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Differential invariants of the metric tensor[1]

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Abstract. The problem of finding differential invariants of arbitrary order, depending on the metric tensor, is solved by the factorization method with respect to a proper subgroup of the differential group acting in the space of differential invariants. It is shown that the domain $T^r_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ of the differential invariants of order r has the structure of a trivial principal K^{r+1}_n-bundle, where K^{r+1}_n is a normal subgroup of the differential group L^{r+1}_n acting on the $T^r_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ from the left. Consequently, any differential invariant with values in a left $GL_n(\mathbb{R})$-manifold factorizes through the projection of this principal fiber bundle.

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1. Introduction

In this paper, representing an extension of the author's preprint [13], we apply the factorization method with respect to a subgroup of differential group, which was used for the first time by Krupka in [7]. It is not a sole method for finding differential invariants, but it allows exact formulation of the problem of finding invariants for general group actions. Using factorization method, we present here complete results; in particular, we find a basis of invariants in our case.

Let X be an n-dimensional smooth manifold, and Met$X$ be the bundle of metrics on X, i.e. the bundle of second order regular symmetric covariant tensors on X. The type fiber of Met$X$ is the left L^1_n-manifold $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$, where $\mathbb{R}^{n*}$ is dual vector space to the vector space $\mathbb{R}^n$. Let $T^r_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ be the prolongation of the left L^1_n-manifold

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$\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$. $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is the set of r-jets with source at the $0 \in \mathbb{R}^n$ and target in the $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ with natural structure of the left L_n^{r+1}-manifold. Then the r-jet prolongation $J^r\mathrm{Met}X$ of $\mathrm{Met}X$ has the structure of a fiber bundle with type fiber $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ associated with bundle of r-frames over X.

Let L_n^r be the r-th differential group. The general theory tells that each r-order differential invariant of the metric tensor is an L_n^r-equivariant mapping defined on the type fiber $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ of a fiber bundle $J^r\mathrm{Met}X$.

At this paper we describe the quotient space of the action of K_n^{r+1} on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$. As a consequence, we get the well known result that every differential invariant of the metric tensor depends only on the metric tensor, the curvature tensor of the Levi-Civita connection and the covariant derivatives of the curvature tensor.

2. Basic structures

Let L_n^r be the r-th differential group with the canonical global coordinate system

$$\left(a^i_j, a^i_{j_1 j_2}, \ldots, a^i_{j_1 j_2 \ldots j_r}\right), \ 1 \le i, j \le n, 1 \le j_1 \cdots \le j_k \le n, 1 \le k \le r.$$

L_n^r is the group of invertible r-jets with source and target at the origin $0 \in \mathbb{R}^n$. The coordinate functions $a^i_{j_1 \ldots j_k}$ are defined by

$$(1) \qquad a^i_{j_1 \ldots j_k}(A) = D_{j_1} D_{j_2} \cdots D_{j_k} \alpha_i(0),$$

where $A \in L_n^r$, $A = J_0^r \alpha$.

In this paper we will also use the second coordinate system

$$\left(b^i_j, b^i_{j_1 j_2}, \ldots, b^i_{j_1 j_2 \ldots j_r}\right), \ 1 \le i, j \le n, 1 \le j_1 \le \cdots \le j_k \le n, 1 \le k \le r,$$

with the coordinates $b^i_{j_1 j_2 \ldots j_k}$, defined by

$$(2) \qquad b^i_{j_1 \ldots j_k}(A) = a^i_{j_1 \ldots j_k}(A^{-1}) = D_{j_1} D_{j_2} \cdots D_{j_k} \alpha_i^{-1}(0),$$

where $A \in L_n^r$, $A = J_0^r \alpha$. It is known that the functions a^i_j, b^k_l satisfy the identity

$$(3) \qquad a^i_k b^k_j = \delta^i_j,$$

where δ^i_j denotes the Kronecker symbol.

Let $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ be the r-th prolongation of the left L_n^1-manifold $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$. $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is the set of r-jets with source at the $0 \in \mathbb{R}^n$ and target in the $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ with the natural structure of the left L_n^{r+1}-manifold.

Let Q be an arbitrary L_n^1-manifold. Let

$$(4) \qquad \pi_n^{r+1,1} : L_n^{r+1} \to L_n^1, \qquad \pi_n^{r+1,1}\left(b^i_j, b^i_{j_1 j_2}, \ldots, b^i_{j_1 j_2 \ldots j_r}\right) = (b^i_j)$$

be the canonical projection homomorphism of differential groups. A mapping

$$(5) \qquad F : T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \to Q$$

is called the *r-th order differential invariant of the metric tensor*, if it satisfies the condition

$$(6) \qquad F(A \cdot J_0^r f) = \pi^{r+1,1}(A) \cdot F(J_0^r f)$$

for each $J_0^r f \in T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$, $A \in L_n^{r+1}$.

Let K_n^{r+1} be the kernel of the homomorphism $\pi_n^{r+1,1}$, K_n^{r+1} is a normal subgroup in the L_n^{r+1} consisting of elements with the coordinates

$$\left(\delta^i_j, b^i_{j_1 j_2}, \dots, b^i_{j_1 j_2 \dots j_r}\right).$$

We can restrict the action of L_n^{r+1} to the subgroup K_n^{r+1} and construct the quotient space $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1}$.

Let us consider the quotient space $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1}$. We define the left action of L_n^1 (which is isomorphic to the L_n^{r+1}/K_n^{r+1}) on $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1}$ by the expression

$$J_0^1\alpha \cdot [w]_{K_n^{r+1}} = \left[\iota^{r+1}(J_0^1\alpha) \cdot w\right]_{K_n^{r+1}}, \tag{7}$$

where $J_0^1\alpha \in L_n^1$, $[w]_{K_n^{r+1}} \in T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1}$ and ι^{r+1} is the homomorphism

$$\iota^{r+1} : L_n^1 \to L_n^{r+1}, \qquad \iota^{r+1}(b^i_j) = (b^i_j, 0, 0, \dots, 0). \tag{8}$$

Formula (7) defines the left action of L_n^1 on $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1}$.

Lemma 1. *Let Q be a left L_n^1-manifold, let*

$$\pi : T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \to T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1}$$

be the canonical projection onto the orbit space. Then every differential invariant

$$F : T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \to Q$$

is of the form

$$F = f \circ \pi, \tag{9}$$

where

$$f : T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1} \to Q$$

is a uniquely determined L_n^1-equivariant mapping.

Proof. Let $F : T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \to Q$ be a differential invariant, let $w \in T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$. We can construct a mapping $f : T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1} \to Q$ by

$$f\left([w]_{K_n^{r+1}}\right) = F(w) \tag{10}$$

(it follows from (9)). Now we must verify that the f is well defined. It is defined on the whole $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1}$, because π is a surjection. We must show that the f is independent on the choice of representative. Let $[w_1]_{K_n^{r+1}} = [w_2]_{K_n^{r+1}}$. Then there exists $B \in K_n^{r+1}$ such that $w_1 = B \cdot w_2$ and

$$\begin{aligned} f\left([w_1]_{K_n^{r+1}}\right) &= F(w_1) = F(B \cdot w_2) = \pi^{r+1,1}(B) \cdot F(w_2) \\ &= \pi^{r+1,1}(B) f\left([w_2]_{K_n^{r+1}}\right) = f\left([w_2]_{K_n^{r+1}}\right) \end{aligned}$$

(the last equality is satisfied because $\pi^{r+1,1}(B) = J_0^1\mathrm{id}$ for each $B \in K_n^{r+1}$).

Now we prove the uniqueness of the f. Suppose that there exist two different mappings which satisfy (9)

$$F = f_1 \circ \pi, \qquad F = f_2 \circ \pi.$$

Then $f_1([w]_{K_n^{r+1}}) = f_2([w]_{K_n^{r+1}})$ for each $w \in T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ and $f_1 = f_2$ on the set

$$\pi\left(T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})\right) = T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1},$$

i.e. $f_1 = f_2$.

Now we must prove that the f is a L_n^1-equivariant mapping.

Let $A \in L_n^1$, $w \in T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$. Then

$$\begin{aligned} f\left(A \cdot [w]_{K_n^{r+1}}\right) &= f\left(\left[\iota^{1,r+1}(A)w\right]_{K_n^{r+1}}\right) = F\left(\iota^{1,r+1}(A)w\right) \\ &= \pi^{r+1,1}\left(\iota^{1,r+1}(A)\right) \cdot F(w) = A \cdot F(w) = A \cdot f\left([w]_{K_n^{r+1}}\right) \end{aligned}$$

and the f is a L_n^1-equivariant mapping.

If we replace the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ by an arbitrary L_n^{r+1}-manifold P, this lemma will be conserved (you can find the proof for example in [2]).

3. The action of L_n^1 on $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$

Let us consider $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$, the space of symmetric covariant second-order tensors on the $\mathbb{R}^n$. Denote by e_i the canonical basis of $\mathbb{R}^n$ and e^i the dual basis of $\mathbb{R}^{n*}$. Each element $g \in \mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ is then uniquely written in the form

$$g = g_{ij}(g)\, e^i \odot e^j,$$

where $1 \le i \le j \le n$.The system of functions

$$(g_{ij}), \quad 1 \le i \le j \le n,$$

define a global coordinate system on the $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$.

Let L_n^1 be the 1-st differential group with the canonical global coordinate system

$$(b^i_j), \quad 1 \le i, j \le n,$$

with the coordinates b^i_j defined by

$$b^i_j(A) = D_j \alpha_i^{-1}(0),$$

where $A \in L_n^1$, $A = J_0^1\alpha$. It is evident that the group L_n^1 can be identified with the general linear group $GL_n(\mathbb{R})$.

Let us consider the left L_n^1-manifold $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$. The standard left action of L_n^1 on $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ is introduced as follows. If $g \in \mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ and $J_0^1\alpha \in L_n^1$ then in the coordinates

$$g_{ij}(J_0^1\alpha \cdot g) = b_i^p(J_0^1\alpha)\, b_j^q(J_0^1\alpha)\, g_{pq}(g).$$

If we use the notation with a bar $\bar{g}_{ij} = g_{ij}(J_0^1\alpha \cdot g)$ without a bar $g_{ij} = g_{ij}(g)$, and $b_i^p = b_i^p(J_0^1\alpha)$ we can rewrite the last expression in the form

$$\bar{g}_{ij} = b_i^p b_j^q g_{pq}. \tag{11}$$

The action of L_n^1 on $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ is given by formula (11).

4. The action of L_n^3 on $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$

Now we will study the special case $r = 2$ in detail (we will find the second order differential invariants of the metric tensor). In this section we will explicitly express the action of L_n^3 on $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$.

Let $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ be the second prolongation of the $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$.

Let $Q \in T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ be a 2-jet and f be a mapping from a neighborhood of the $0 \in \mathbb{R}^n$ to the $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ such that $Q = J_0^2 f$. There exists a canonical global coordinate system

$$(g_{ij}, g_{ij,k}, g_{ij,kl}), \quad 1 \le i \le j \le n, 1 \le k \le l \le n,$$

on the $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$, defined by

$$\begin{aligned} & g_{ij}(Q) = g_{ij}(f(0)), \\ (12) \qquad & g_{ij,k}(Q) = D_k(g_{ij} f)(0), \\ & g_{ij,kl}(Q) = D_k D_l(g_{ij} f)(0). \end{aligned}$$

If we prolong the left action of L_n^1 on $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$, we obtain the left action of L_n^3 on $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$. Let $J_0^3\alpha \in L_n^3$, $J_0^2 f \in T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$, then

$$J_0^3\alpha \cdot J_0^2 f = J_0^2\Phi,$$

where Φ is a mapping from a neighborhood of the $0 \in \mathbb{R}^n$ to the $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ defined by

$$(13) \qquad \Phi(x) = J_0^1\left(t_x\alpha t_{-\alpha^{-1}(x)}\right) \cdot f\left(\alpha^{-1}(x)\right),$$

where t_x denotes the translation of $\mathbb{R}^n$, which transfer the point $x \in \mathbb{R}^n$ to the point $0 \in \mathbb{R}^n$ and dot means the action of L_n^1 on $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$. We express the formula (13) in the coordinates

$$g_{ij}\left(\Phi(x)\right) = D_i\left(t_x\alpha t_{-\alpha^{-1}(x)}\right)_p^{-1}(0) D_j\left(t_x\alpha t_{-\alpha^{-1}(x)}\right)_q^{-1}(0) g_{pq}\left(f\alpha^{-1}(x)\right).$$

It is easy to show that it is the same as $g_{ij}(\Phi(x)) = D_i\alpha_p^{-1}(0) D_j\alpha_q^{-1}(0) g_{pq}(f\alpha^{-1}(x))$. Now we can calculate. From now on we will denote for short

$$\begin{aligned} & b_i^p = b_i^p(J_0^1\alpha), \\ & \bar{g}_{ij} = g_{ij}(J_0^2\Phi), \qquad \bar{g}_{ij,k} = g_{ij,k}(J_0^2\Phi), \qquad \bar{g}_{ij,kl} = g_{ij,kl}(J_0^2\Phi), \\ & g_{ij} = g_{ij}(J_0^2 f), \qquad g_{ij,k} = g_{ij,k}(J_0^2 f), \qquad g_{ij,kl} = g_{ij,kl}(J_0^2 f). \end{aligned}$$

In the coordinates the left action of L_n^3 on $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is given by

$$\begin{aligned} & \bar{g}_{ij} = b_i^p b_j^q g_{pq}, \\ (14) \qquad & \bar{g}_{ij,k} = b_i^p b_j^q b_k^r g_{pq,r} + \left(b_{ki}^p b_j^q + b_i^p b_{kj}^q\right) g_{pq}, \\ & \bar{g}_{ij,kl} = b_i^p b_j^q b_k^r b_l^s g_{pq,rs} + \left(b_{li}^p b_j^q b_k^r + b_i^p b_{lj}^q b_k^r + b_i^p b_j^q b_{lk}^r + b_{ki}^p b_j^q b_l^r\right. \\ & \qquad \left. + b_i^p b_{kj}^q b_l^r\right) g_{pq,r} + \left(b_{lki}^p b_j^q + b_{ki}^p b_{lj}^q + b_{li}^p b_{kj}^q + b_i^p b_{lkj}^q\right) g_{pq}. \end{aligned}$$

Let K_n^3 be the kernel of the homomorphism $\pi_n^{3,1}$, then K_n^3 is a normal subgroup in the L_n^3 consisting of elements with the coordinates

$$\left(\delta_j^i, b_{j_1 j_2}^i, b_{j_1 j_2 j_3}^i\right).$$

We can restrict the action of L_n^3 to the subgroup K_n^3 and construct the quotient space $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^3$. In formulas (14) we can put $b_j^i = \delta_j^i$.

In the canonical coordinates the action of K_n^3 on $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is given by

$$
\begin{aligned}
\bar{g}_{ij} &= g_{ij},\\
\bar{g}_{ij,k} &= g_{ij,k} + b^p_{ki} g_{pj} + b^p_{kj} g_{ip},\\
\bar{g}_{ij,kl} &= g_{ij,kl} + b^p_{lk} g_{ij,p} + b^p_{li} g_{pj,k} + b^p_{lj} g_{ip,k} + b^p_{ki} g_{pj,l} + b^p_{kj} g_{ip,l}\\
&\quad + \big(b^p_{ki} b^q_{lj} + b^p_{li} b^q_{kj}\big) g_{pq} + b^p_{lki} g_{pj} + b^p_{lkj} g_{ip}.
\end{aligned} \tag{15}
$$

Now we can see from (15) that it is very hard to characterize the quotient space $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^3$ in the canonical coordinates $(g_{ij}, g_{ij,k}, g_{ij,kl})$. So, we will define the new coordinates, which we obtain by the following process.

Recall that the point $J_0^2 f \in T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$, $J_0^2 f = (g_{ij}, g_{ij,k}, g_{ij,kl})$, is called regular if $\det(g_{ij}) \neq 0$. Denote by g^{ij} the functions on a neighborhood of this regular point which satisfy the equality

$$
g^{ik} g_{kl} = \delta^i_j, \tag{16}
$$

where δ^i_j denotes the Kronecker symbol.

Now we will restrict our attention to the open subspace in $T_n^2(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ consisting of all regular points. We shall denote this subspace as $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ (no special notation). Remark that this subspace is L_n^3-equivariant. On this subspace we can change the coordinates from $(g_{ij}, g_{ij,k}, g_{ij,kl})$ to $(g_{ij}, \Gamma_{i,jk}, R_{ijkl}, \Gamma_{i,jkl})$ by the formulas

$$
\begin{aligned}
g_{ij} &= g_{ij},\\
\Gamma_{i,jk} &= \tfrac{1}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right),\\
R_{ijkl} &= \tfrac{1}{2}\big(g_{il,jk} + g_{jk,il} - g_{ik,jl} - g_{jl,ik}\big)\\
&\quad + \tfrac{1}{4} g^{mp}\big((g_{mj,k} + g_{mk,j} - g_{jk,m})(g_{pi,l} + g_{pl,i} - g_{il,p})\\
&\quad - (g_{mj,l} + g_{ml,j} - g_{jl,m})(g_{pi,k} + g_{pk,i} - g_{ik,p})\big),\\
\Gamma_{i,jkl} &= \tfrac{1}{3}\left(g_{ij,kl} + g_{il,jk} + g_{ik,lj}\right) - \tfrac{1}{6}\left(g_{jk,li} + g_{lj,ki} + g_{kl,ji}\right).
\end{aligned} \tag{17}
$$

Here $\Gamma_{i,jk}$ are symmetric in the last two indices and define the Levi-Civita connection, $\Gamma_{i,jkl}$ are symmetric in the last three indices, and R_{ijkl} satisfy the identities

$$
\begin{aligned}
&R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij},\\
&R_{ijkl} + R_{iljk} + R_{iklj} = 0
\end{aligned} \tag{18}
$$

and define the curvature tensor.

The inverse coordinate transformation is given by

$$
\begin{aligned}
g_{ij} &= g_{ij},\\
g_{ij,k} &= \Gamma_{i,jk} + \Gamma_{j,ik},\\
g_{ij,kl} &= \Gamma_{i,jkl} + \Gamma_{j,ikl} - \tfrac{1}{3}\left(R_{ikjl} + R_{jkil}\right)\\
&\quad + \tfrac{1}{3} g^{pq}\left(\Gamma_{p,il}\Gamma_{q,kj} + \Gamma_{p,jl}\Gamma_{q,ki} - 2\Gamma_{p,ij}\Gamma_{q,kl}\right).
\end{aligned} \tag{19}
$$

The new coordinate system

$$
\big(g_{ij}, \Gamma_{i,jk}, R_{ijkl}, \Gamma_{i,jkl}\big)
$$

is called an *adapted coordinate system*. Now we simplify the formulas (14) by using the new coordinates.

If we use (14), transformations (17) and (19), we can formulate

Proposition 1. *In the new coordinates* $(g_{ij}, \Gamma_{i,jk}, R_{ijkl}, \Gamma_{i,jkl})$ *the action of* L^3_n *on* $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ *is given by*

$$
\begin{aligned}
&\bar{g}_{ij} = b^p_i b^q_j g_{pq},\\
&\bar{\Gamma}_{i,jk} = b^p_i b^q_j b^r_k \Gamma_{p,qr} + b^p_i b^q_{jk} g_{pq},\\
&\bar{R}_{ijkl} = b^p_i b^q_j b^r_k b^s_l R_{pqrs},\\
&\bar{\Gamma}_{i,jkl} = b^p_i b^q_j b^r_k b^s_l \Gamma_{p,qrs} + \big[b^p_i\big(b^q_j b^r_{kl} + b^q_k b^r_{lj} + b^q_l b^r_{kj}\big)\\
&\quad + \tfrac{1}{3} b^q_i \big(b^p_{kj} b^r_l + b^p_{lj} b^r_k + b^p_{kl} b^r_j\big) + \tfrac{1}{3}\big(b^p_{ki} b^q_j b^r_l + b^p_{ji} b^q_k b^r_l\\
&\quad + b^p_{li} b^q_j b^r_k\big)\big]\, \Gamma_{p,qr} + \big[b^p_i b^q_{jkl} + \tfrac{1}{3}\big(b^p_{ki} b^q_{jl} + b^p_{li} b^q_{jk} + b^p_{ji} b^q_{lk}\big)\big] g_{pq}
\end{aligned}
\tag{20}
$$

and the action of K^3_n *on* $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ *is given by*

$$
\begin{aligned}
&\bar{g}_{ij} = g_{ij},\\
&\bar{\Gamma}_{i,jk} = \Gamma_{i,jk} + b^p_{jk} g_{ip},\\
&\bar{R}_{ijkl} = R_{ijkl},\\
&\bar{\Gamma}_{i,jkl} = \Gamma_{i,jkl} + b^p_{kl}\Gamma_{i,jp} + b^p_{lj}\Gamma_{i,kp} + b^p_{kj}\Gamma_{i,lp} + \tfrac{1}{3}\big(b^p_{kj}\Gamma_{p,il} + b^p_{lj}\Gamma_{p,ik}\\
&\quad + b^p_{kl}\Gamma_{p,ij}\big) + \tfrac{1}{3}\big(b^p_{ki}\Gamma_{p,jl} + b^p_{ji}\Gamma_{p,kl} + b^p_{li}\Gamma_{p,jk}\big) + \tfrac{1}{3}\big(b^p_{ki} b^q_{jl} + b^p_{li} b^q_{jk}\\
&\quad + b^p_{ji} b^q_{lk}\big)\, g_{pq} + b^p_{jkl} g_{ip}.
\end{aligned}
\tag{21}
$$

5. Second order differential invariants of the metric tensor

In this section we will find the differential invariants

$$F : T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \to Q,$$

where Q is any left L^1_n-manifold. Recall that the differential invariant F with values in a left L^1_n-manifold satisfies the condition

$$F(A \cdot J^2_0 f) = \pi^{3,1}_n(A) \cdot F(J^2_0 f), \tag{22}$$

where $J^2_0 f \in T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$, $A \in L^3_n$.

Let us consider the space $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n$. For each class $[J^2_0 f]_{K^3_n}$ from the $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n$ we can take the same values g_{ij}, R_{ijkl} as for its representative, i.e. we put

$$
\begin{aligned}
&g_{ij}\big([J^2_0 f]_{K^3_n}\big) = g_{ij}(J^2_0 f),\\
&R_{ijkl}\big([J^2_0 f]_{K^3_n}\big) = R_{ijkl}(J^2_0 f).
\end{aligned}
\tag{23}
$$

It follows from (15) that these expressions are independent on the choice of representatives and that two different classes have different systems of numbers g_{ij}, R_{ijkl}. We define a coordinate system on the $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n$ by (23). Now we can express the factor projection

$$\pi : T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \to T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n$$

in the form

$$\pi = (g_{ij}, R_{ijkl}). \tag{24}$$

The group L^1_n (which is isomorphic to the L^3_n/K^3_n) acts on $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ by

$$A \cdot [J^2_0 f]_{K^3_n} = \left[\iota^3(A) \cdot J^2_0 f\right]_{K^3_n}, \tag{25}$$

where $\iota^3(A) = (a^i_j, 0, 0)$. The manifold $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n$ has the structure of a left L^1_n-manifold.

Let P_n be the subspace of tensor space

$$(\mathbb{R}^{n*} \wedge \mathbb{R}^{n*}) \odot (\mathbb{R}^{n*} \wedge \mathbb{R}^{n*})$$

which is in the canonical coordinates R_{ijkl} defined by

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0. \tag{26}$$

The dimension of P_n is $\frac{1}{12}n^2(n^2-1)$, it is equal to the number of coordinates R_{ijkl} on the space $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$. We can write the following theorem.

Theorem 1. *The L^3_n-manifold $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ has the structure of a left principal K^3_n-bundle. This left principal K^3_n-bundle is trivial, and its base $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n$ is diffeomorphic to the $(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \times P_n$.*

Proof. It is well known that is enough to prove that the graph of equivalence of the relation "there exists $A \in K^3_n$, such that $A \cdot J^2_0 f_1 = J^2_0 f_2$" is a closed submanifold of the

$$T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \times T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$$

and that the action of K^3_n on $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is free.

Let us to prove the first condition. Let us consider the system of coordinates

$$g_{ij}, \Gamma_{i,jk}, R_{ijkl}, \Gamma_{i,jkl}, \bar{g}_{ij}, \bar{\Gamma}_{i,jk}, \bar{R}_{ijkl}, \bar{\Gamma}_{i,jkl}, \tag{27}$$

on

$$T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \times T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}).$$

It follows from (21) that the graph of the above mentioned relation is determined by the equations

$$\bar{g}_{ij} = g_{ij}, \qquad \bar{R}_{ijkl} = R_{ijkl}, \quad 1 \le s \le r$$

and is therefore closed.

Let us prove that the action of K^3_n on $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is free. Suppose that

$$A \cdot J^2_0 f = J^2_0 f,$$

where $A \in K^3_n$ and $J^2_0 f \in T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$. In the coordinates we write

$$\left(\bar{g}_{ij}, \bar{\Gamma}_{i,jk}, \bar{R}_{ijkl}, \bar{\Gamma}_{i,jkl}\right) = \left(g_{ij}, \Gamma_{i,jk}, R_{ijkl}.\Gamma_{i,jkl}\right). \tag{28}$$

It follows from (15) that

$$b^r_{jk} = g^{ri}\left(\bar{\Gamma}_{i,jk} - \Gamma_{i,jk}\right), \qquad (29)$$
$$b^r_{jkl} = g^{ri}\left(\bar{\Gamma}_{i,jkl} - \Gamma_{i,jkl}\right) - B^r_{jkl},$$

where

$$\begin{aligned} B^r_{jkl} = g^{ri}\big[&b^p_{kl}\Gamma_{i,jp} + b^p_{lj}\Gamma_{i,kp} + b^p_{kj}\Gamma_{i,lp} \\ &+ \tfrac{1}{3}\big(b^p_{kj}\Gamma_{p,il} + b^p_{lj}\Gamma_{p,ik} + b^p_{kl}\Gamma_{p,ij}\big) \\ &+ \tfrac{1}{3}\big(b^p_{ki}\Gamma_{p,jl} + b^p_{ji}\Gamma_{p,kl} + b^p_{li}\Gamma_{p,jk}\big) \\ &+ \tfrac{1}{3}\big(b^p_{ki}b^q_{jl} + b^p_{li}b^q_{jk} + b^p_{ji}b^q_{lk}\big)g_{pq}\big] \end{aligned} \qquad (30)$$

depends only on b^p_{rs}, g_{pq}, $\Gamma_{p,rs}$. If we use (28) in (29), then we obtain $b^r_{jk} = 0$, $b^r_{jkl} = 0$ for each $1 \le r, j, k, l \le n$. That is why the $A = (\delta^i_j, 0, 0)$ is the unit element of the group K^3_n and the action is free.

Finally we have to introduce a diffeomorphism which maps the base $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n$ to the $(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \times P_n$. Let us consider the diffeomorphism which maps class from the $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n$ with the coordinates (g_{ij}, R_{ijkl}) to the element of the $(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \times P_n$ which has the same coordinates (g_{ij}, R_{ijkl}).

This completes the proof of Theorem 1.

Theorem 2. *Every differential invariant from the left L^3_n-manifold $T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ to any left L^1_n-manifold Q depends only on g_{ij} and R_{ijkl}.*

Proof. Let

$$\pi : T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \to T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n$$

be the canonical projection. Let Q be a left L^1_n-manifold. Suppose that

$$F : T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \to Q$$

is a differential invariant. By Lemma 1 there exists a uniquely determined L^1_n-equivariant mapping

$$f : T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K^3_n \to Q$$

which satisfies the condition (9)

$$F = f \circ \pi.$$

This mapping f is defined by (10)

$$f\big([p]_{K^3_n}\big) = F(p)$$

for each $p \in T^2_n(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$.

From the uniqueness of the f follows that the F depends only on g_{ij} and R_{ijkl}. We say that

$$\pi = (g_{ij}, R_{ijkl})$$

is the *basis of the invariants of metric with values in a left L^1_n-manifold.*

This completes the proof of Theorem 2.

Remark. There exists no nontrivial first order differential invariant of the metric tensor, because the space $T_n^1(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^2$ is isomorphic to the $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ and every invariant from the $T_n^1(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ to any left L_n^1-manifold depends only on g_{ij}.

6. r-th order differential invariants of the metric tensor

Let L_n^r be the r-th differential group with the canonical global coordinate system

$$\left(b^i_j, b^i_{j_1 j_2}, \dots, b^i_{j_1 j_2 \dots j_r}\right), \ 1 \le i, j \le n, 1 \le j_1 \le \cdots \le j_k \le n, 1 \le k \le r,$$

with the coordinates $b^i_{j_1 j_2 \dots j_k}$ defined by (2)

$$b^i_{j_1 \dots j_k}(A) = a^i_{j_1 \dots j_k}(A^{-1}) = D_{j_1} D_{j_2} \cdots D_{j_k} \alpha_i^{-1}(0),$$

where $A \in L_n^r$, $A = J_0^r \alpha$.

Let $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ be the r-th prolongation of the left L_n^1-manifold $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$. $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is the set of r-jets with source at the $0 \in \mathbb{R}^n$ and target in the $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ with the natural structure of the left L_n^{r+1}-manifold. Let

$$(g_{ij}), \quad 1 \le i \le j \le n,$$

be the canonical global coordinate system on the $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$. Then the canonical global coordinate system

$$\left(g_{ij}, g_{ij,k_1}, g_{ij,k_1k_2}, \dots, g_{ij,k_1k_2\dots k_r}\right),$$
$$1 \le i \le j \le n, \ 1 \le k_1 \le \cdots \le k_r \le n,$$

on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is defined analogically as for case $r = 2$ (see (12)).

The group action of L_n^{r+1} on $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is induced by the prolongation of the group action (11) of L_n^1 on $\mathbb{R}^{n*} \odot \mathbb{R}^{n*}$. Corresponding equations are obtained by s-th formal differentiation of (11) for $s = 1, 2, \dots, r$ (see [4]). The action of L_n^{r+1} on $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is given by (14) for $r = 2$ in the canonical coordinates.

Let Q be an arbitrary L_n^1-manifold. Let $\pi_n^{r+1,1} : L_n^{r+1} \to L_n^1$ be the projection homomorphism of differential groups (see (4)). Recall that mapping

$$F : T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \to Q \tag{31}$$

is called *the r-th order differential invariant of the metric tensor*, if it satisfies the condition

$$F(A \cdot J_0^r f) = \pi^{r+1,1}(A) \cdot F(J_0^r f) \tag{32}$$

for each $J_0^r f \in T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$, $A \in L_n^r$.

Let K_n^{r+1} be the kernel of the homomorphism $\pi_n^{r+1,1}$, K_n^{r+1} is a normal subgroup in the L_n^{r+1} consisting of elements with the coordinates

$$(\delta^i_j, b^i_{j_1 j_2}, \dots, b^i_{j_1 j_2 \dots j_r}).$$

We can restrict the action of L_n^{r+1} to the subgroup K_n^{r+1} and construct the quotient space $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1}$.

In the canonical coordinates the action of K_n^{r+1} on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is given by

$$
\begin{aligned}
&\bar{g}_{ij} = g_{ij},\\
&\bar{g}_{ij,k_1} = g_{ij,k_1} + b^p_{k_1 i} g_{pj} + b^p_{k_1 j} g_{ip},\\
&\bar{g}_{ij,k_1k_2} = g_{ij,k_1k_2} + b^p_{k_2k_1} g_{ij,p} + b^p_{k_2 i} g_{pj,k_1} + b^p_{k_2 j} g_{ip,k_1} + b^p_{k_1 i} g_{pj,k_2}\\
&\quad + b^p_{k_1 j} g_{ip,k_2} + b^p_{k_2k_1 i} g_{pj} + \big(b^p_{k_1 i} b^q_{k_2 j} + b^p_{k_2 i} b^q_{k_1 j}\big) g_{pq} + b^p_{k_2k_1 j} g_{ip},\\
&\dots
\end{aligned}
\tag{33}
$$

Now we can see from (33) that it is very hard to characterize the quotient space $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})/K_n^{r+1}$ in the canonical coordinates

$$\big(g_{ij}, g_{ij,k_1}, g_{ij,k_1k_2}, \dots, g_{ij,k_1k_2\dots k_r}\big).$$

So, we will define the coordinates, which we obtain by the following process.

On the space $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ let us consider the functions

$$\Gamma_{i,jk} = \tfrac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i}) \tag{34}$$

and the functions

$$\Gamma_{i,jk,m_1\dots m_s} \tag{35}$$

defined as the s-th derivative of (34). Using these functions, we can consider the following functions

$$\Gamma_{i,j_1j_2}, \Gamma_{i,j_1j_2j_3} \dots, \Gamma_{i,j_1j_2j_3\dots j_{r+1}}, \tag{36}$$

$$R_{ijkl}, R_{ijkl;m_1}, \dots, R_{ijkl;m_1;\dots;m_{r-2}}, \tag{37}$$

where $\Gamma_{i,j_1j_2j_3\dots j_s} = \Gamma_{i,(j_1j_2,j_3\dots j_s)}$ (the symmetrization in indices $j_1, j_2, j_3, \dots, j_s$) and $R_{ijkl;m_1;\dots;m_s}$ denotes the s-th covariant derivative of the curvature tensor

$$R_{ijkl} = \Gamma_{i,jk,l} - \Gamma_{i,jl,k} + g^{pq}\left(\Gamma_{i,pl}\Gamma_{q,jk} - \Gamma_{i,pk}\Gamma_{q,jl}\right). \tag{38}$$

The first covariant derivative of the curvature tensor is the system of functions

$$
\begin{aligned}
R_{ijkl;m} = R_{ijkl,m} - g^{pq}\big(&\Gamma_{p,mi} R_{qjkl} + \Gamma_{p,mj} R_{iqkl}\\
&+ \Gamma_{p,mk} R_{ijql} + \Gamma_{p,ml} R_{ijkq}\big),
\end{aligned}
\tag{39}
$$

where

$$R_{ijkl,m} = \left(\frac{\partial R_{ijkl}}{\partial g_{pq}} g_{pq,m} + \frac{\partial R_{ijkl}}{\partial g_{pq,r}} g_{pq,rm} + \frac{\partial R_{ijkl}}{\partial g_{pq,rs}} g_{pq,rsm}\right). \tag{40}$$

Lemma 2. *The system of functions* g_{ij}, (36) *and* (37) *contains a subsystem defining a coordinate system on the* $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$.

Proof. For each s, $1 \le s \le r-2$, consider the canonical coordinates $g_{ij,k_1k_2\dots k_{s+2}}$. We have the decomposition

$$
\begin{aligned}
g_{ij,k_1k_2\dots k_{s+2}} &= \Gamma_{i,jk_1\dots k_{s+2}} + \Gamma_{j,ik_1\dots k_{s+2}}\\
&\quad + \big(g_{ij,k_1k_2\dots k_{s+2}} - \Gamma_{i,jk_1\dots k_{s+2}} - \Gamma_{j,ik_1\dots k_{s+2}}\big).
\end{aligned}
\tag{41}
$$

It is seen that the expression in the bracket may be rewritten as a sum of terms of the form

$$\Delta_{i,pqrj_1\dots j_s} = g_{ip,qrj_1\dots j_s} - g_{ir,qpj_1\dots j_{s+1}}. \tag{42}$$

Consider the systems

$$G_s = (g_{ij,k_1k_2\dots k_{s+2}}), \quad 1 \le i \le j \le n, k_1 \le k_2 \le \dots \le k_{s+2},$$
$$\Gamma_s = (\Gamma_{i,j_1j_2\dots j_{s+3}}), \quad 1 \le i \le n, j_1 \le j_2 \le \dots \le j_{s+3},$$
$$\Delta_{i,p_1p_2\dots p_{s+2}},$$

and the linear mapping $G_s \to (\Gamma_s, \Delta_s)$. We can write

$$\begin{pmatrix} \Gamma_s \\ \Delta_s \end{pmatrix} = C_s \cdot G_s,$$

where C_s is the matrix of the linear mapping. Relations (41) show that there exists a matrix $\bar{C}_s$ such that $\bar{C}_s \cdot C_s = I$ (the identity matrix). This implies

$$\operatorname{rank} C_s = \operatorname{rank} \bar{C}_s = \binom{n+1}{2}\binom{n+s+1}{s+2},$$

where the right hand side expression is the number of the coordinates $g_{ij,k_1k_2\dots k_{s+2}}$. Choose a squared submatrix C_s^0 of C_s such that $\operatorname{rank} C_s^0 = \operatorname{rank} C_s$. It is clear that the system of functions

$$g_{ij}, \quad i \le j, \qquad \Gamma_{i,jk}, \quad j \le k, \qquad C_s^0 \cdot G_s, \quad 0 \le s \le r-2,$$

defines a coordinate system on $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$.

Now consider the s-th formal covariant derivative of R_{ijkl}. By definition

$$R_{ijkl;m_1;\dots;m_s} = \tfrac{1}{2}(\Delta_{i,kjlm_1\dots m_s} - \Delta_{j,kilm_1\dots m_s}) + P_{i,jklm_1\dots m_s},$$

where $P_{i,jklm_1\dots m_s}$ is a polynomial in the canonical coordinates, independent on the coordinates $g_{ij,k_1k_2\dots k_{s+2}}$. Combining this fact with the above assertion about the coordinate system g_{ij}, $\Gamma_{i,jk}$, $C_s^0 \cdot G_s$ on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ we obtain the subsystem of g_{ij}, (36) and (37) required.

Each coordinate system on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ defined by Lemma 2 will be called an *adapted coordinate system*. The functions belonging to an adapted coordinate system will be called *adapted coordinates*.

Using formal differentiation of (20) and the transformation formulas for (36) and (37), we can see that in the adapted coordinates the action of L_n^{r+1} on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is given by the formulas

$$\bar{g}_{ij} = b_i^p b_j^q g_{pq}, \tag{43}$$

$$\bar{\Gamma}_{i,j_1j_2\dots j_{s+1}} = b_i^p b_{j_1}^{q_1} b_{j_2}^{q_2} \cdots b_{j_{s+1}}^{q_{s+1}} \Gamma_{p,q_1q_2\dots q_{s+1}} + B^i_{j_1j_2\dots j_{s+1}} + g_{pq} b_i^p b^p_{j_1j_2\dots j_{s+1}}, \tag{44}$$

$$\bar{R}_{ijkl;m_1;\dots;m_{s-2}} = b_i^p b_j^q b_k^u b_l^v b_{m_1}^{t_1} \cdots b_{m_{s-2}}^{t_{s-2}} R_{p,quv;t_1;\dots;t_{s-2}}, \tag{45}$$

where $1 \le s \le r$ and $B^i_{j_1j_2\dots j_{s+1}}$ is a polynomial in the canonical coordinates on the L_n^s and in the adapted coordinates on the $T_n^{s-1}(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$.

Proposition 2. *The action of K_n^{r+1} on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ in the adapted coordinates on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ and the canonical coordinates on the K_n^{r+1} is given by the formulas*

$$\begin{aligned} &\bar{g}_{ij} = g_{ij}, \\ (46) \qquad &\bar{\Gamma}_{i,j_1 j_2 \ldots j_{s+1}} = \Gamma_{i,j_1 j_2 \ldots j_{s+1}} + B^i_{j_1 j_2 \ldots j_{s+1}} + g_{ip} b^p_{j_1 j_2 \ldots j_{s+1}}, \\ &\bar{R}_{ijkl;m_1;\ldots;m_{s-2}} = R_{ijkl;m_1;\ldots;m_{s-2}}, \end{aligned}$$

where $1 \leq s \leq r$ and $B^i_{j_1 j_2 \ldots j_{s+1}}$ is a polynomial in the canonical coordinates on the K_n^s and in the adapted coordinates on the $T_n^{s-1}(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$.

Now we have the formulas (46), which will help us to prove the following theorem.

Theorem 3. *The L_n^{r+1}-manifold $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ has the structure of a left principal K_n^{r+1}-bundle. This left principal K_n^{r+1}-bundle is trivial, and its base is diffeomorphic to some Euclidean space.*

Proof. It is well known that it is sufficient to prove that the graph of equivalence relation "there exists $A \in K_n^{r+1}$ such that $A \cdot J_0^r f_1 = J_0^r f_2$" is a closed submanifold of

$$T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \times T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$$

and that the action of K_n^{r+1} on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is free.

Let us prove the first condition. Let us consider the system of coordinates

$$g_{ij}, \Gamma^i_{j_1 j_2 \ldots j_{s+1}}, R^i_{jkl;m_1;\ldots;m_{s-2}}, \bar{g}_{ij}, \bar{\Gamma}^i_{j_1 j_2 \ldots j_{s+1}}, \bar{R}^i_{jkl;m_1;\ldots;m_{s-2}}$$

on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*}) \times T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$. From (46) it follows that the graph of the above mentioned relation is determined by the equations

$$\bar{g}_{ij} = g_{ij}, \qquad \bar{R}^i_{jkl;m_1;\ldots;m_{s-2}} = R^i_{jkl;m_1;\ldots;m_{s-2}}, \quad 1 \leq s \leq r$$

and is therefore closed.

Let us prove that the action of K_n^{r+1} on the $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ is free. From the condition $A \cdot J_0^r f_1 = J_0^r f_2$; in the coordinates it can be written as

$$\left(\bar{g}_{ij}, \bar{\Gamma}^i_{j_1 j_2 \ldots j_{s+1}}, \bar{R}^i_{jkl;m_1;\ldots;m_{s-2}}\right) = \left(g_{ij}, \Gamma^i_{j_1 j_2 \ldots j_{s+1}}, R^i_{jkl;m_1;\ldots;m_{s-2}}\right),$$

$1 \leq s \leq n$, follows, using (46), that for every indices $i, j_1, j_2, \ldots, j_{s+1}$ is $b^i_{j_1 j_2 \ldots j_{s+1}} = 0$. It is satisfied only for the unit element $A = (\delta^i_j, 0, 0, \ldots, 0)$ and the action is free.

This completes the proof of Theorem 3.

Theorem 4. *Every differential invariant from the left L_n^{r+1}-manifold $T_n^r(\mathbb{R}^{n*} \odot \mathbb{R}^{n*})$ to any left L_n^1-manifold Q depends only on g_{ij} and*

$$R_{ijkl}, R_{ijkl;m}, R_{ijkl;m_1;m_2}, \ldots, R_{ijkl;m_1;\ldots;m_{r-2}}.$$

Proof. It is consequence of Theorem 3 and Lemma 1.

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