Proceedings of the Seminar on Differential Geometry
Mathematical Publications Volume 2
Silesian University in Opava, Opava, 2000, 131–144

Geometric aspects of S-integrability[1]

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Abstract. This is a report on results on zero-curvature representations and Bäcklund transformations of S-integrable partial differential equations recently obtained within the framework of the Vinogradov diffiety theory.

Keywords and phrases. Diffiety, S-integrability, zero-curvature representation.

MS classification. 58A20, 58J10, 58J72.

1. Introduction

Soliton systems of nonlinear partial differential equations (PDE) continue to attract attention of researchers since the Korteweg–de Vries equation (KdV) became the first nonlinear PDE solved by the Inverse Spectral Transform (IST) in 1967 ([12, 13]). The same method has been found applicable to a whole class of nonlinear systems ([1, 51]), now called *S-integrable*, as opposed to *C-integrable* systems, which are transformable to a linear system. S-integrable equations share common typical properties that make them objects of wide interest in several areas of mathematics and mathematical physics.

In the end of seventies, Zakharov and Shabat [51] related S-integrability—at least in dimension 2—to the existence of a zero-curvature representation (ZCR) and since then ZCR's occupy central position in soliton theory ([50, 2, 38]). In particular, knowing a ZCR depending on a "non-removable" parameter (spectral parameter) opens a way to solution (e.g., via a Riemann–Hilbert problem [38]). Since the early days, methods

[1] Research supported by Grants CEZ:J10/98:192400002 and VS 96003 "Global Analysis" of the Czech Ministry of Education, Youth and Sports, and also by Grants 201/98/0853 and 201/00/0724 of the Czech Grant Agency.

This paper is in final form and no part of it will be published elsewhere.

have been designed to generate integrable equations with prescribed (meromorphic) dependence on the spectral parameter. But even without dependence on parameters, zero-curvature representations are relevant for geometry, see Tenenblat [39] and references therein.

To find a ZCR for a given equation, the widely known "prolongation procedure" of Wahlquist and Estabrook [48] may be employed. The procedure is considered algorithmic for a wide class of equations (Dodd and Fordy [8]), but still some cases remain difficult if not impossible: the method is sensitive to explicit dependence on independent variables (first treated by Molino [27]) and only incomplete answers have been obtained so far in the case of dependence on higher order derivatives (Finley and McIver [9]). Likewise, classification problems are rather out of the scope—adding to motivation for the recent progress in symmetry analysis [25, 26, 34] and Painlevé analysis [49, 28]). Among works studying zero-curvature representations as structures we point out [42, 19].

Within their diffiety theory, Krasil'shchik and Vinogradov [17] introduced the concept of a *covering*, substantially generalizing the Wahlquist–Estabrook prolongation structures. Isomorphism classes of coverings are computable. Zero-curvature representations form a rather narrow class of coverings, but with a rich structure in the background. In the papers surveyed here, we built a cohomology theory accompanying any zero-curvature representation, with or without the spectral parameter. We also suggested a method to compute ZCR's with values in a given Lie algebra $\mathfrak{g}$, even though practical results were obtained only for $\mathfrak{g} = \mathfrak{sl}_2$. In a forthcoming paper ([23]) we focus on 1-parametric families of ZCR's and obstructions to removability of the parameter.

S-integrability also involves the presence of Bäcklund (and Darboux) transformations (see [31, 24] or the survey [10]), accompanied by nonlinear superposition principles, They provide a link to remarkable last-century's discoveries in differential geometry (see, e.g., [30]). Within the diffiety theory Bäcklund transformations have a very clear geometric description as a pair of coverings with a common total space. Krasil'shchik [14] then presented intriguing ideas to explain permutability of Bäcklund transformations. The omnipresence of permutability and nonlinear superposition principles contributed to the common belief that these two properties are interrelated.

2. Prerequisites

Diffieties. We use the Vinogradov category $\mathcal{DE}$ of diffieties [44, 45, 46]—geometric objects that naturally represent PDEs. Diffieties provide a convenient language to deal with nonlinear PDE's in full generality. By definition, a diffiety is an infinite-dimensional smooth manifold $\mathcal{E}$ endowed with a finite-dimensional involutive distribution $\mathcal{C}$, while morphisms of the category are smooth maps that preserve the distribution. PDE's related by a contact transformation are isomorphic as diffieties. It is the diffiety structure what determines symmetries, conservation laws and other invariant objects of interest in the current research of nonlinear equations.

However, for simplicity we use a subcategory $\mathcal{DE}_M$ consisting of diffieties $\mathcal{E}$ fibered over a fixed finite-dimensional base manifold M and such that the tangent spaces $T_z\mathcal{E}$ decompose as $T_z\mathcal{E} = \mathcal{C}_z \oplus V_z\mathcal{E}$, with $V\mathcal{E}$ being the vertical vector bundle with respect to the projection $\mathcal{E} \to M$. In this case, $\mathcal{C}$ is simply a connection, and the involutivity condition means that $\mathcal{C}$ is flat. Isomorphisms in $\mathcal{DE}_M$ represent invertible transformations of dependent variables of PDE's. Also, according to [20], $\mathcal{DE}_M$ has a well-understood categorical property, namely, cotripleability.

PDEs become diffieties in the following way. Suppose we are given a finite system of finite-order equations (indexed by l),

(1) $$F^l(x^i, u^k, \dots, u^k_{i_1\dots i_r}, \dots) = 0,$$

in independent variables x^i and dependent variables u^k, which may be interpreted as base and fibre coordinates, respectively, of some finite-dimensional fibred manifold $Y \to M$. Then $u^k_{i_1\dots i_r}$ means the partial derivative $\partial^r u^k/\partial x^{i_1} \cdots \partial x^{i_r}$ and may be interpreted as a local coordinate along the fibres of the infinite jet prolongation $j^\infty Y \to Y$. Differential operators $D_i = \partial/\partial x^i + \sum_{r,I} u^k_{Ii}\, \partial/\partial u^k_I$ on $j^\infty Y$ (with $I = i_1 \dots i_r$ denoting a symmetric multiindex, possibly void) are called *total derivatives*. Viewed as vector fields they commute and span the involutive finite-dimensional *Cartan distribution* on $j^\infty Y$, which makes $j^\infty Y$ into an object of the category $\mathcal{DE}_M$.

Treating F^l's as functions on $j^\infty Y$, we assume that the system (1) along with all its differential consequences $D_I F^l = 0$ determines a submanifold $\mathcal{E} \subseteq j^\infty Y$ (typically infinite-dimensional). Endowed with the distribution spanned by the restricted fields $D_i = D_i|_{\mathcal{E}}$ (the vector fields D_i are obviously tangent to $\mathcal{E}$), the manifold $\mathcal{E}$ is an object of $\mathcal{DE}_M$, namely the diffiety corresponding to the system (1). Although diffieties are more general than PDE's, in the sequel we often use these words interchangeably.

A remark on smooth (C^∞) structures is due. There is no general consent on what is a smooth infinite-dimensional manifold and therefore the diffiety theory can have various realizations depending on the choice of the underlying infinite-dimensional smooth structure. Our preferred choice is that of [4]: topological manifolds are modelled on $\mathbb{R}^\infty$, with smooth functions defined as locally depending on only a finite number of arguments, and then smoothly in the standard sense.

The $\mathcal{C}$-spectral sequence. Let us recall from Vinogradov [45] that the diffiety structure determines the *$\mathcal{C}$-spectral sequence*. In the subcategory $\mathcal{DE}_M$ it is associated with the *variational bicomplex*, formed by $C^\infty\mathcal{E}$-modules

$$\Lambda^{p,q}\mathcal{E} = \bigwedge\nolimits^p \Lambda^{1,0}\mathcal{E} \otimes \bigwedge\nolimits^q \Lambda^{0,1}\mathcal{E},$$

where $\Lambda^{1,0}\mathcal{E} = \operatorname{Ann}\mathcal{C}$ and $\Lambda^{0,1}\mathcal{E} = \operatorname{Ann} V\mathcal{E}$ are the $C^\infty\mathcal{E}$-modules of contact forms and horizontal 1-forms, respectively. Here $C^\infty\mathcal{E}$ is the ring of C^∞ functions on $\mathcal{E}$. In this way the decomposition $T\mathcal{E} = \mathcal{C} \oplus V\mathcal{E}$ induces a decomposition of the $C^\infty\mathcal{E}$-module of exterior r-forms as $\Lambda^r\mathcal{E} = \bigoplus_{p+q=r} \Lambda^{p,q}\mathcal{E}$. Accordingly, the exterior differential $d : \Lambda^n\mathcal{E} \to \Lambda^{n+1}\mathcal{E}$ splits into the *horizontal* differential $\bar{d} : \Lambda^{p,q}\mathcal{E} \to \Lambda^{p,q+1}\mathcal{E}$ and the *vertical* differential $\ell : \Lambda^{p,q}\mathcal{E} \to \Lambda^{p+1,q}\mathcal{E}$ of the variational bicomplex. The $\mathcal{C}$-spectral sequence is associated to this bicomplex by the identification $E_0^{p,q} = \Lambda^{p,q}\mathcal{E}$ and $d_0 = \bar{d}$. All groups $E_r^{p,q}$ are important geometric invariants of PDE's. In particular, $\bar{H}^{m-1} := E_1^{0,m-1}$ is the group of conservation laws, while $\bar{H}^1 := E_1^{0,1}$ is the group of abelian coverings.

Efficient methods to compute the $\mathcal{C}$-spectral sequence have been given by Vinogradov [45], Tsujishita [40], and Verbovetsky [43]. As the starting point, Tsujishita and Verbovetsky use the Janet sequence [40, Sect. 5.2 and 5.3] (also known as the compatibility complex)

$$P_0 \xrightarrow{\phi_1} P_1 \xrightarrow{\phi_2} P_2 \to \cdots \to P_{m-1} \xrightarrow{\phi_m} P_m \to 0$$

associated with the system (1). Here each P_j is a vector bundle over the diffiety. Always $P_j = 0$ for all $j > m$, while for non-overdetermined systems we have even $P_j = 0$ for all $j > 1$. The differential ϕ_1 is easily identified with the so-called *universal linearization*. Each of the subsequent differentials ϕ_k, $k > 1$, expresses the integrability conditions for the previous one. Considering the formally adjoint complex

(2) $$P_0^* \xleftarrow{\phi_1^*} P_1^* \xleftarrow{\phi_2^*} P_2^* \leftarrow \cdots \leftarrow P_{m-1}^* \xleftarrow{\phi_m^*} P_m^* \leftarrow 0$$

the main result of Tsujishita [40, Theorem 5.3.1] states that

(3) $$E_1^{1,m-q} \cong \frac{\operatorname{Ker}\phi_q^*}{\operatorname{Im}\phi_{q-1}^*}.$$

3. Zero-curvature representations

In this section we report on our investigation [21] of the gauge cohomology related to ZCR's (in a rather loose analogy with the $\mathcal{C}$-spectral sequence) leading to an alternative method to compute ZCR's. We show that gauge equivalence classes of ZCR's are computable from a determining system of differential equations in total derivatives, which has as many equations as it has unknowns. Even though only few explicit computational examples have been published so far, the "direct" method, as we call it now, has been found applicable to classification problems even in combination with completely arbitrary dependences. Independently, in the context of evolution equations, and without cohomological interpretation, Sakovich introduced essentially the same characteristic element (see below) and put it in the core of a completely different method to compute ZCR's [32] of evolution systems.

The gauge cohomology generalizes the groups $E_1^{1,q}$ of the $\mathcal{C}$-spectral sequence and formula (3). Recently, Verbovetsky [43] interpreted the gauge cohomology in terms of the $\mathcal{C}$-spectral sequence with coefficients in a $\mathcal{C}$-module (as one of the three motivating examples).

Let $\mathcal{E}$ be the diffiety corresponding to the system (1). Let G be a matrix Lie group, let $\mathfrak{g}$ be the corresponding matrix Lie algebra. A $\mathfrak{g}$-valued *zero-curvature representation* (ZCR) for system (1) is a $\mathfrak{g}$-valued horizontal form $\alpha = A_i\, dx^i$ on the diffiety, such that $D_j A_i - D_i A_j + [A_i, A_j] = 0$ for $i \neq j$ or, more compactly,

(4) $$\bar{d}\alpha = \tfrac{1}{2}[\alpha, \alpha].$$

For any G-matrix H, the form

$$\alpha^H = \bar{d}H \cdot H^{-1} + H \cdot \alpha \cdot H^{-1}$$

is a ZCR again. The mapping $\alpha \mapsto \alpha^H$ is called the *gauge transformation*; it is a group action. The ZCR α^H is said to be *gauge equivalent* to α. A ZCR gauge equivalent to the zero form $\alpha = 0$ is said to be *trivial*. We call a $\mathfrak{g}$-valued ZCR α *irreducible* if neither of the gauge equivalent forms α^H falls into a proper subalgebra of $\mathfrak{g}$. Otherwise α is called *reducible*.

The problem is to decide whether a given system (1) admits a nontrivial ZCR in a given Lie algebra $\mathfrak{g}$, and if so, to compute all its ZCR's modulo gauge equivalence. Of special interest are irreducible ZCR's with coefficients in a non-solvable Lie algebra, and 1-parameter families of them.

Turning to the material of [21], consider the tensor product of the Lie algebra $\mathfrak{g}$ with the variational bicomplex $\Lambda^{p,q}\mathcal{E}$. Given a ZCR α, we introduce operators

$$\bar\partial_\alpha = \bar d - \mathrm{ad}_\alpha,$$

where $\mathrm{ad}_\alpha \rho = [\alpha, \rho]$ for any $\rho \in \Lambda^{p,q}\mathcal{E} \otimes \mathfrak{g}$. We have $\bar\partial_\alpha \circ \bar\partial_\alpha = 0$ as a consequence of (4), hence the pth *linear gauge complex*

$$\Lambda^{p,0}\mathcal{E} \otimes \mathfrak{g} \xrightarrow{\bar\partial_\alpha} \Lambda^{p,1}\mathcal{E} \otimes \mathfrak{g} \xrightarrow{\bar\partial_\alpha} \Lambda^{p,2}\mathcal{E} \otimes \mathfrak{g} \to \cdots \to \Lambda^{p,m}\mathcal{E} \otimes \mathfrak{g} \to 0.$$

The groups

$$H^{p,q}_\alpha(\mathcal{E}, \mathfrak{g}) = \frac{\mathrm{Ker}\big(\Lambda^{p,q}\mathcal{E} \otimes \mathfrak{g} \xrightarrow{\bar\partial_\alpha} \Lambda^{p,q+1}\mathcal{E} \otimes \mathfrak{g}\big)}{\mathrm{Im}\big(\Lambda^{p,q-1}\mathcal{E} \otimes \mathfrak{g} \xrightarrow{\bar\partial_\alpha} \Lambda^{p,q}\mathcal{E} \otimes \mathfrak{g}\big)}$$

are called the *gauge cohomology groups* with respect to the ZCR α.

If $p = 1$, then in analogy with (3) we have

Proposition ([21, eq. (2.10)]).

$$H^{1,m-q}_\alpha \cong \frac{\mathrm{Ker}\,\widehat{\phi}^*_q}{\mathrm{Im}\,\widehat{\phi}^*_{q-1}}, \tag{5}$$

where for a differential operator $\phi : P \to Q$ in total derivatives ($\mathcal{C}$-differential operator) we denote by $\widehat{\phi}^* : P \otimes \mathfrak{g} \to Q \otimes \mathfrak{g}$ the operator obtained by replacing every occurrence of the total derivative D_i in ϕ^* with the operator $\widehat{D}_i = D_i - \mathrm{ad}_{A_i}$.

Here ϕ^*_q are differentials of the complex (2). Consequently, for $m > 2$ (three and more independent variables) we have $H^{1,1}_\alpha(\mathcal{E}, \mathfrak{g}) = 0$ unless the system (1) is overdetermined.

Let us consider the element $\ell(\alpha) \in \Lambda^{1,1}\mathcal{E} \otimes \mathfrak{g}$, which is easily seen to be $\bar\partial_\alpha$-closed. Then we have the corresponding 1st cohomology class $[\ell(\alpha)] \in H^{1,1}_\alpha(\mathcal{E}, \mathfrak{g})$, whose image in $P^*_{m-1} \otimes \mathfrak{g}$ under the isomorphism (5) is called the *characteristic element* of α and denoted by χ_α.

Proposition ([21, Prop. 4.2]). *If $H^{2,0}_\alpha = 0$ and $\chi_\alpha = 0$, then α is trivial (gauge equivalent to zero).*

For any matrix function $S : \mathcal{E} \to G$ we have the *conjugation* $\mathrm{Ad}_S : \bar\Lambda^q\mathcal{E} \otimes \mathfrak{g} \to \bar\Lambda^q\mathcal{E} \otimes \mathfrak{g}$ by $\gamma \mapsto S \cdot \gamma \cdot S^{-1}$. Then $\bar\partial_{\alpha^S} \circ \mathrm{Ad}_S = \mathrm{Ad}_S \circ \bar\partial_\alpha$ so that Ad_S is a morphism between the horizontal gauge complexes for α and the gauge equivalent α^S. But Ad_S is invertible with $\mathrm{Ad}_{S^{-1}}$ as the inverse, hence

$$H^{1,q}_\alpha(\mathcal{E}, \mathfrak{g}) \cong H^{1,q}_{\alpha^S}(\mathcal{E}, \mathfrak{g}).$$

Summing up the above, we have

Proposition. 1° ([21, Prop. 3.9], [32]) *Gauge equivalent ZCR's have conjugate characteristic elements;*

2° ([21, eq. 2.11, Prop. 2.7]) χ *satisfies*

$$\sum_{I,\ell}(-1)^{|I|}\widehat{D}_I\left(\frac{\partial F^\ell}{\partial u^k_I}\chi_\ell\right)=0.$$

3° ([21, Prop. 4.2, 4.3]) *For non-overdetermined systems,* $\chi \neq 0$ *unless* α *is trivial.*

The converse of 1° is not true in general: two ZCR's with conjugate characteristic elements may still be gauge inequivalent. Examples are provided by numerous S-integrable equations of mathematical physics (e.g., KdV, mKdV, sine-Gordon, etc.) that have their characteristic elements independent of the spectral parameter.

Note also that for $\mathfrak{g} = \mathbb{C}$ and $m = 2$ a ZCR reduces to a conservation law, while the characteristic element is just the n-tuple of generating functions in the sense of [45]. Similar reduction takes place for any abelian algebra $\mathfrak{g}$. Note that for $\mathfrak{g} = \mathbb{C}$, eq. 2° is exactly the determining condition for generating functions of conservation laws.

A procedure to compute ZCR's of non-overdetermined systems (1) follows. The input are eq. (1) and the Lie algebra $\mathfrak{g}$.

Procedure ([22]). Solve the *determining system*

$$\bar d\bar\alpha=\tfrac{1}{2}[\bar\alpha,\bar\alpha], \tag{6}$$

$$0=\sum_{I,\ell}(-1)^{|I|}\widehat{D}_I\left(\frac{\partial F^\ell}{\partial u^k_I}\bar\chi_\ell\right) \tag{7}$$

for unknowns $\bar\chi, \bar\alpha$ for $(\bar\chi, \bar\alpha)$ running through possible normal forms for couples (χ, α), $\chi \in P^*_{m-1} \otimes \mathfrak{g}$, $\chi \neq 0$, with respect to the the group action

$$(\chi, \alpha) \mapsto (S\chi S^{-1}, \alpha^S).$$

The unknown ZCR $\bar\alpha$ enters eqs. (6)–(7) via the coefficients of $\widehat{D}_i$. Components of $\bar\chi$ come as auxiliary unknowns; note that (7) is linear in them. In case of general position the system in question has $(n+1)\dim\mathfrak{g}$ equations for the same number of unknowns, where n is the number of equations in the system (1).

Possible normal forms $\bar\chi, \bar\alpha$ depend only on the algebra $\mathfrak{g}$. In [22] we analysed the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$.

Proposition ([21, Prop. 4.2]). *Let* $\alpha = A\,dx + B\,dy$ *be an irreducible* $\mathfrak{sl}_2$*-valued ZCR, let* $\chi \neq 0$ *be its characteristic matrix. Then we have one of the following two normal forms for (the first nonzero matrix in)* χ *and* A*:*

– *Nilpotent case*

$$\bar\chi=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\quad \bar A=\begin{pmatrix}0 & a_2\\ a_3 & 0\end{pmatrix}. \tag{8}$$

– *Diagonal case*

$$\bar\chi=\begin{pmatrix}r & 0\\ 0 & -r\end{pmatrix},\quad \bar A=\begin{pmatrix}a_1 & 1\\ a_3 & -a_1\end{pmatrix}. \tag{9}$$

No further reduction of B *is possible, i.e.,* $\bar B = B$.

Thus, in the case of $\mathfrak{g} = \mathfrak{sl}_2$ the classification turns out to be very simple. Let us note in this context that the problem of classification of normal forms of pairs of matrices with respect to conjugation is "wild" (see Sergeichuk [33]).

The following construction was, in the particular case of evolution equations, discovered by Sakovich [32].

Proposition. Let $\mathfrak{g}$-matrices C_ℓ^I satisfy

$$\bar{d}\alpha - \tfrac{1}{2}[\alpha, \alpha] = \sum_{\ell,I} D_I F^\ell \cdot C_\ell^I. \tag{10}$$

Put

$$\bar{C}_\ell = \sum_I (-\widehat{D})_I C_\ell^I. \tag{11}$$

Then $(\bar{C}_\ell)$, restricted to the equation manifold, is the *characteristic element* for α.

Examples. (1) In [20], irreducible $\mathfrak{sl}_2$-valued ZCR's were computed for equations of the form $u_{xy} = f(u)$. The well-known result ([35], see also [2, Sect. 3.2d]) that f must be $\mu e^{cu} + \nu e^{-cu}$ ($c, \mu, \nu =$ const) was reestablished.

(2) In [21], an incomplete classification of third-order evolution equations of the form $u_t = u_{xxx} + F(t, x, u, u_x)$ possessing an irreducible $\mathfrak{sl}_2$-valued ZCR's was given. We restricted ourself to the so called "nilpotent case"; among the resulting equations were integrable equations such as KdV and cKdV. All S-integrable equations of this class were found reducible to KdV by a point transformation. ZCR's depending on higher-order derivatives of u were also considered, but all reduced to second-order ones. Both S- and C-integrable equations of the same form have been already classified by the symmetry analysis in [25, 34].

(3) In [16] we found four S-integrable cases among reduced Gauss–Mainardi–Codazzi equations for surfaces immersed in E^3, in geodesic and Chebyshev coordinates separately. One of the resulting classes of equations was well known (linear Weingarten equations); the remaining three were given by coordinate-dependent conditions, hence non-geometric.

4. Application: insertion of the spectral parameter

For a long time it is known that for many integrable systems the parameter can be successfully inserted by action of a finite symmetry([31, 8, 17]). In the framework of the Wahlquist–Estabrook theory, the action of symmetries has been explained in [29], see also [31]. Even more instructive is the action of symmetries on coverings, see [17, Sect. 3.6]. Believing that the parameter can always be inserted that way, Levi, Sym and Tu [18] suggested to put point symmetries in the basis for an effective algorithm. Later on Cieśliński ([5, 7]) discovered a counterexample to show that local point symmetries may be insufficient for inserting the spectral parameter. He also suggested an extension of the symmetry method ([5, 7]) to overcome the problem. Yet different method to insert the parameter has been applied by Bandos [3].

However, in many cases the whole 1-parametric family of ZCR's has one and the same characteristic element. We call such parameters *passive*. Many equations of mathematical physics (including "model" integrable equations such as KdV, mKdV, sine-Gordon [1]) have a passive spectral parameter.

Passive parameters are very easy to insert—it suffices to reconstruct the ZCR from its characteristic element χ using eq. (7). Recall that $\bar{\alpha}$ enters eq. (7) via the coefficients of operators $\widehat{D}$.

Experience shows that the reconstruction is best possible for ZCR's with coefficients in a semisimple algebra $\mathfrak{g}$, and is more or less hindered when the ZCR admits a reduction to a solvable subalgebra. If the algebra is abelian, then eq. (7) does not explicitly contain α, whence the reconstruction from eq. (7) is impossible at all (in this case the ZCR reduces to conservation laws, which still may be reconstructed from generating functions—see [45]).

Some care is needed to avoid parameters removable by gauge transformation. Consider a one-parametric ZCR $A(\lambda)$, $B(\lambda)$, where λ belongs to an open interval. The parameter λ is said to be *removable* if $A(\lambda)$, $B(\lambda)$ are mutually gauge equivalent for all values of λ, otherwise it is *non-removable*. A removable parameter λ may be "removed" by gauge action with respect to an appropriate λ-dependent matrix $H(\lambda)$. Only non-removable parameters are relevant for complete integrability theory.

In [22, Remark 1] we show that when the characteristic element is $\chi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ one always obtains a removable parameter. E.g., the reconstructed ZCR for the KdV equation comes out as depending on two parameters, and just one of them is the non-removable spectral parameter (see [22]).

Example. Consider the non-homogeneous non-linear Schrödinger equation

$$
\begin{aligned}
q_t &= \mathrm{i}(qf)_{xx} + 2\mathrm{i}qr, \\
r_x &= -(|q|^2)_x f - 2|q|^2 f_x \,,
\end{aligned}
\tag{12}
$$

where q is complex, r is real, and f is real and given. We follow the exposition by Cieśliński [5, 7]. For every given function $f = f(t, x)$ the system admits an $\mathfrak{sl}_2$-valued ZCR (A, B) with

$$
A = \begin{pmatrix} 0 & q \\ -p & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \mathrm{i}r & \mathrm{i}(qf)_x \\ \mathrm{i}(pf)_x & -\mathrm{i}r \end{pmatrix}.
$$

In [5, 7] this ZCR served as a counterexample to the local symmetry method. It is known that the system (12) is completely integrable for any f linear in x (see loc. cit. and references therein). Here we demonstrate that it is easy to insert the parameter by reconstruction, assuming that it is passive. The condition of f being linear in x reappears again.

To start with, we rewrite equation (12) as

$$
\begin{aligned}
0 &= q_t - \mathrm{i}(qf)_{xx} - 2\mathrm{i}qr, \\
0 &= p_t + \mathrm{i}(pf)_{xx} + 2\mathrm{i}pr, \\
0 &= r_x - (pq)_x f - 2pqf_x \,,
\end{aligned}
$$

where $p = \bar{q}$. Using eq. (11) we obtain the characteristic element as the triple of matrices

$$
\chi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \chi_3 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix},
$$

which is already in normal form with respect to conjugation. Upon inserting χ_1, χ_2, χ_3 and yet unknown $\mathfrak{sl}_2$-matrices

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}.$$

into formula (7) we get a system of linear algebraic equations on unknowns a_2, a_3, b_1, b_2, b_3:

$$\begin{aligned}
&b_2 = \mathrm{i} f D_x a_2 + 2\mathrm{i} a_2 f a_1 + \mathrm{i} q\, \partial f/\partial x,\\
&b_1 = \mathrm{i} r + \mathrm{i} f D_x a_1 + 2\mathrm{i} f a_1^2 + \mathrm{i} f a_2 a_3 - \mathrm{i} f q a_3,\\
&b_3 = -\mathrm{i} f D_x a_3 + 2\mathrm{i} a_3 f a_1 + \mathrm{i} p\, \partial f/\partial x,\\
&b_1 = -\mathrm{i} f D_x a_1 + I f a_2 a_3 + 2\mathrm{i} f a_1^2 + \mathrm{i} r + \mathrm{i} f p a_2,\\
&a_2 = q,\\
&a_3 = -p.
\end{aligned}$$

By comparing the two expressions for b_1 we immediately get a necessary condition

(13) $$D_x a_1 = 0.$$

Then

$$A = \begin{pmatrix} a_1 & q \\ -p & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} \mathrm{i} r + 2\mathrm{i} f a_1^2 & \mathrm{i}(q f)_x + 2\mathrm{i} q f a_1 \\ \mathrm{i}(p f)_x - 2\mathrm{i} p f a_1 & -\mathrm{i} r - 2\mathrm{i} f a_1^2 \end{pmatrix},$$

and the condition $A_t - B_x + [A, B] = 0$ becomes

(14) $$D_t a_1 = 2\mathrm{i} f_x a_1^2.$$

By cross-differentiating (13) and (14) we get $0 = D_x(f_x a_1^2)$, i.e., $0 = f_{xx}$ (unless $a_1 = 0$, which would lead us back to the initial non-parametric ZCR), and therefore

$$f = c_1 x + c_0,$$

where c_1, c_0 are arbitrary functions of t. Finally, by (14)

$$a_1 = -1/(2\mathrm{i} c_1 t + \lambda),$$

which is the same result as in [5, 7].

One can easily prove that the parameter λ is non-removable. Indeed, assuming the converse we see that the gauge matrix H must commute with each of the three matrices χ_i, $i = 1, 2, 3$. But then H is a scalar multiple of the identity matrix, and as such it cannot remove λ by gauge action.

5. Bäcklund transformations

In this section we report on the work [22]. Bäcklund transformations usually accompany S-integrable equations [2, 31]. We introduce two local properties, effectivity and normality, which arise very naturally within the Krasil'shchik and Vinogradov's [17] theory of coverings. These properties allow us to understand geometrically the concept of "generating power" of Bäcklund transformations and also imply the existence of a nonlinear superposition principle (independently of the permutability).

Adapted to the category $\mathcal{DE}_M$, basic definitions of Krasil'shchik and Vinogradov [17] are: Let $E, \tilde{E} \in \mathcal{DE}_M$ be diffieties over a common base manifold M. A morphism $p : \tilde{E} \to E$ is said to be a *covering* if $\tilde{E}$ is a fibered manifold over E with respect to p. A covering is said to be n-dimensional if the fibre dimension of the fibered manifold is n. A *Bäcklund transformation* is a diagram consisting of two coverings:

$$\begin{array}{ccc} & \tilde{E} & \\ {}^{p}\swarrow & & \searrow^{q} \\ E_1 & & E_2 \end{array} \tag{15}$$

The case $E_1 = E_2$ is referred to as a *Bäcklund autotransformation*.

The Bäcklund transformation is a tool to generate new solutions from known ones. On the level of jets of solutions (= points of diffieties) the process is as follows: Assume that the covering $\tilde{E} \to E_1$ is n-dimensional, $n < \infty$. Given a point $u \in E_1$, the preimage $p^{-1}u \subset \tilde{E}$ is an n-dimensional submanifold. Points in the image $qp^{-1}u \subset E_2$ are said to be related to u by the Bäcklund transformation (15).

Quite naturally, if $qp^{-1}u$ is a manifold, then its dimension l will be called the *generating power* of the Bäcklund transformation. It is easy to give an infinitesimal criterion to ensure that $l = n$:

Definition ([22]). A Bäcklund transformation (15) is called *effective* if no nonzero vector in $T\tilde{E}$ is vertical with respect to both p and q.

Proposition ([22]). *The Bäcklund transformation (15) is effective if and only if for every point $u \in E_1$ the mapping $q|_{p^{-1}u} : p^{-1}u \to E_2$ is an immersion.*

Under effectivity, the image $qp^{-1}u$ is locally a submanifold of dimension n, and the generating power l is equal to n. Then one can induce *natural coordinates* on the fibres $p^{-1}u$ from any natural coordinates in E_2. Given two Bäcklund transformations between E_1 and E_2, it is then a matter of routine to decide whether they are identical: it suffices to express both in one and the same natural system of coordinates.

Effectivity is equivalent to $\Xi = \operatorname{Ker} Tp \cap \operatorname{Ker} Tq$ being zero. In general, $\Xi \subset T\tilde{E}$ is a distribution. If $\Xi \neq 0$ and the factor $r : \tilde{E} \to \tilde{E}/\Xi$ exists, then one can factorize p, q through r to obtain an effective Bäcklund transformation with the same generating properties. In [22] an example of this factorization has been given for the Bäcklund transformation for the Tzitzéica equation [41], which in original formulation has the dimension $n = 4$, but whose generating power is only three.

The next natural question is what happens if $u \in E_1$ is replaced with a submanifold $K \subset E_1$ of dimension $k < \infty$. We have $\dim p^{-1}K = k + n$, but we may still have $\dim qp^{-1}K < k + n$ even if the Bäcklund transformation is effective.

Definition ([22]). Consider a Bäcklund transformation (15). Let Ξ, H denote the involutive distributions $\Xi_x = \operatorname{Ker} T_x p$, $H_x = \operatorname{Ker} T_x q$ on $\tilde{E}$, respectively. Denote

$$r = \inf_{\eta \in H \setminus \{0\}} \dim \mathcal{L}\langle \Xi, \eta \rangle,$$

where $\mathcal{L}\langle \Xi, \eta \rangle$ stands for the commutator of the distribution generated by Ξ and $\eta \in H$. The Bäcklund transformation is said to be *r-normal* if $r < \infty$, and *normal* if $r = \infty$.

Obviously, if $\eta \in \Xi$, then $r = \dim \Xi = k$. Thus, one needs effectivity to have normality with $r > k$.

Proposition ([22]). *Let the Bäcklund transformation (15) be r-normal with $r > n + k$. Then there exists an open dense subset $N^0 \subseteq p^{-1}K$ such that $q|_{N^0} : N^0 \to E_2$ is an immersion.*

Considering a normal Bäcklund autotransformation of generating power l, we conclude that the generating power of its ith iteration is il. In particular, a normal Bäcklund autotransformation never fails to generate a new solution (outside K).

Using normality, one can also explain the nonlinear superposition principles, usually atributed to permutability. Consider two successive Bäcklund transformations: $E_1 \leftarrow \tilde{E}_1 \to E_2$ and $E_2 \leftarrow \tilde{E}_2 \to E_3$. Assuming normality and regularity, we have:

Proposition ([22]). *For $u \in E_1$ arbitrary, let $P_u \subset \tilde{E}_1 \times_{E_2} \tilde{E}_2$ denote the set of all triples (u, v, w) where v is related to u by the first and w is related to v by the second Bäcklund transformation. Then there exists an open dense subset $P_u^0 \subseteq P_u$ such that the mapping $P_u^0 \to E_3$, $(u, v, w) \mapsto w$, is an immersion.*

The proposition implies that v can be computed from u and w by the implicit function theorem (without integration).

An explicit check of normality has been performed for three well-known S-intagrable equations:

Proposition ([22]). *The pKdV equation $u_t = -u_{xxx} - 3u_x^2$, the sine-Gordon equation $u_{tx} = \sin u$, and the Tzitzéica equation $u_{tx} = e^u - e^{-2u}$ have normal Bäcklund transformations.*

To prove normality (after appropriate factorization in the third case), we substantially exploited the concept of pseudosymmetry due to Sokolov [36].

Finally, one is naturally lead to a conjecture that normality is a typical property shared by BT's of all S-integrable equations.

Acknowledgements

The influence of I.S. Krasil'shchik on the development of this work is gratefully acknowledged. Thanks are also due to V.V. Sokolov and V.V. Sergeichuk for discussions and correspondence on various aspects of the subject, and to Artur Sergyeyev for improvements to readability of the manuscript.

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Received 10 November 2000