with $m > k$, and any given density or flux depends only on a finite number of $\mathbf{u}_r$ and of nonlocal variables $\omega_\alpha^{(j)}$. The nontriviality of a conservation law $D_t^{(k)}(\rho) = D_x^{(k)}(\sigma)$ from $\mathrm{CL}_F^{(k)}$ means that $\rho$ cannot be represented in the form $D_x^{(k)}(f)$ for some $f = f(x, t, \vec{\omega}^{(1)}, \ldots, \vec{\omega}^{(k)}, \mathbf{u}, \mathbf{u}_1, \ldots)$.

We employed here the notation

$$D_x^{(k)} = D_x^{(k-1)} + \sum_{\alpha\in\mathcal{I}_k} \rho_\alpha^{(k-1)}\partial/\partial\omega_\alpha^{(k)}, \quad D_t^{(k)} = D_t^{(k-1)} + \sum_{\alpha\in\mathcal{I}_k} \sigma_\alpha^{(k-1)}\partial/\partial\omega_\alpha^{(k)},$$

and the notation $\vec{\omega}^{(k)}$ for the totality of variables $\omega_\alpha^{(k)}$, $\alpha \in \mathcal{I}_k$. We shall also denote by $\vec{\omega}$ the totality of variables $\omega_\alpha^{(j)}$ for all $j$ and $\alpha$.

The operators of total derivatives on the space of functions of $x, t, \vec{\omega}, \mathbf{u}, \mathbf{u}_1, \ldots$ are

$$D \equiv D_x = D_x^{(0)} + \sum_{j=1}^{\infty} \sum_{\alpha\in\mathcal{I}_j} \rho_\alpha^{(j-1)}\partial/\partial\omega_\alpha^{(j)},$$
$$D_t = D_t^{(0)} + \sum_{j=1}^{\infty} \sum_{\alpha\in\mathcal{I}_j} \sigma_\alpha^{(j-1)}\partial/\partial\omega_\alpha^{(j)}.$$

The relations $D_t^{(k)}(\rho_\alpha^{(k)}) = D_x^{(k)}(\sigma_\alpha^{(k)})$ imply the compatibility of (3) and (4). In turn, the consequence of the latter and of the equality $[D_x^{(0)}, D_t^{(0)}] = 0$ are the relations

$$[D_x^{(k)}, D_t^{(k)}] = 0, k = 1, 2, \ldots,$$
$$[D_x, D_t] = 0.$$

We shall say (cf. [8, 10, 11]) that a function $f = f(x, t, \vec{\omega}, \mathbf{u}, \mathbf{u}_1, \ldots)$ is a *nonlocal UAC* function, if a) $f$ depends only on a finite number of variables $\omega_\alpha^{(j)}$ and $\mathbf{u}_k$ and b) $f$

is a locally analytic function of its arguments. We shall call $f$ a nonlocal UAC function of level $k$, if $f$ is a nonlocal UAC function independent of $\omega_\alpha^{(j)}$ for $j > k$.

Since the kernel of $D$ in the space of nonlocal UAC functions is exhausted by functions of $t$ (see Appendix A for the proof), it is easy to verify that our definition (cf. [9]) of nontriviality of a conservation law is in fact equivalent to the standard one [12, 16].

Let us stress that $\rho_\alpha^{(k-1)}$ are defined up to the addition of the terms from $\operatorname{Im} D_x^{(k-1)}$, and $\sigma_\alpha^{(k-1)}$ are defined up to the addition of the terms from $\ker D_x^{(k-1)}$, i.e., they should be considered as equivalence classes modulo $\operatorname{Im} D_x^{(k-1)}$ and $\ker D_x^{(k-1)}$, respectively. This means, in particular, that the nonlocal variables $\omega_\alpha^{(k)}$ are defined up to the addition of arbitrary nonlocal UAC functions of level $k-1$.

Making different choices of representatives in these equivalence classes yields different coverings over (1), but these coverings are equivalent in the sense of definition from [16, Ch. 6], and in the sequel we shall assume that we deal with a fixed representative of the respective equivalence class, because constantly operating with the whole class in the explicit computations is extremely inconvenient. However, the results obtained below are obviously independent of this choice, and thus hold true for the whole class of equivalent coverings, which result from the above construction.

**2.2. Nonlocal UAC symmetries.** We shall call (cf. [1, 11, 12, 16]) an $s$-component nonlocal UAC vector function $\mathbf{G}$ a *nonlocal UAC symmetry* of (1), if the evolution system $\partial\mathbf{u}/\partial\tau = \mathbf{G}$ is compatible with (1), i.e., $\partial^2\mathbf{u}/\partial t\partial\tau = \partial^2\mathbf{u}/\partial\tau\partial t$, where the derivatives with respect to $t$ and $\tau$ are computed with usage of (1), (4) and $\partial\mathbf{u}/\partial\tau = \mathbf{G}$, respectively. We shall denote the set of all nonlocal UAC symmetries of (1) by $\mathrm{NS}_F(\mathcal{U})$. If $\partial\mathbf{G}/\partial\vec{\omega} = 0$, then $\mathbf{G}$ is called (local) *generalized* (or *higher local*, or just *local*) symmetry of (1), see e.g. [8, 10, 12]. With $\mathbf{G}$ being a nonlocal UAC vector function, the compatibility condition for (1) and $\partial\mathbf{u}/\partial\tau = \mathbf{G}$ takes the form

$$D_t(\mathbf{G}) = \mathbf{F}'[\mathbf{G}], \tag{5}$$

where $\mathbf{F}' = \sum_{i=0}^n \partial\mathbf{F}/\partial\mathbf{u}_i D^i$.

Let us mention that nonlocal UAC symmetries are nothing but a particular case of general nonlocal symmetries, considered e.g. in [1, 2]. Indeed, the determining equations for the latter, given in [1, 2], are nothing but the compatibility conditions for (1) and $\mathbf{u}_\tau = \mathbf{G}$, and these conditions reduce to (5), if $\mathbf{G}$ is a nonlocal UAC vector function.

The set $\mathrm{NS}_F(\mathcal{U})$ of nonlocal UAC symmetries is interesting and important. In particular, our results imply that for nearly all known examples the action of the recursion operator is well defined on $\mathrm{NS}_F(\mathcal{U})$ and leaves it invariant. Hence, $\mathrm{NS}_F(\mathcal{U})$ contains all elements of the hereditary algebra (see [1, 2, 3] for its precise definition) of time-dependent symmetries for (1), if they are generated (cf. e.g. [1]) by means of the repeated application of the recursion operator to the scaling symmetry of (1) and to a time-independent local generalized symmetry of (1) (e.g. to $\mathbf{F}$, if $\partial\mathbf{F}/\partial t = 0$). Note that $\mathrm{NS}_F(\mathcal{U})$ is a Lie algebra with respect to the so-called Lie bracket (see Appendix B below for the proof). Therefore, if the hereditary algebra for (1) is generated by (a finite or infinite number of) nonlocal UAC symmetries, then *all* its elements are nonlocal UAC symmetries. Note that there also exist (see e.g. [8, 15]) integrable systems (1) that possess only a finite number of local generalized symmetries, but have infinite hierarchies of nonlocal symmetries, and these nonlocal symmetries turn out to be nonlocal UAC ones.

To avoid possible confusion, let us stress that the definition of nonlocal symmetries used in [6, 16] is different from the ours, but by Theorem 3.1 from [6] we always can recover from $\mathbf{G} \in \mathrm{NS}_F(\mathcal{U})$ the nonlocal symmetry in the sense of [6, 16].

## 3. On the action of recursion operators

In this section we prove for the case of systems (1) and recursion operators of the form (7) the conjecture of Khor'kova [6] stating that for any $(1+1)$-dimensional system of PDEs its recursion operator $\mathfrak{R}$ can be extended to a well-defined operator $\tilde{\mathfrak{R}}$ on the space $\mathrm{NS}_F(\mathcal{U})$ of nonlocal UAC symmetries and leaves this space invariant.

In complete analogy with the case of local functions, see e.g. [12, Ch. V, §5.3], we call the operator $\mathfrak{B}^\dagger = \sum\limits_{i=0}^{q} (-D)^i \circ b_i^T$ a *formal adjoint* of $\mathfrak{B} = \sum\limits_{i=0}^{q} b_i D^i$. Here $b_i$ are some $r \times r$ matrix-valued nonlocal UAC functions, $^T$ denotes the matrix transposition and $\circ$ stands for the composition of operators.

An $s$-component nonlocal UAC vector function $\gamma$ is called a *cosymmetry* [1, 18] of (1), if it satisfies the equation

$$D_t(\gamma) + (\mathbf{F}')^\dagger[\gamma] = 0. \tag{6}$$

Nearly all known today recursion operators for systems (1) have the form (cf. [18])

$$\mathfrak{R} = \sum_{i=0}^{k} a_i D^i + \sum_{j=1}^{p} \mathbf{G}_j \otimes D^{-1} \circ \gamma_j, \tag{7}$$

where $a_i$ are $s \times s$ matrix-valued local functions, and $\mathbf{G}_j$ and $\gamma_j$ are local symmetries and cosymmetries of (1), respectively.

Note that the action of recursion operators of the form (7) is initially defined only on *local* generalized symmetries (see e.g. [12]), but the formula (7) together with the subsequent definition of $D^{-1}$ enable us to construct an extension $\tilde{\mathfrak{R}}$ of $\mathfrak{R}$ to the space $\mathrm{NS}_F(\mathcal{U})$. However, it is not clear *a priori* whether $\tilde{\mathfrak{R}}$ is a well-defined operator on $\mathrm{NS}_F(\mathcal{U})$ and whether $\tilde{\mathfrak{R}}(\mathbf{G})$ is a nonlocal UAC symmetry of (1), provided so is $\mathbf{G}$.

Let us mention the following result of Wang [18]. Suppose that $\mathfrak{R}$ (7), with $\mathbf{G}_j$ and $\gamma_j$ being arbitrary $s$-component time-independent local vector functions, is a recursion operator for (1), both $\mathfrak{R}$ and system (1) are homogeneous with respect to a scaling of $x, t$ and $\mathbf{u}$, and $\partial \mathbf{F}/\partial t = 0$ and $\partial \mathfrak{R}/\partial t = 0$. Then under minor restrictions on $\mathbf{F}$ and $\mathfrak{R}$ the functions $\mathbf{G}_j$ and $\gamma_j$ indeed are symmetries and cosymmetries for (1), respectively. Obviously, this result remains valid when $a_i$, $\mathbf{G}_j$ and $\gamma_j$ are nonlocal UAC functions.

In order to proceed, we should agree how to interpret the action of $D^{-1}$. For our purposes it suffices to adopt the following definition, which is a particular case of the general construction of Guthrie [4]:

**Definition 1.** *Let $P$ be a nonlocal UAC function such that $D_t(P) = D(Q)$ for some nonlocal UAC function $Q$ (i.e., $P$ is a conserved density).*

*Then we shall understand under $R = D^{-1}(P)$ a solution of the system*

$$\begin{aligned} D(R) &= P, \\ D_t(R) &= Q. \end{aligned} \tag{8}$$

The existence of nonlocal UAC solution $R$ of (8), provided $D_t(P) = D(Q)$, is one of the fundamental properties of universal Abelian covering, proved in [6]. Note that according to the above definition we have $\omega_\alpha^{(j)} = D^{-1}(\rho_\alpha^{(j-1)})$, as it would be natural to expect. As we show in Appendix A, the kernel of $D$ in the space of nonlocal UAC functions is exhausted by functions of $t$, whence it is immediate that the solution $R$ of (8) for given $P$ and $Q$ is unique up to the addition of an arbitrary constant. So, the "integral" $D^{-1}(P)$, defined above, is not an integral of $P$ itself, but rather of a conservation law $D_t(P) = D(Q)$. It is also clear that the integrals $D^{-1}(P)$ evaluated for different $Q$ in general will differ by a function of $t$.

In order to prove that we can extend the action of $\mathfrak{R}$ to any nonlocal UAC symmetry $\mathbf{H}$ of (1), we have to show that for any $\mathbf{H} \in \mathrm{NS}_F(\mathcal{U})$ we have $D_t(\gamma_j \mathbf{H}) = D(\zeta_j)$ for some nonlocal UAC functions $\zeta_j$. Indeed, then the integrals $D^{-1}(\gamma_j \mathbf{H})$, interpreted in the sense of the above definition with $Q = \zeta_j$, are nonlocal UAC functions defined up to the addition of arbitrary constants. Note that the expressions like **ab** stand here and below for the scalar product of $s$-component vectors **a** and **b**.

By (5) and (6) we have $D_t(\mathbf{H}) = \mathbf{F}'[\mathbf{H}]$ and $D_t(\gamma_j) = -(\mathbf{F}')^\dagger[\gamma_j]$, whence

$$D_t(\gamma_j \mathbf{H}) = -(\mathbf{F}')^\dagger[\gamma_j]\mathbf{H} + \gamma_j \mathbf{F}'[\mathbf{H}]. \tag{9}$$

There is an obvious generalization (cf. e.g. [12, Ch. V, §5.3]) of the well-known Lagrange identity from theory of ordinary differential equations, namely

$$\vec{f}\mathfrak{B}(\vec{g}) - \mathfrak{B}^\dagger(\vec{f})\vec{g} = D(\eta), \quad \eta = \sum_{i=1}^{q} \sum_{j=0}^{i-1} (-D)^j (b_i^T \vec{f}) D^{i-j-1}(\vec{g}), \tag{10}$$

valid for any differential operator $\mathfrak{B} = \sum_{i=0}^{q} b_i D^i$ and for any $r$-component nonlocal UAC vector functions $\vec{f}$ and $\vec{g}$, provided $b_i$ are $r \times r$ matrix-valued nonlocal UAC functions.

Using (10) for $\mathfrak{B} = \mathbf{F}'$, $\vec{f} = \gamma_j$, $\vec{g} = \mathbf{H}$, we conclude (cf. [5]) that $D_t(\gamma_j \mathbf{H})$ indeed can be represented in the form $D(\zeta_j)$ for the nonlocal UAC function

$$\zeta_j = \sum_{i=1}^{n} \sum_{m=0}^{i-1} (-D)^m ((\partial \mathbf{F}/\partial \mathbf{u}_i)^T \gamma_j) D^{i-m-1}(\mathbf{H}). \tag{11}$$

Hence, for $P = \gamma_j \mathbf{H}$ we can always make a 'canonical' choice $Q = \zeta_j$ while computing $D^{-1}(P)$ according to Definition 1. With this choice and the above definition of $D^{-1}$, the recursion operator $\mathfrak{R}$ (7) is, in essence, replaced by a new operator $\tilde{\mathfrak{R}}$, which is easily seen to be a recursion operator in the sense of Guthrie [4].

Many important properties and definitions (for instance, that of hereditarity) can be readily transferred from $\mathfrak{R}$ to $\tilde{\mathfrak{R}}$. The operator $\tilde{\mathfrak{R}}$ is free [4] of the pathologies caused by naïve definition of $D^{-1}$, cf. [13] for an alternative way of overcoming these difficulties. In particular, it is easy to see that $\tilde{\mathfrak{R}}$ always maps nonlocal UAC symmetries to nonlocal UAC symmetries (cf. [4]), because $\mathfrak{R}$ satisfies the equation $[D_t - \mathbf{F}', \mathfrak{R}] = 0$ [12].

Thus, we have proved

**Proposition 2.** *Any recursion operator $\mathfrak{R}$ (7) for (1), with $a_i$ being $s \times s$ matrix-valued nonlocal UAC functions, and $\mathbf{G}_j$ and $\gamma_j$ being nonlocal UAC symmetries and cosymmetries for (1), respectively, can be extended to a well-defined operator $\tilde{\mathfrak{R}}$ that acts on the whole space* $\mathrm{NS}_F(\mathcal{U})$ *of nonlocal UAC symmetries for (1) and leaves this space invariant.*

Now let us consider what happens if $a_i$, $\mathbf{G}_j$, $\gamma_j$ are local and we apply $\tilde{\mathfrak{R}}$ to a local generalized symmetry $\mathbf{H}$ of (1). The above reasoning indeed holds true. Moreover, we see that $\gamma_j\mathbf{H}$ are *local* conserved densities for (1) and the respective $\zeta_j$ are local as well. Hence, the application of $\tilde{\mathfrak{R}}$ to local generalized symmetries of (1) yields nonlocal UAC symmetries of the form (cf. [6] and [16, Ch. 6])

$$\mathbf{H} = \mathbf{H}_0 + \sum_{\alpha\in\mathcal{I}_H} \mathbf{H}_\alpha \omega_\alpha^{(1)}, \tag{12}$$

where $\mathbf{H}_0$ and $\mathbf{H}_\alpha$ are $s$-component local vector functions, and $\mathcal{I}_H$ is a finite subset of $\mathcal{I}_1$.

We shall call an $r$-component nonlocal UAC vector function *weakly nonlocal UAC* vector function, if it can be represented in the form (12) with $\mathbf{H}_0$ and $\mathbf{H}_\alpha$ being $r$-component local vector functions. We shall denote by $\mathrm{WNLS}_F(\mathcal{U})$ the set of all weakly nonlocal UAC symmetries for (1). For the majority of integrable systems (1) their master symmetries are $s$-component weakly nonlocal UAC vector functions.

Note that if $\mathbf{H}$ is a local generalized symmetry of (1) and

$$\gamma_j\mathbf{H} = D(\xi_j), \tag{13}$$

where $\xi_j$ are local functions, then by the above $D^{-1}(\gamma_j\mathbf{H})$ can differ from $\xi_j$ only by a function of $t$. Hence, $D^{-1}(\gamma_j\mathbf{H})$ is a local function, and thus $\tilde{\mathfrak{R}}(\mathbf{H})$ is a *local* generalized symmetry for (1). In other words, the application of $\tilde{\mathfrak{R}}$ to local generalized symmetries of (1) satisfying (13) again yields local generalized symmetries of (1). Below we shall assume (obviously without loss of generality) that $\mathbf{G}_j$ in (7) are linearly independent. Then it is easy to see that the conditions (13) for $j = 1, \ldots, p$ are *equivalent* to the requirement that $\tilde{\mathfrak{R}}(\mathbf{H})$ is a local generalized symmetry, provided so is $\mathbf{H}$. Let us mention that Theorems 6–8 and 6–9 of Wang [18] provide an easy way to verify the conditions (13) for large families of time-independent local generalized symmetries of (1).

Since $\mathbf{F}$ and the coefficients of $D_t^{(1)}$ are independent of $\omega_\alpha^{(j)}$ for all $\alpha$ and $j$, it is immediate that for any nonlocal UAC symmetry $\mathbf{G}$ of level one for (1) the quantities $\partial\mathbf{G}/\partial\omega_\alpha^{(1)}$ satisfy the determining equation (5) and hence also are nonlocal UAC symmetries of level one for (1). In particular, for any $\mathbf{H}$ of the form (12) the quantities $\mathbf{H}_\alpha = \partial\mathbf{H}/\partial\omega_\alpha^{(1)}$ are in fact local generalized symmetries of (1).

Using this result, let us show that $\tilde{\mathbf{H}} = \tilde{\mathfrak{R}}(\mathbf{H}) \in \mathrm{WNLS}_F(\mathcal{U})$ for any $\mathbf{H}$ of the form (12), provided $\tilde{\mathfrak{R}}(\mathbf{H}_\alpha)$ are *local* generalized symmetries of (1) (or, equivalently, $\gamma_j\mathbf{H}_\alpha = D(\xi_{j,\alpha})$ for all $j = 1, \ldots, p$ and all $\alpha \in \mathcal{I}_H$, where $\xi_{j,\alpha}$ are local functions).

As $D^i(\omega_\alpha^{(1)}) = D^{i-1}(\rho_\alpha^{(0)})$ are local functions for $i \geq 1$, it is clear that $\tilde{\mathfrak{R}}(\mathbf{H}) \in \mathrm{WNLS}_F(\mathcal{U})$, if there exist scalar weakly nonlocal UAC functions $R_j$ such that

$$D(R_j) = \gamma_j\mathbf{H}, \tag{14}$$

$$D_t(R_j) = \zeta_j, \tag{15}$$

where $\zeta_j$ are given by (11). Indeed, then $\tilde{\mathfrak{R}}(\mathbf{H})$ is a weakly nonlocal UAC vector function, and by Proposition 2 $\tilde{\mathfrak{R}}(\mathbf{H}) \in \mathrm{NS}_F(\mathcal{U})$, hence $\tilde{\mathfrak{R}}(\mathbf{H}) \in \mathrm{WNLS}_F(\mathcal{U})$.

Let $\tilde{R}_j = R_j - \sum_{\alpha\in\mathcal{I}_H} \xi_{j,\alpha}\omega_\alpha^{(1)}$. Using (14), (15), we obtain

$$D(\tilde{R}_j) = \gamma_j\mathbf{H}_0 - \sum_{\alpha\in\mathcal{I}_H} \xi_{j,\alpha}\rho_\alpha^{(0)} \equiv \psi_j,$$

$$D_t(\tilde{R}_j) = \zeta_j - \sum_{\alpha\in\mathcal{I}_H} D_t(\xi_{j,\alpha}\omega_\alpha^{(1)}) \equiv \chi_j.$$

It is clear that $\psi_j$ are local functions and that $D_t(\psi_j) = D(\chi_j)$. If we show that $\chi_j$ are local as well, then $D^{-1}(\psi_j)$ obviously are linear combinations of $\omega_\alpha^{(1)}$ (modulo local functions), so $\tilde{R}_j$ are weakly nonlocal UAC functions, and the result follows.

As $\chi_j$ may depend on the nonlocal variables of the first level $\omega_\alpha^{(1)}$ at most, we only have to check that $\partial\chi_j/\partial\omega_\alpha^{(1)} = 0$, i.e., $\partial\zeta_j/\partial\omega_\alpha^{(1)} = D_t(\xi_{j,\alpha})$. Obviously, the only nonlocal terms in $\zeta_j$ are

$$\sum_{\alpha\in\mathcal{I}_H}\left(\sum_{i=1}^{n}\sum_{m=0}^{i-1}(-D)^m((\partial\mathbf{F}/\partial\mathbf{u}_i)^T\gamma_j)D^{i-m-1}(\mathbf{H}_\alpha)\right)\omega_\alpha^{(1)}.$$

Hence, in order to prove our result it remains to show that

$$\zeta_{j,\alpha} \equiv \sum_{i=1}^{n}\sum_{m=0}^{i-1}(-D)^m((\partial\mathbf{F}/\partial\mathbf{u}_i)^T\gamma_j)D^{i-m-1}(\mathbf{H}_\alpha) = D_t(\xi_{j,\alpha}).$$

Comparing this equality with (11) and bearing in mind that $\mathbf{H}_\alpha$ are local generalized symmetries of (1), we see that

$$D(\zeta_{j,\alpha}) = D_t(\gamma_j\mathbf{H}_\alpha) = D_t(D(\xi_{j,\alpha})) = D(D_t(\xi_{j,\alpha})).$$

Using Proposition 5 from Appendix A, we find

$$\zeta_{j,\alpha} = c_{j,\alpha}(t) + D_t(\xi_{j,\alpha}),$$

where $c_{j,\alpha}(t)$ is arbitrary function of $t$.

But it is clear that the function $\xi_{j,\alpha}$, determined from the relation $D(\xi_{j,\alpha}) = \gamma_j\mathbf{H}_\alpha$, is defined only up to the addition of arbitrary element of ker $D$, i.e., an arbitrary function $b_{j,\alpha}(t)$ of $t$. Hence, replacing $\xi_{j,\alpha}$ by $\xi_{j,\alpha} - \int_{t_0}^{t} c_{j,\alpha}(\tau)d\tau$, we can assume without loss of generality that $c_{j,\alpha}(t) = 0$, and thus $\zeta_{j,\alpha} = D_t(\xi_{j,\alpha})$, as required.

Thus, we have proved the following result, generalizing Proposition 4.1 from [6]:

**Proposition 3.** *Let (1) have a recursion operator $\mathfrak{R}$ (7), where $a_i$,$\mathbf{G}_j$,$\gamma_j$ are local. Then for any $\mathbf{H} \in \mathrm{WNLS}_F(\mathcal{U})$ the quantity $\tilde{\mathfrak{R}}(\mathbf{H})$ is well defined and $\tilde{\mathfrak{R}}(\mathbf{H}) \in \mathrm{WNLS}_F(\mathcal{U})$, provided $\tilde{\mathfrak{R}}(\partial\mathbf{H}/\partial\omega_\alpha^{(1)})$ are well-defined* local *generalized symmetries of (1) for all $\alpha \in \mathcal{I}_1$.*

If $\mathbf{H} = \tilde{\mathfrak{R}}(\mathbf{G})$ for some local generalized symmetry $\mathbf{G}$, then $\mathbf{H}_\alpha$ in (12) are linear combinations of the symmetries $\mathbf{G}_j$ that enter into $\mathfrak{R}$. We can easily see that the coefficients $\tilde{\mathbf{H}}_\alpha = \partial\tilde{\mathbf{H}}/\partial\omega_\alpha^{(1)}$ at $\omega_\alpha^{(1)}$ in the representation (12) for $\tilde{\mathbf{H}} = \tilde{\mathfrak{R}}^2(\mathbf{G})$ are linear combinations of $\tilde{\mathfrak{R}}(\mathbf{H}_\alpha)$ and of $\mathbf{G}_j$, and hence are in fact linear combinations of $\mathbf{G}_j$ and $\tilde{\mathfrak{R}}(\mathbf{G}_j)$ only. Thus, $\tilde{\mathbf{H}} \in \mathrm{WNLS}_F(\mathcal{U})$, provided $\tilde{\mathfrak{R}}(\mathbf{G}_j)$ are well-defined local generalized symmetries of (1). This is *equivalent* (cf. above) to the requirement that $\gamma_j\tilde{\mathfrak{R}}(\mathbf{G}_i) = D(\xi_{i,j})$, where $\xi_{i,j}$ are local functions, for all $i, j = 1, \ldots, p$.

Iterating this reasoning, we conclude that $\tilde{\mathfrak{R}}(\mathbf{Q})$ can be represented in the form (12) for any nonlocal UAC symmetry $\mathbf{Q}$ obtained by the repeated application of the recursion operator $\tilde{\mathfrak{R}}$ to local generalized symmetries, provided $\gamma_j\tilde{\mathfrak{R}}^d(\mathbf{G}_i) = D(\xi_{i,j,d})$, where $\xi_{i,j,d}$ are local functions, for all $i, j = 1, \ldots, p$ and all $d = 0, 1, 2, 3, \ldots$. These conditions are equivalent (see above) to the requirement that $\tilde{\mathfrak{R}}^d(\mathbf{G}_i)$ are well-defined *local* generalized symmetries of (1) for all $i = 1, \ldots, p$ and $d = 1, 2, 3, \ldots$.

In particular, if these conditions are satisfied, then for any (time-dependent) local generalized symmetry $\mathbf{G}$ of (1) we have $\tilde{\mathfrak{R}}^j(\mathbf{G}) \in \mathrm{WNLS}_F(\mathcal{U})$ for all $j = 1, 2, \ldots$. Hence, if the hereditary algebra (see e.g. [1] for its definition) of time-dependent symmetries for (1) is generated by the repeated application of the extension $\tilde{\mathfrak{R}}$ of a recursion operator $\mathfrak{R}$ (7) to some local generalized symmetries of (1), and $a_i$, $\mathbf{G}_j$ and $\gamma_j$ are local and satisfy the above conditions, then all elements of this algebra belong to $\mathrm{WNLS}_F(\mathcal{U})$.

## 4. Generalization to non-evolution systems

The above results can be applied to any $(1+1)$-dimensional systems of PDEs transformable into the evolutionary form (1) by the appropriate change of variables. This set includes, in particular, all systems transformable into Cauchy–Kovalevskaya form

$$\frac{\partial^{r_I} u^I}{\partial t^{r_I}} = \Phi_I\big(x, t, u^1, \ldots, u^q, \ldots, \partial^{\alpha+\beta} u^J / \partial t^\alpha \partial x^\beta, \ldots\big), \quad I = 1, \ldots, q, \tag{16}$$

where $\Phi_I$ may depend only on $x, t, u^1, \ldots, u^q$ and

$$\left\{\partial^{\alpha+\beta} u_J / \partial t^\alpha \partial x^\beta | \alpha \le r_J - 1, \beta \le k\right\}, \quad J = 1, \ldots, q.$$

The system (16) can be further transformed into an evolution system of the form (1) by introducing new dependent variables $v_\alpha^I = \partial^\alpha u^I / \partial t^\alpha$ for $\alpha = 1, \ldots, r_I - 1$. Indeed, combining the variables $u^1, \ldots, u^q$ and $v_\alpha^I$ into a single vector $\mathbf{v}$, we see that (16) together with the equations

$$\partial^\alpha u^I / \partial t^\alpha = v_\alpha^I, \ \alpha = 1, \ldots, r_I - 1, I = 1, \ldots, q,$$

forms the evolution system of exactly the same form as (1):

$$\partial \mathbf{v} / \partial t = \mathbf{K}(x, t, \mathbf{v}, \partial \mathbf{v} / \partial x, \partial^2 \mathbf{v} / \partial x^2, \ldots, \partial^k \mathbf{v} / \partial x^k). \tag{17}$$

The class of systems of PDEs transformable into the form (16) and hence into the evolutionary form (17) is very large. In particular, it includes [12, Ch. 2] all analytic locally solvable $(1+1)$-dimensional systems of PDEs possessing at least one noncharacteristic direction. The majority of known examples of non-evolutionary integrable $(1+1)$-dimensional systems are indeed transformable into the form (16) by the (appropriate modification of) above change of variables.

Hence, Khor'kova's conjecture stating that the (extended) recursion operators are well defined on nonlocal UAC symmetries holds true not only for the evolution systems (1) with the recursion operators (7), but also for any systems transformable into Cauchy–Kovalevskaya form (16) and then into (17), provided the recursion operator for transformed system (17) has the form (7). Indeed, making the inverse change of variables we can readily see that the (extended) recursion operator of original system is also well defined on its nonlocal UAC symmetries.

For instance, it is well known that the sine-Gordon equation $u_{\xi\eta} = \sin u$ can be transformed into $u_{tt} - u_{xx} = \sin u$ by setting $x = \xi - \eta$, $t = \xi + \eta$. Then, introducing a new dependent variable $v = u_t$, we obtain the evolution system (see e.g. [18] and references therein), equivalent to the SG equation:

$$u_t = v, \ v_t = u_{xx} + \sin u. \tag{18}$$

The recursion operator $\mathfrak{R}$ for (18) is (see e.g. [18]) of the form (7), so by Proposition 2 the action of $\tilde{\mathfrak{R}}$ on nonlocal UAC symmetries of (18) is well defined and leaves the space of these symmetries invariant. Returning to the original variables, we conclude that the same is true for the recursion operator (rewritten as a recursion operator in the sense of Guthrie [4]) of SG equation, so we recover the result of Khor'kova [6], initially obtained by straightforward computation.

## 5. Applications

Consider, for instance, the well known integrable Harry Dym equation $u_t = u^3 u_3$. Its recursion operator (see e.g. [18]) $\mathfrak{R} = u^2 D^2 - u u_1 D + u u_2 + u^3 u_3 D^{-1} \circ u^{-2}$ indeed has the form (7), and $u^{-2}$ is a cosymmetry and $u^3 u_3$ is an (obvious) symmetry for this equation. Using (11), we find that $D_t(u^{-2}G) = D(D^2(uG) - 3u_1 D(G))$ for any nonlocal UAC symmetry $G$ of HD equation, and hence $D^{-1}(u^{-2}G)$ is a nonlocal UAC function. Thus, by Proposition 2 the extension $\tilde{\mathfrak{R}}$ of the above $\mathfrak{R}$ is well defined on the space of nonlocal UAC symmetries of HD equation and leaves this space invariant.

It is possible to give a lot of other examples where Proposition 2 ensures that the (extended) recursion operators are well defined on the space of nonlocal UAC symmetries and leave it invariant. This fact often allows to draw a number of useful conclusions. For instance, provided the hereditary algebra [1, 2] of time-dependent symmetries is generated by the repeated application of the extension $\tilde{\mathfrak{R}}$ of a recursion operator $\mathfrak{R}$ of the form (7) to some *local* generalized symmetries, all elements of this algebra are nothing but nonlocal UAC symmetries. Moreover, if $\tilde{\mathfrak{R}}^k(\mathbf{G}_j)$ are well-defined *local* generalized symmetries for all $k \in \mathbb{N}$ and $j = 1, \ldots, p$, then by Proposition 3 all elements of this algebra are weakly nonlocal UAC symmetries, that is, they depend only on the nonlocal variables $\omega_\alpha^{(1)}$, i.e., on the "integrals" of nontrivial local conserved densities for (1), and are linear in $\omega_\alpha^{(1)}$. The hereditary algebra contains time-dependent symmetries of arbitrarily high order, hence their directional derivatives are formal symmetries of arbitrarily high order (see e.g. [10] for definition of formal symmetry) for (1). Thus, the evolution systems (1) possessing the hereditary algebra have time-dependent formal symmetries of arbitrarily high (and hence of infinite) order, and the coefficients of these formal symmetries are usually nonlocal (more precisely, weakly nonlocal) UAC functions.

To conclude, let us mention that it would be very interesting to generalize the results of the present paper to the evolution equations with constraints, introduced in [11].

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## Appendix A: On the structure of $\ker D$

The aim of this appendix is to prove that the kernel of operator $D$ in the space $\mathrm{NL}_F(\mathcal{U})$ of nonlocal UAC functions consists solely of functions of $t$, cf. [11].

Let $\mathcal{A}_0$ be the algebra of all scalar local functions under the standard mulplication.

We shall call an algebra $\mathcal{A}$ of scalar nonlocal UAC functions (under standard multiplication) *admissible*, if it has the following properties:

- for any locally analytic function $h(y_1, \dots, y_p)$ and any $a_j \in \mathcal{A}$ we have $h(a_1, \dots, a_p) \in \mathcal{A}$;
- $\mathcal{A}$ is closed under the action of $D$ and $D_t$;
- $\mathcal{A}$ is obtained from the algebra $\mathcal{A}_0$ by means of a finite sequence of extensions.

The third property means that there exists a finite chain of admissible algebras $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m = \mathcal{A}$ such that $\mathcal{A}_j$ is generated by the elements of $\mathcal{A}_{j-1}$ and just one new nonlocal variable $\zeta_j = D^{-1}(\eta_j)$, where $\eta_j \in \mathcal{A}_{j-1}$ is such that $\eta_j \notin \operatorname{Im} D|_{\mathcal{A}_{j-1}}$ and $D_t(\eta_j) \in \operatorname{Im} D|_{\mathcal{A}_{j-1}}$.

Consider a nonlocal UAC function $f$. It may depend only on a finite number $m$ of variables $\omega_\alpha^{(j)}$, and it is easy to see that there exists a minimal (i.e., obtained from $\mathcal{A}_0$ by means of the minimal possible number of extensions) admissible algebra $\mathcal{K}$ of scalar nonlocal UAC functions which contains $f$.

It is clear that in order to prove that $f \in \ker D$ implies that $f$ depends on $t$ only it suffices to prove that $\ker D|_{\mathcal{K}}$ consists solely of functions of $t$.

In order to proceed, we shall need the following

**Lemma 4.** *Let $\mathcal{A}$ be an admissible algebra,* $\ker D|_{\mathcal{A}}$ *consist solely of functions of $t$, and $\tilde{\mathcal{A}}$ be the extension of $\mathcal{A}$ obtained by adding the nonlocal variable $\zeta = D^{-1}(\gamma)$, where $\gamma \in \mathcal{A}$ is such that $\gamma \notin$* $\operatorname{Im} D|_{\mathcal{A}}$ *and $D_t(\gamma) \in$* $\operatorname{Im} D|_{\mathcal{A}}$.

*Then $\tilde{\mathcal{A}}$ is admissible and* $\ker D|_{\tilde{\mathcal{A}}}$ *also consists solely of functions of $t$.*

**Remark 1.** The conditions $\gamma \notin \operatorname{Im} D|_{\mathcal{A}}$ and $D_t(\gamma) \in \operatorname{Im} D|_{\mathcal{A}}$ imply that $\gamma$ is a linear combination of $\rho_\alpha^{(j)}$ (modulo the terms from $\operatorname{Im} D|_{\mathcal{A}}$).

**Remark 2.** This lemma is a natural generalization of Proposition 1.1 from [11] to the case of time-dependent nonlocal UAC functions, and its proof relies on the same ideas.

**Proof of the lemma.** The admissibility of $\tilde{\mathcal{A}}$ is obvious from the above, so it remains to describe $\ker D|_{\tilde{\mathcal{A}}}$. By definition, the elements of $\mathcal{A}$ may depend only on a finite number of nonlocal variables $\zeta_1, \dots, \zeta_m$.

Let $\mathcal{B}_0 \subset \mathcal{A}$ be the algebra of all locally analytic functions of $x, t, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_p$, $\zeta_1, \dots, \zeta_m$, where $p$ is the minimal number such that $\mathcal{B}_0$ contains $\gamma$ and $D(\zeta_i)$ for $i = 1, \dots, m$. It is straightforward to check that such $p$ does exist.

Consider the following chain of subalgebras of $\mathcal{A}$:

$$\mathcal{B}_{j+1} = \{h \in \mathcal{B}_j \mid D|_{\mathcal{A}}(h) = g\gamma, g \in \mathcal{B}_j\},\ j = 0, 1, 2, \dots.$$

Any locally analytic function of elements of $\mathcal{B}_j$ obviously belongs to $\mathcal{B}_j$.

As $\mathcal{B}_0$ is generated by $s(p+1)+m+2$ elements $x, t, u^I, u_1^I, \dots, u_p^I, \zeta_1, \dots, \zeta_m$, where $I = 1, \dots, s$, we conclude that $\mathcal{B}_{j+1} = \mathcal{B}_j$ for $j \geq s(p+1)+m+2$ (cf. [11]). Indeed, by construction $\mathcal{B}_{j+1} \subset \mathcal{B}_j$, and hence the functional dimension $d_{j+1}$ of $\mathcal{B}_{j+1}$ does not exceed that of $\mathcal{B}_j$. If these dimensions coincide, then we have $\mathcal{B}_{j+1} = \mathcal{B}_j$, and otherwise $d_{j+1} \leq d_j - 1$. Since $d_0 = s(p+1)+m+2$, it is clear that $d \equiv d_{s(p+1)+m+2} \leq 1$ provided $\mathcal{B}_{j+1} \neq \mathcal{B}_j$ for $j = 0, \dots, s(p+1)+m+1$. On the other hand, $d \geq 1$, because in any case $\mathcal{B}_{s(p+1)+m+2}$ contains the algebra of functions of $t$, and the result follows. Thus, $\mathcal{B}_{s(p+1)+m+2}$ is the algebra of all locally analytic functions of some its elements $z_1, \ldots, z_d$, i.e., it is generated by $z_1, \ldots, z_d$.

Let $f = f(x, t, \mathbf{u}, \ldots, \mathbf{u}_q, \zeta_1, \ldots, \zeta_m, \zeta) \in \ker D|_{\tilde{\mathcal{A}}}$, $\partial f/\partial\zeta \neq 0$. Differentiating the equality $D(f) = 0$ with respect to $\mathbf{u}_j$, $j > p$, we readily obtain $\partial f/\partial \mathbf{u}_j = 0$ for $j > p$. Therefore, $f \in \mathcal{B}_0$ for any fixed value of $\zeta$.

We have

$$D(f) = D|_{\mathcal{A}}(f) + \gamma\partial f/\partial\zeta = 0. \tag{19}$$

But (19) implies that $f \in \mathcal{B}_1$ for any fixed value of $\zeta$. Then, again by virtue of (19), we have $f \in \mathcal{B}_2$ for any fixed $\zeta$, and so on.

Thus, $f \in \mathcal{B}_{s(p+1)+m+2}$ for any fixed value of $\zeta$ and $D|_{\mathcal{A}}(f) \neq 0$. Hence, the operator $D|_{\mathcal{B}_{s(p+1)+m+2}}$ is nonzero and $D|_{\mathcal{B}_{s(p+1)+m+2}} = \gamma X$, where $X$ is a nonzero vector field on the space of variables $z_1, \ldots, z_d$.

Let $w \in \mathcal{B}_{s(p+1)+m+2}$ be a solution of equation $X(w) = 1$. Then $D|_{\mathcal{A}}(w) = \gamma$, what contradicts the assumption that $\gamma \notin \operatorname{Im} D|_{\mathcal{A}}$. The contradiction proves the lemma. □

The desired result about $\ker D|_{\mathcal{K}}$ readily follows, if we successively apply the above lemma for $\mathcal{A} = \mathcal{A}_0$ and $\tilde{\mathcal{A}} = \mathcal{A}_1$, then for $\mathcal{A} = \mathcal{A}_1$ and $\tilde{\mathcal{A}} = \mathcal{A}_2$, and so on, until we see that $\ker D|_{\mathcal{A}_m}$, $\mathcal{A}_m = \mathcal{K}$, consists solely of functions of $t$.

Our reasoning applies to any nonlocal UAC function $f$, so we have proved

**Proposition 5.** *The kernel of the operator $D$ in the space $\mathrm{NL}_F(\mathcal{U})$ of nonlocal UAC functions consists solely of functions of $t$.*

From this result it is immediate that the intersection $\ker D \cap \ker D_t$ in the space of nonlocal UAC functions consists solely of constants, and hence by Proposition 1.4 from [16, Ch. 6,§1] universal Abelian covering over (1) is locally irreducible.

Let us stress that the above proposition is not valid for the functions that depend on the *infinite* number of variables $\mathbf{u}_j$ and $\omega_\alpha^{(k)}$ at once.

Indeed, consider the well-known Burgers equation $u_t = u_2 + uu_1$, whose only nontrivial local conserved density is $u$ (see e.g. [16] for proof). Let $\psi = \psi(x, t, u, u_1, \ldots)$ be an arbitrary infinitely differentiable local function. Then it is straightforward to check that the function $\Psi = \sum_{j=0}^{\infty} \dfrac{\Upsilon^j(\psi)\omega^j}{j!}$, where $\omega = D^{-1}(u)$ and $\Upsilon = -(1/u)D$, belongs to $\ker D$. It is clear that $\Psi$ depends on an infinite number of variables $u_j = \partial^j u/\partial x^j$, $j = 0, 1, 2, \ldots$, provided $D(\psi) \neq 0$.

## Appendix B: Lie algebra structure of $\mathrm{NS}_F(\mathcal{U})$

In this appendix we prove that the set $\mathrm{NS}_F(\mathcal{U})$ is a Lie algebra with respect to the so-called Lie bracket (see e.g. [1]), defined as

$$[\mathbf{G}, \mathbf{H}] = \mathbf{H}'[\mathbf{G}] - \mathbf{G}'[\mathbf{H}]. \tag{20}$$

We employed here the notation $f'[\mathbf{H}] = (df(x, t, \mathbf{u} + \epsilon\mathbf{H}, \mathbf{u}_1 + \epsilon D(\mathbf{H}), \ldots)/d\epsilon)|_{\epsilon=0}$ for the directional derivative of any (smooth nonlocal) function $f$ along $\mathbf{H}$, see e.g. [1]. The bracket (20) is obviously skew-symmetric. It satisfies the Jacobi identity by virtue of properties of the directional derivative, see e.g. [1] and references therein.

If $f$ is a local function, then $f'[\mathbf{H}]=\sum\limits_{i=0}^{\infty}\dfrac{\partial f}{\partial \mathbf{u}_i}D^i(\mathbf{H})$ is a well-defined nonlocal UAC function for any $s$-component nonlocal UAC vector function $\mathbf{H}$, and $f'=\sum\limits_{i=0}^{\infty}\dfrac{\partial f}{\partial \mathbf{u}_i}D^i$ is a differential operator, cf. e.g. [8]–[14].

If $f$ is a nonlocal UAC function, then we have

$$f'[\mathbf{H}]=\sum_{k=0}^{\infty}\frac{\partial f}{\partial \mathbf{u}_k}D^k(\mathbf{H})+\sum_{j=1}^{\infty}\sum_{\alpha\in\mathcal{I}_j}\frac{\partial f}{\partial \omega_\alpha^{(j)}}\omega'^{(j)}_\alpha[\mathbf{H}]. \tag{21}$$

Hence, in order to show that $f'[\mathbf{H}]$ is a nonlocal UAC function for any $\mathbf{H}\in \mathrm{NS}_F(\mathcal{U})$, we should define $\omega'^{(j)}_\alpha[\mathbf{H}]$ and show that $\omega'^{(j)}_\alpha[\mathbf{H}]$ are nonlocal UAC functions.

From now on we assume that the integration constants arising while computing $D^{-1}$ according to Definition 1 are chosen so that for any constant $c$ we have $\tilde{R}\equiv D^{-1}(\tilde{P})=cR\equiv cD^{-1}(P)$, where $\tilde{P}=cP$ (and $\tilde{Q}=cQ$).

Using the interpretation of $\omega_\alpha^{(j)}$ as $D^{-1}(\rho_\alpha^{(j-1)})$, given above (cf. [1, 2, 11]), we set $\omega'^{(j)}_\alpha[\mathbf{H}]=D^{-1}(\rho'^{(j-1)}_\alpha[\mathbf{H}])$. The quantities $\rho'^{(j-1)}_\alpha[\mathbf{H}]$ can be computed inductively. Indeed, $\rho_\alpha^{(0)}$ are local functions, so we know the formula for $\rho'^{(0)}_\alpha[\mathbf{H}]$ (see above), and hence we can evaluate $\omega'^{(1)}_\alpha[\mathbf{H}]$. Next, as $\rho_\alpha^{(1)}$ involve only $\omega_\alpha^{(1)}$ and local variables $x,t,\mathbf{u},\mathbf{u}_1,\dots$, we can find $\rho'^{(1)}_\alpha[\mathbf{H}]$, using (21). Then we find $\rho'^{(2)}_\alpha[\mathbf{H}]$, and so on.

According to our definition of $D^{-1}$, in order to guarantee that $D^{-1}(\rho'^{(j)}_\alpha[\mathbf{H}])$ are well defined we have to show that there exist nonlocal UAC functions $\zeta_\alpha^{(j)}$ such that

$$D_t(\rho'^{(j)}_\alpha[\mathbf{H}])=D(\zeta_\alpha^{(j)}). \tag{22}$$

As $D_t(\rho_\alpha^{(j)})=D(\sigma_\alpha^{(j)})$, we have $(D_t(\rho_\alpha^{(j)}))'[\mathbf{H}]=(D(\sigma_\alpha^{(j)}))'[\mathbf{H}]$. Since $\mathbf{H}\in\mathrm{NS}_F(\mathcal{U})$, we readily obtain from (5) that $(D_t(\rho_\alpha^{(j)}))'[\mathbf{H}]=D_t(\rho'^{(j)}_\alpha[\mathbf{H}])$. It is also easy to see that $(D(\sigma_\alpha^{(j)}))'[\mathbf{H}]=D(\sigma'^{(j)}_\alpha[\mathbf{H}])$. Hence, $D_t(\rho'^{(j)}_\alpha[\mathbf{H}])=D(\sigma'^{(j)}_\alpha[\mathbf{H}])$, and (22) holds, if we take $\sigma'^{(j)}_\alpha[\mathbf{H}]$ for $\zeta_\alpha^{(j)}$. Using this result, we always can make a 'canonical' choice $Q=\sigma'^{(j)}_\alpha[\mathbf{H}]$ while computing $D^{-1}(P)$ for $P=\rho'^{(j)}_\alpha[\mathbf{H}]$ according to Definition 1, and thus get rid of possible pathologies.

It is clear from the above that if $\mathbf{G},\mathbf{H}\in\mathrm{NS}_F(\mathcal{U})$, then $\mathbf{G}'[\mathbf{H}]$ and $\mathbf{H}'[\mathbf{G}]$ are nonlocal UAC vector functions, and thus $[\mathbf{G},\mathbf{H}]$ is a nonlocal UAC vector function. Using (5) for $\mathbf{G}$ and $\mathbf{H}$, we can easily show that $[\mathbf{G},\mathbf{H}]\in\mathrm{NS}_F(\mathcal{U})$, so we have proved the following

**Proposition 6.** *The set* $\mathrm{NS}_F(\mathcal{U})$ *is a Lie algebra under the commutator (20).*

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