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Differential systems in higher-order mechanics[1]

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Abstract. The aim of this paper is to provide a geometric description and classification of general systems of ordinary differential equations of order $s \geq 1$ on fibered manifolds, and to investigate geometric properties of solutions of these equations. The present setting covers Lagrangian systems, higher order semisprays, and constrained systems (equations subject to constraints modeled by exterior differential systems). Emphasis is put on singular systems.

Keywords and phrases. System of higher-order ordinary differential equations, distribution of nonconstant rank, geometric classification of ODE, regular, semiregular, weakly regular, singular ODE, constraint algorithm, higher order semispray problem; Lepage 2-form, dynamical form, dynamical distribution, mechanical system, Hamiltonian system, Lagrangian system, semispray connection; constraint ideal, holonomic, semiholonomic and nonholonomic first and higher-order constraints, constrained equations, constrained distribution; regular, Lagrangian constrained systems.

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1. Introduction

The subject of this work are ordinary differential equations on fibered manifolds, namely, systems of m ODE of order $s \geq 1$ for sections γ of a fibered manifold (over a one-dimensional base),

$$E_\sigma\left(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}, \dots, \frac{d^s\gamma^\nu}{dt^s}\right) = 0, \quad 1 \leq \nu \leq m,$$

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where γ^ν are components of γ. Such equations are modeled by *dynamical forms* defined on final jet prolongations of the underlying fibered manifold. We propose a geometric framework for study these equations in terms of the so called *Lepage classes of dynamical forms* and their *dynamical distributions*. On the basis of the properties of dynamical distributions we provide a *geometric classification of ordinary differential equations* reflecting properties of their solutions.

Emphasis is put on *singular* equations. In this case the dynamical distribution need not have a constant rank, and consequently, need not be spanned by a system of continuous vector fields (which means that the corresponding dynamics generally cannot be described by a vector field, or by a system of vector fields). Dealing with singular systems we are faced with a new non-trivial question—to clarify *the structure of solutions* of the dynamical distributions, i.e., the "dynamical picture". For regular systems this question is solved trivially—the dynamics is represented by a one-dimensional foliation of the "phase space" (i.e., the manifold where the dynamical distribution is defined). To answer this question in general it is reasonable first to *classify* dynamical distributions with respect to their geometrical properties. In this way one could distinguish between systems with relatively simple dynamical distributions (e.g. horizontal, of a constant rank, etc.), and systems the dynamics of which cannot be studied easily with help of Frobenius theorem. Based on this classification, an integration algorithm (the so called *constraint algorithm*) for highly singular dynamical distributions is proposed. Important classes of equations, namely *variational equations* (i.e., equations which can be identified with Euler–Lagrange equations of certain Lagrangians), and equations which can be modeled by *semisprays* are incorporated within this scheme, and their specific properties are discussed. In this respect the paper generalizes some recent works on second and higher order ODE, e.g. [31], [36], [39], [49], [50], [52], [53], [62–64]. The present setting covers also the so called *higher order constrained systems*, which arise as a generalization of systems with *holonomic* and *nonholonomic constraints* of classical mechanics (cf. eg. [15], [35], [37], [41], [46], [47], [50], [51], [55] and references therein for the first order, and [11], [19], [38], [42], [48], [58], [59], [61] for higher order constraints). The main point is that constraints of order r are modeled as a *fibered submanifold* of the fibered manifold J^rY, *endowed with a distribution* (of a constant rank, generally non-involutive). Equivalently, the constraint structure is represented by means of a (regularly *not closed*) *ideal* of differential forms on a fibered manifold. Differential equations subject to constraints are then represented, roughly speaking, by means of *classes of Lepagean* 2*-forms modulo the constraint ideal*. *Unconstrained* systems and *holonomic* systems then correspond to the case when the constraint ideal is *trivial*.

The present work is both a review and research paper. The exposition summarizes and develops concepts and results from [30], [32], and [35–38]. The plan of the paper is as follows. In Sec. 2 we introduce notations and briefly recall basics on jet prolongations of fibered manifolds, and vector fields and differential forms related with the fibered structure. We also recall basic concepts from the theory of distributions of nonconstant rank, frequently used throughout the paper. Secs. 3 and 4 present a general framework for the theory of systems of higher order ordinary differential equations on fibered manifolds. Similarly as in [38] we introduce the main objects—*Lepage classes of dynamical forms* and related *dynamical distributions*, globally representing the family of ODE of any finite order. Within this setting, the equations are naturally locally extended to the so called *Hamiltonian systems*. In this way, the problem of study the solutions (paths) of the equations is transferred to the geometrically well-defined problem of study the corresponding dynamical *distributions*. On this basis, a *geometric classification* of the equations, as well as basic properties of the solutions arising from this classification are presented in Secs. 5 and 6, and an algorithm for finding *one-dimensional* integral manifolds of generally *noncontinuous* distributions of a *nonconstant rank* (possibly greater than 1) is proposed in Sec. 7. This part of the paper is a direct generalization to *arbitrary* ODE of the corresponding results on *variational* equations, presented in [36] (cf. also [30], [32]). *Lagrangian* (variational) systems are then subject of Sec. 8, where also some additional properties connected with the existence of Lagrangians are discussed. Sec. 9 brings concrete examples of application of the constraint algorithm for solving "highly singular" distributions. In Secs. 10 and 11 the theory of higher order *"equations with external constraints"*, developed in [38], is briefly recalled. The constraint structure on a fibered manifold, generalizing the well known concepts of holonomic, semiholonomic, and (general) nonholonomic constraints from classical mechanics, is introduced. It is shown that with help of the arising *constraint ideal* the theory of (higher order) equations subject to (higher order) constraints can be set and processed as a *particular* case within the general setting of the geometric theory of higher order ODE presented in the preceeding part of the paper. It should be stressed, that the theory works in the general situation, i.e., neither the unconstrained nor the constrained equations must be supposed to have a particular structure (such as semispray and/or Lagrangian), and no additional restrictions on the constraints have to be assumed. This enables us to study directly different properties of the constraints and constrained systems, e.g., regularity conditions, Hamiltonian extensions, if (and when) a constrained system is Lagrangian, and other important questions arising within the theory of constrained systems.

2. Basic structures

If not otherwise stated, the manifolds and mappings throughout the text are smooth, and the summation convention is used. We use the symbols T for the tangent functor, J^r for the r-jet prolongation functor, id for the identity mapping, * for the pull-back, d for the exterior derivative, and i for the inner product. Other notations are explained when first used.

We use final jet prolongations of fibered manifolds over one-dimensional bases, and the corresponding calculus of horizontal and contact forms (see e.g. [25], [26], [36] or [54] for review). In this section we recall basic concepts and fix notations.

Let $\pi : Y \to X$ be a fibered manifold, $\dim X = 1$, $\dim Y = m + 1$. In what follows, fiber coordinates will be denoted by (t, q^σ), where $1 \leq \sigma \leq m$. A mapping $\gamma : X \to Y$ defined on an open subset $U \subset X$ is called a *section* of the fibered manifold π if the composite mapping $\pi \circ \gamma$ is the identity mapping of U. For $r \geq 1$ we denote by $\pi_r : J^r Y \to X$, the *r-jet prolongation* of the fibered manifold π, and by $\pi_{r,k} : J^r Y \to J^k Y, k \geq 0, k < r$, the corresponding canonical projections (here we identify $J^0 Y$ with Y). The *r-jet prolongation* of a section $\gamma : U \to Y$ of π is denoted by $J^r\gamma$; it is a section of π_r. Clearly, not every section of π_r is of the form of the r-jet prolongation of a section of π. We say that a section δ of π_r is *holonomic* if there exists a section γ of π such that $\delta = J^r\gamma$. If (t, q^σ) are fiber coordinates on

$V \subset Y$, we have the *associated coordinates* $(t, q^\sigma, q^\sigma_1, \dots, q^\sigma_r)$ on $V_r = \pi^{-1}_{r,0} V$ defined by $q^\sigma_k(J^r_x\gamma) = (d^k\gamma^\sigma/dt^k)_x$, $1 \le k \le r$.

A jet prolongation of a fibered manifold π_r, $r > 0$, is often called an *anholonomic prolongation* of π. In this paper we shall work with the first jet prolongation of the fibered manifold π_{s-1}, $s \ge 2$; then the associated fibered coordinates on $J^1(J^{s-1}Y)$ will be denoted by $(t, q^\sigma_j, q^\sigma_{j,1})$, $1 \le \sigma \le m$, $0 \le j \le s-1$.

A vector field ξ on Y is called π*-projectable* if there exists a vector field ξ_0 on X such that $T\pi.\xi = \xi_0 \circ \pi$. If, in particular, $\xi_0 = 0$ then ξ is called π*-vertical*. For a π-projectable (respectively, π-vertical) vector field ξ on Y one has in fiber coordinates

$$\xi = \xi^0 \frac{\partial}{\partial t} + \xi^\sigma \frac{\partial}{\partial q^\sigma}$$

where ξ^σ are functions of (t, q^ν), and the ξ^0 depend only on t (respectively, $\xi^0 = 0$). The r*-jet prolongation* of a π-projectable vector field ξ on Y is a vector field $J^r\xi$ on J^rY,

$$J^r\xi = \xi^0 \frac{\partial}{\partial t} + \sum_{i=0}^{r} \xi^\sigma_i \frac{\partial}{\partial q^\sigma_i},$$

where the components ξ^σ_i, $i = 1, \dots, r$, are defined by the recurrent formula

$$\xi^\sigma_i = \frac{d\xi^\sigma_{i-1}}{dt} - \frac{d\xi^0}{dt} q^\sigma_i.$$

Let η be a p-form on J^rY. We say that η is π_r*-horizontal* if $i_\xi\eta = 0$ for every π_r-vertical vector field ξ on J^rY. Similarly, η is said to be $\pi_{r,k}$-horizontal, $0 \le k < r$, if $i_\xi\eta = 0$ for every $\pi_{r,k}$-vertical vector field on J^rY. From these definitions we get that, in particular, a 1-form η is π_r-horizontal if its representation in every fiber chart reads $\eta = f(t, q^\sigma, \dots, q^\sigma_r)\, dt$, and a 2-form η is $\pi_{r,0}$-horizontal if its representation in every fiber chart contains only the wedge products of dt and dq^ν's with the components possibly dependent upon $t, q^\sigma, \dots, q^\sigma_r$.

Let η be a q-form, $q \ge 1$, on J^rY . For every point $y = J^{r+1}_x\gamma \in J^{r+1}Y$, and every system of vector fields $\xi_1, \dots, \xi_q \in T_yJ^{r+1}Y$ we set

$$\begin{aligned} &h\eta(J^{r+1}_x\gamma)(\xi_1, \dots, \xi_q) \\ &\quad = \eta(J^r_x\gamma)(T_xJ^r\gamma \cdot T\pi_{r+1} \cdot \xi_1, \dots, T_xJ^r\gamma \cdot T\pi_{r+1} \cdot \xi_q). \end{aligned}$$

Evidently, $h\eta$ is a π_{r+1}-horizontal q-form on $J^{r+1}Y$. If f is a function on J^sY we set

$$hf(J^{r+1}_x\gamma) = f(J^r_x\gamma).$$

The mapping h is called *horizontalization* with respect to the projection π. For computing horizontalization one can utilize the following formulas:

$$h\,dt = dt, \quad h\,dq^\sigma_j = q^\sigma_{j+1}\,dt, \quad 0 \le j \le r, \quad hf = f \circ \pi_{r+1,r},$$

$$h\,df = \frac{df}{dt}\,dt, \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{j=0}^{r} \frac{\partial f}{\partial q^\sigma_j} q^\sigma_{j+1}.$$

Apparently, if η is a 1-form on J^rY then $J^r\gamma^*\eta = J^{r+1}\gamma^*h\eta$ for every section γ of π.

A form η on J^rY is called *contact* if $J^r\gamma^*\eta = 0$ for every section γ of π. Notice that every q-form for $q > \dim X$ is contact. In particular, a 2-form η is called 1-*contact* if for every π_r-vertical vector field ξ the 1-form $i_\xi\eta$ is horizontal, and it is called 2-*contact* if $i_\xi\eta$ is contact.

Put

$$\omega_j^\sigma = dq_j^\sigma - q_{j+1}^\sigma\, dt, \quad 1 \le \sigma \le m, 0 \le j \le r. \tag{2.1}$$

The 1-forms (2.1) are obviously contact, and they form a basis of contact 1-forms on $J^{r+1}Y$. Notice that the forms $dt, \omega^\sigma, \dots, \omega_{r-1}^\sigma, dq_r^\sigma$ form a *basis of* 1-*forms* on J^rY adapted to the contact structure. We shall frequently make use of this basis.

Every 1-form η on J^rY admits a unique decomposition into a sum of a horizontal and contact form. In fibered coordinates where $\eta = f\,dt + f_\sigma^0\,dq^\sigma + \cdots + f_\sigma^r\,dq_r^\sigma$ this decomposition reads

$$\pi_{r+1,r}^*\eta = \left(f + f_\sigma^0 q_1^\sigma + \cdots f_\sigma^r q_{r+1}^\sigma\right)dt + f_\sigma^0\,\omega^\sigma + \cdots + f_\sigma^r\,\omega_r^\sigma.$$

We denote by $p\eta$ the contact part of η, i.e., $p\eta = f_\sigma^0\,\omega^\sigma + \cdots + f_\sigma^r\,\omega_r^\sigma$. Similarly, every 2-form η on J^rY admits a unique decomposition into a sum of a 1-contact and 2-contact form. We denote by $p_1\eta$ and $p_2\eta$ the 1-contact and 2-contact part of η, respectively. Thus, for a 2-form η on J^rY we write

$$\pi_{r+1,r}^*\eta = p_1\eta + p_2\eta.$$

Non-vertical vector fields annihilating (all) contact 1-forms on J^rY are called *semisprays* of order r. In fibered coordinates they take the form

$$\zeta = \zeta_0\left(\frac{\partial}{\partial t} + \sum_{j=0}^{r-1} q_{j+1}^\sigma \frac{\partial}{\partial q_j^\sigma} + \zeta_r^\sigma \frac{\partial}{\partial q_r^\sigma}\right),$$

where at each point $\zeta_0 \neq 0$, and ζ_r^σ, $1 \le \sigma \le m$, are functions on an open subset of J^rY.

Finally let us recall basic concepts of the theory of distributions which will be frequently used throughout this paper. For more details we refer, e.g., to [2], [36], or [43]. Let M be a smooth manifold. By a *distribution* on M we mean a mapping Δ assigning to every point $x \in M$ a vector subspace Δ_x of the vector space T_xM. The function $\operatorname{rank}\Delta : M \to R$, assigning to every point $x \in M$ the number $\dim\Delta_x$ will be called *rank* of the distribution Δ. If this function is constant (respectively, constant on each connected component of M) we say that the distribution is of a *constant* (respectively, *locally constant*) *rank*. A distribution can be defined by a system of (local) vector fields, at each point spanning the vector space Δ_x, or by a system of annihilating 1-forms. If Δ is a distribution we denote its *annihilator* by Δ^0; the mapping $\Delta^0 : x \ni M \to \Delta_x^0 \subset T^*M$ is said to be a *codistribution* on M.

We say that Δ is *continuous* (respectively, *smooth*) if it can be spanned by a system of continuous (respectively, smooth) vector fields. Since the rank of a continuous distribution is a lower semicontinuous function on M, a smooth distribution of a non-constant rank *cannot* be defined by means of *continuous* one-forms, and conversely, a distribution of a non-constant rank defined by means of smooth one-forms *is not continuous*.

An immersion $f : Q \to M$ is called an *integral mapping* of Δ if

$$f^*\omega = 0 \quad \text{for all } \omega \in \Delta^0.$$

A connected submanifold Q of M of dimension q is called an *integral manifold of* Δ, if the canonical inclusion $i : Q \to M$ is an integral mapping of Δ. A distribution Δ on M is called *completely integrable* if through every point $x \in M$ there passes an integral manifold of Δ of maximal dimension (i.e., such that $\dim Q(x) = \operatorname{rank} \Delta(x)$).

The geometric structure of solutions of *completely integrable* distributions is described by *foliations*. More precisely, maximal integral manifolds of a completely integrable distribution on M form a foliation of M, and the leaves of this foliation are *immersed* submanifolds of M. If the distribution has a constant rank then all the leaves of the corresponding foliation are manifolds of the same dimension, in general however the dimensions of the leaves may differ. The existence of a foliation means in particular that (i) through every point of M there passes a unique maximal integral manifold of Δ, (ii) if $f : Q \to M$ is an integral mapping of *maximal dimension* then $f(Q)$ is a local diffeomorphism onto an open subset in a leaf, (iii) if $f : Q \to M$ is an integral mapping and $\operatorname{rank} \Delta = \text{const}$ then f is an immersion of Q into a leaf of the foliation. One should note, however, that a completely integrable distribution of a *non-constant* rank may possess integral mappings (of dimension less than maximal) which intersect different leaves.

For distributions of a *constant* (locally constant) rank one has the famous *Frobenius Theorem* stating necessary and sufficient conditions of complete integrability. As a consequence, for a completely integrable distribution Δ of a constant rank, k, in a neighborhood of every point $x \in M$ there exists an adapted chart to the associated foliation. In this chart, maximal integral manifolds of Δ are given by the equations

$$x^{k+1} = c^{k+1}, \ldots, x^n = c^n,$$

where $c^{k+1}, \ldots, c^n$ are constant functions. Maximal integral manifolds of a distribution of a constant rank can be found by means of *symmetries* (for different geometric integration methods see e.g. [14], [31], [36], [39], [44], [62-64]).

In this paper we shall deal mostly with *distributions of nonconstant rank defined by smooth one-forms*. The structure of solutions of such distributions is rather complicated, and no general integrability theory is available. We shall show in Sec. 7 that local solutions can be effectively found by means of the so called *constraint algorithm*.

An important example of distributions is connected with smooth 2-forms. Let α be a (smooth) 2-form on M. Denote by $\mathcal{V}(M)$ the system of all vector fields on M. There arises a distribution, $\mathcal{D}$, called the *characteristic distribution* of α, defined by means of the system of (local, generally not continuous) vector fields such that $i_\zeta \alpha = 0$. The annihilator $\mathcal{D}^0$ of $\mathcal{D}$ is spanned by the (smooth) 1-forms

$$i_\xi \alpha, \quad \text{where } \xi \text{ runs over } \mathcal{V}(M),$$

and is called the *associated system of* α. Notice that at each point $x \in M$, $\operatorname{rank} \alpha(x) = \operatorname{corank} \mathcal{D}(x)$.

Two distributions Δ_1 and Δ_2 on M are called *complementary* if at each point $x \in M$, $\Delta_1(x) \oplus \Delta_2(x) = T_x M$.

A distribution on a *fibered manifold* is called *vertical* if it is spanned by vertical vector fields only. A distribution complementary to a vertical distribution is called *weakly horizontal*.

3. Lepage classes and dynamical distributions

A 2-form E on $J^sY, s \geq 1$, is called a *dynamical form* if it is 1-contact and $\pi_{s,0}$-horizontal. This means that E is a dynamical form if in every fiber chart

$$E = E_\sigma(t, q^\nu, q_1^\nu, \dots, q_s^\nu)\, dq^\sigma \wedge dt. \tag{3.1}$$

A section γ of π is called a *path* of E if $E \circ J^s\gamma = 0$. In fiber coordinates this equation represents a *system of m ordinary differential equations of order s*,

$$E_\sigma\left(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}, \dots, \frac{d^s\gamma^\nu}{dt^s}\right) = 0 \tag{3.2}$$

for the components $\gamma^\nu(t)$, $1 \leq \nu \leq m$, of γ.

Denote by $\Lambda^2_{\mathrm{af}}(J^sY)$ the set of dynamical forms on J^sY with the components E_σ *affine in the derivatives of order s*, i.e., in every fiber chart given as follows:

$$E_\sigma = A_\sigma(t, q^\rho, \dots, q^\rho_{s-1}) + B_{\sigma\nu}(t, q^\rho, \dots, q^\rho_{s-1})\, q_s^\nu. \tag{3.3}$$

3.1. Remark. (i) One can easily check that $\Lambda^2_{\mathrm{af}}(J^sY)$ is correctly defined. Namely, if (V, ψ) and $(\bar{V}, \bar{\psi})$ are two overlapping fiber charts, and if the restriction of E to $\pi^{-1}_{s,0}(V)$ belongs to $\Lambda^2_{\mathrm{af}}(\pi^{-1}_{s,0}(V))$ then the restriction of E to $\pi^{-1}_{s,0}(\bar{V})$ belongs to $\Lambda^2_{\mathrm{af}}(\pi^{-1}_{s,0}(\bar{V}))$.

(ii) *Throughout this paper we shall always suppose that a dynamical form E is of the form* (3.3). Notice, however, that restricting to dynamical forms affine in the highest derivatives means in fact *no loss of generality*. Indeed, if a dynamical form E on J^sY does not belong to $\Lambda^2_{af}(J^sY)$ then it belongs to $\Lambda^2_{af}(J^{s+1}Y)$, i.e., it is “affine in the highest derivatives” if considered as a form on $J^{s+1}Y$.

Let us turn to a geometric description of the equations for paths of a dynamical form of order s in terms of (generally local) distributions on $J^{s-1}Y$. To this purpose we shall first assign to every dynamical form $E \in \Lambda^2_{\mathrm{af}}(J^sY)$ local *classes* of 2-forms of order $s-1$.

Let $W \subset J^rY$ be open. We say that a 2-form α on W is a (generalized) *Lepage 2-form* of order r if its 1-contact part $p_1\alpha$ is a dynamical form. Two Lepage 2-forms α' and α with the same domain of definition W will be called *equivalent* if $p_1\alpha' = p_1\alpha$ (note that forms equivalent in this sense are equivalent also in the sense of the Krupka variational sequence [27–29]). For a Lepage 2-form α on $W \subset J^rY$ put

$$\Delta^0_\alpha = \mathrm{span}\{i_\xi\alpha \mid \xi \text{ runs over the set of all } \pi_r\text{-vertical vector fields on } W\}. \tag{3.4}$$

Δ^0_α is a codistribution on W, it will be called the *dynamical codistribution* of α. The corresponding distribution Δ_α will be then called the *dynamical distribution* of α.

Let $s \geq 2$. Every dynamical form $E \in \Lambda^2_{\mathrm{af}}(J^sY)$ can be represented by local equivalence classes of Lepage 2-forms *of order* $s-1$. The family $[\alpha]$ of Lepage 2-forms defined on open sets of $J^{s-1}Y$ such that for every $\alpha \in [\alpha]$, $p_1\alpha = E$ on the domain of definition of α, is called the *Lepage class of E* of order $s-1$ [38]. In fiber coordinates, the Lepage class of E is represented as follows: If (V, ψ), $\psi = (t, q^\sigma)$, is a fiber chart on Y then on $V_{s-1} = \pi^{-1}_{s-1,0}(V)$,

$$\alpha = \omega^\sigma \wedge (A_\sigma dt + B_{\sigma\nu} dq^\nu_{s-1}) + \eta, \tag{3.5}$$

where

$$\eta = \sum_{i,j=0}^{s-2} F^{ij}_{\sigma\nu} \omega^\sigma_i \wedge \omega^\nu_j \tag{3.6}$$

is an *arbitrary* 2-contact 2-form. The corresponding dynamical codistributions Δ^0_α are locally spanned by the following 1-forms:

$$A_\sigma\, dt + \sum_{j=0}^{s-2} 2F^{0j}_{\sigma\nu}\, \omega^\nu_j + B_{\sigma\nu}\, dq^\nu_{s-1}, \quad B_{\sigma\nu}\, \omega^\nu, \quad \sum_{j=0}^{s-2} F^{ij}_{\sigma\nu}\, \omega^\nu_j, \quad 1 \le i \le s-2. \tag{3.7}$$

Notice that Δ_α need not be of a constant rank, but rank $\Delta_\alpha \geq 1$ at each point of V_{s-1}.

For a dynamical form E of order 1 the above construction provides us with a Lepage class defined on J^1Y, which obviously is of the form

$$\alpha = \omega^\sigma \wedge (A_\sigma dt + B_{\sigma\nu} d\dot{q}^\nu) + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu.$$

However, the situation may even simplify as follows:

3.2. Theorem. *Let $E \in \Lambda^2_{\mathrm{af}}(J^1Y)$ be a dynamical form. There exists a Lepage class of E of order zero (i.e., defined on Y) if and only if (in every fiber chart) the matrix*

$$\mathrm{B} = (B_{\sigma\nu}) = \left(\frac{\partial E_\sigma}{\partial q^\nu_1}\right) \tag{3.8}$$

is antisymmetric. The Lepage class of E of order zero consists of a unique Lepage 2-form α on Y. In fiber coordinates where $E = (A_\sigma + B_{\sigma\nu}\dot{q}^\nu)\, dq^\sigma \wedge dt$, one has

$$\alpha = A_\sigma\, dq^\sigma \wedge dt + \tfrac{1}{2} B_{\sigma\nu}\, dq^\sigma \wedge dq^\nu. \tag{3.9}$$

Proof. Let $E \in \Lambda^2_{\mathrm{af}}(J^1Y)$, and denote $E = (A_\sigma + B_{\sigma\nu}\dot{q}^\nu)\, dq^\sigma \wedge dt$ in a fiber chart V on Y. Let α be a 2-form on V, $\alpha = \alpha_\sigma dq^\sigma \wedge dt + \alpha_{\sigma\nu} dq^\sigma \wedge dq^\nu$. Since $p_1\alpha = (\alpha_\sigma + (\alpha_{\sigma\nu} - \alpha_{\nu\sigma})\dot{q}^\nu)\, dq^\sigma \wedge dt$, the condition $E = p_1\alpha$ gives

$$\alpha_\sigma = A_\sigma, \qquad B_{\sigma\nu} = \alpha_{\sigma\nu} - \alpha_{\nu\sigma}.$$

Hence, the matrix B is antisymmetric. Obviously, if B is antisymmetric in a fiber chart on Y, it is antisymmetric in every fiber chart on Y. Conversely, if the matrix (3.8) is antisymmetric we obtain by (3.9) a Lepage class of E.

Let E satisfy the antisymmetry condition. Then putting (3.9) in a fiber chart V on Y, one gets a Lepage 2-form of E over V, which is obviously *unique*. Moreover, transformation properties of the functions $B_{\sigma\nu}$ show that the form $p_2\alpha$ is well-defined, proving the global existence of α. □

3.3. Remark. For simplicity of notations, *from now on we shall suppose that s denotes the "true order" of a dynamical form E* in the sense that *the lowest possible order for the corresponding Lepage class of E is $s-1$*. More precisely,

(i) for $s \geq 3$, $E \in \Lambda^2_{\mathrm{af}}(J^sY)$ means that $E \notin \Lambda^2_{\mathrm{af}}(J^{s-1}Y)$,

(ii) $E \in \Lambda^2_{\mathrm{af}}(J^2Y)$ means that either $E \notin \Lambda^2_{\mathrm{af}}(J^1Y)$, or $E \in \Lambda^2_{\mathrm{af}}(J^1Y)$ but its Lepage class is not projectable onto Y (E does not satisfy the antisymmetry condition of Theorem 3.2), and

(iii) $E \in \Lambda^2_{\mathrm{af}}(J^1Y)$ means that E is representable by a unique Lepage form on Y (cf. Theorem 3.2.).

In what follows, the Lepage class $[\alpha]$ associated with a dynamical form $E \in \Lambda^2_{\mathrm{af}}(J^sY)$ is called a *mechanical system*, and the number $s-1$ is called *the order* of the mechanical system $[\alpha]$. The manifold Y is called the *configuration space*, accordingly, $J^{s-1}Y$ is called the *phase space* for E. The class of the corresponding dynamical distributions is denoted by $[\Delta_\alpha]$. Note that every mechanical system of order zero is representable by a *unique* and *global* Lepage 2-form on Y.

We can immediately see that *for equivalent Lepage* 2*-forms the holonomic integral sections of their dynamical distributions coincide*. Moreover, if $[\alpha]$ is the Lepage class of E then *the set of holonomic integral sections of any dynamical distribution* Δ_α *locally coincides with the set of paths of* E. Summarizing, we easily get the following equations for paths of a dynamical form.

3.4. Theorem. *Let E be a dynamical form on J^sY where $s \geq 1$, $[\alpha]$ its Lepage class on $J^{s-1}Y$. Let $\gamma : I \to Y$ be a section of π, defined on an open set $I \subset X$. The following conditions are equivalent:*

(1) *γ is a path of E.*

(2) *E vanishes along $J^s\gamma$, i.e., $E \circ J^s\gamma = 0$.*

(3) *For every π_{s-1}-vertical vector field ξ on $J^{s-1}Y$ and every element α of* $[\alpha]$ *such that $J^{s-1}\gamma(I) \cap \operatorname{dom}\alpha \neq \emptyset$, γ satisfies the equation*

$$J^{s-1}\gamma^*\, i_\xi\, \alpha = 0. \tag{3.10}$$

(4) *For every π_{s-1}-projectable vector field ξ on $J^{s-1}Y$ and every $\alpha \in [\alpha]$ such that $J^{s-1}\gamma(I) \cap \operatorname{dom}\alpha \neq \emptyset$, γ satisfies the equation $J^{s-1}\gamma^*\, i_\xi\, \alpha = 0$.*

(5) *For every vector field ξ on $J^{s-1}Y$ and every $\alpha \in [\alpha]$ such that $J^{s-1}\gamma(I) \cap \operatorname{dom}\alpha \neq \emptyset$, γ satisfies the equation $J^{s-1}\gamma^*\, i_\xi\, \alpha = 0$.*

(6) *In every fiber chart, γ satisfies the system* (3.2) *of m ordinary differential equations of order s for the components γ^ν, $1 \leq \nu \leq m$, of γ.*

4. Hamiltonian extensions of a mechanical system

By Theorem 3.4, paths of a dynamical form E can be interpreted as *holonomic* integral sections of the *dynamical distributions* associated to E (item (3)), or, *equivalently*, as *holonomic* integral sections of the *characteristic distributions* of the corresponding Lepage 2-forms (item(5)). Moreover, locally the paths do *not* depend on the choice of the 2-form α from the Lepage class $[\alpha]$ of E.

The set of *holonomic* integral sections of the dynamical (respectively, characteristic) distribution of a Lepage 2-form is a subset of the set of *all* its integral sections, and these sets are generally different. Moreover, distributions corresponding to different equivalent Lepage 2-forms possess different sets of integral sections. Thus, inspired by the classical mechanics, we can introduce the following concepts.

Let $[\alpha]$ be a mechanical system of order $s-1$, let α be a representative of $[\alpha]$, $\operatorname{dom}\alpha = U \subset J^{s-1}Y$, and Δ_α the corresponding dynamical distribution. The 2-form α will be called a *Hamiltonian system* related to E. Integral sections of the distribution Δ_α, i.e., local sections δ of the fibered manifold π_{s-1} passing in U and such that

$$\delta^* i_\xi \alpha = 0 \tag{4.1}$$

for every π_{s-1}-vertical vector field ξ on U, will be called *Hamilton paths* of E. Equations (4.1), which are first-order ODE for δ will be called *generalized Hamilton equations*.

Let us now consider the 1-jet prolongation of the fibered manifold $\pi_{s-1} : J^{s-1}Y \to X$, i.e., the fibered manifold $(\pi_{s-1})_1 : J^1(J^{s-1}Y) \to X$. To avoid confusion, we shall use the notation $\tilde{h}$ (resp. $\tilde{p}$, resp. $\tilde{p}_1$) for the horizontalization (resp. contactization, resp. 1-contactization) with respect to the projection π_{s-1}. According to Theorem 3.2, the 2-form α on $U \subset J^{s-1}Y$ is the unique representative of a Lepage class of order zero (with respect to the projection π_{s-1}). This Lepage class is associated with the dynamical form

$$\mathcal{H}_\alpha = \tilde{p}_1 \alpha, \tag{4.2}$$

which is defined on $(\pi_{s-1})^{-1}_{1,0}(U) \subset J^1(J^{s-1}Y)$. The form $\mathcal{H}_\alpha$ will be called a *Hamilton form* related to E.

Directly from the definition of Hamilton form we obtain the following:

4.1. Proposition. *Let E be a dynamical form on J^sY, α a representative of its Lepage class defined on $U \subset J^{s-1}Y$. Then for every π_{s-1}-vertical vector field ξ on U,*

$$i_{J^1\xi}\, \mathcal{H}_\alpha = \tilde{h}\, i_\xi\, \alpha. \tag{4.3}$$

Conversely, if for a 2-form η the relation $i_{J^1\xi}\, \eta = \tilde{h}\, i_\xi\, \alpha$ for every π_{s-1}-vertical vector field ξ holds, then $\eta = \mathcal{H}_\alpha$.

Using a fiber-chart expression of α, and denoting the associated coordinates on $J^1(J^{s-1}Y)$ by $(t, q^\sigma, \dots, q^\sigma_{s-1}, q^\sigma_{,1}, \dots, q^\sigma_{s-1,1})$, we get for the Hamilton form the following formula:

$$\begin{aligned} H_\alpha &= \sum_{i=0}^{s-1} H^i_\sigma(\alpha)\, dq^\sigma_i \wedge dt, \\ H^i_\sigma(\alpha) &= E_\sigma\, \delta^{0i} + \sum_{k=0}^{s-1-i} 2F^{ik}_{\sigma\nu}(q^\nu_{k,1} - q^\nu_{k+1}). \end{aligned} \tag{4.4}$$

Theorem 3.4 applied to the dynamical form $\mathcal{H}_\alpha$ gives equivalent expressions for the generalized Hamilton equations:

4.2. Theorem. *Let $E \in \Lambda^2_{\mathrm{af}}(J^sY)$ be a dynamical form, $[\alpha]$ its Lepage class of order $s-1$. Let $\mathcal{H}_\alpha = \tilde{p}_1\alpha$ be a related Hamilton form defined on an open subset of $J^1(J^{s-1}Y)$. Let δ be a (local) section of π_{s-1} passing through* dom α. *The following conditions are equivalent:*

(1) *δ is a Hamilton path of E.*
(2) *The Hamilton form $\mathcal{H}_\alpha$ vanishes along $J^1\delta$, i.e., $\mathcal{H}_\alpha \circ J^1\delta = 0$.*
(3) *For every π_{s-1}-vertical vector field ξ on $J^{s-1}Y$, $\delta^*\, i_\xi\, \alpha = 0$.*
(4) *For every π_{s-1}-projectable vector field ξ on $J^{s-1}Y$, $\delta^*\, i_\xi\, \alpha = 0$.*
(5) *For every vector field ξ on $J^{s-1}Y$, $\delta^*\, i_\xi\, \alpha = 0$.*
(6) *In every fiber chart, δ satisfies the system of ms first-order ODE,*

$$H^i_\sigma(\alpha) \circ J^1\delta = 0, \quad 0 \le i \le s-1,\, 1 \le \sigma \le m, \tag{4.5}$$

where $H^i_\sigma(\alpha)$ are given by (4.4).

In order to understand relations between a given mechanical system and an associated Hamiltonian system it is necessary to clarify relations between paths and Hamilton paths. Obviously, in general, paths are not in one-to-one correspondence with Hamilton paths, since among Hamilton paths there may appear sections which do not correspond to any solution of the original equations. However, apparently, a set of Hamilton paths contains all (prolonged) paths; more precisely, the following assertion holds.

4.3. Proposition. *Let E be a dynamical form of order s, α an associated Hamiltonian system defined on an open set $U \subset J^{s-1}Y$. Then the paths of E passing in U are in one-to-one correspondence with the holonomic Hamilton paths.*

In view of the above proposition we shall also say that equations for paths describe *proper dynamics* and generalized Hamilton equations describe *extended dynamics*. Accordingly, each (local) Hamiltonian system associated with E will be called a (local) *Hamiltonian extension* of E.

5. Geometric classification of Hamiltonian extensions of ODE

In the previous section we have associated with a mechanical system (represented by a dynamical form E of order s, or by a Lepage class $[\alpha]$ of order $s-1$) a *family* of different Hamiltonian systems, represented by local dynamical forms of order 1 with respect to the projection π_{s-1}. This means that now we have the possibility to study instead of the original dynamics (represented by local sections of π) certain *new dynamics* given by Hamilton paths which are local sections of π_{s-1}. According to the definition, generalized Hamilton equations represent a system of sm first-order ODE of the form

$$\mathrm{F}\dot{x} = b, \tag{5.1}$$

where x stands for the components of a section δ of π_{s-1} (i.e., $x_k^\nu = q_k^\nu \circ \delta$), and F is a matrix which generally *need not be regular*. If F is *regular* then these equations become equations for integral sections of a (nowhere zero smooth) *vector field*, hence are completely integrable (in the sense of the Frobenius Theorem) and their solutions are obtained if in a neighborhood of each point sm independent first integrals are found. Moreover, in this case, obviously, paths and Hamilton paths locally coincide, i.e., equations for paths and the corresponding equations for Hamilton paths are *equivalent*. Unfortunately, in general, the situation is much more complicated. If the matrix F is *not regular* one has to find integral *sections* of a *distribution of generally a nonconstant rank greater than* 1, *spanned by a system of (smooth)* 1*-forms*. In such an unpleasant case, however, a distribution cannot be spanned by a system of continuous vector fields, i.e., Hamilton equations can no more be interpreted as equations for integral sections of a vector field. Consequently, *no integrability theory is available* (neither Frobenius–Sussmann–Viflyantsev theorem can be applied to obtain integrability conditions, nor first integrals can be used for finding the solutions).

In order to be able to deal with the general situation, it is convenient to provide a geometric classification of higher order ordinary differential equations, based on geometric properties of the corresponding dynamical and characteristic distributions, as follows.

Consider a dynamical form E on J^sY, $s \geq 1$, and let α be its Hamiltonian extension defined on an open subset $W \subset J^{s-1}Y$. We denote by $\mathcal{D}_\alpha$ the *characteristic distribution*, i.e., the distribution on W annihilated by means of the one-forms

$$(5.2) \qquad i_\xi\,\alpha, \quad \text{where } \xi \text{ runs over the set of } \textit{all} \text{ vector fields on } W,$$

or equivalently, spanned by vector fields ζ such that

$$(5.3) \qquad i_\zeta\,\alpha = 0.$$

It is easy to see that the annihilator of $\mathcal{D}_\alpha$ takes the form

$$(5.4) \qquad \mathcal{D}^0_\alpha = \operatorname{span}\left\{\eta_0, \eta_\sigma, \eta^i_\sigma, 1 \leq i \leq s-1, 1 \leq \sigma \leq m\right\},$$

where

$$(5.5) \qquad \begin{aligned} \eta_0 &= \left(A_\sigma - \sum_{k=0}^{s-2} 2F^{0k}_{\sigma\nu}\, q^\nu_{k+1}\right) dq^\sigma - \sum_{i=1}^{s-1}\sum_{k=0}^{s-1} 2F^{ik}_{\sigma\nu}\, q^\nu_{k+1}\, dq^\sigma_i, \\ \eta_\sigma &= A_\sigma\, dt + \sum_{k=0}^{s-2} 2F^{0k}_{\sigma\nu}\, \omega^\nu_k + B_{\sigma\nu}\, dq^\nu_{s-1}, \\ \eta^i_\sigma &= \sum_{k=0}^{s-1} 2F^{ik}_{\sigma\nu}\, \omega^\nu_k, \quad 1 \leq i \leq s-1, 1 \leq \sigma \leq m. \end{aligned}$$

The rank of the distribution $\mathcal{D}_\alpha$ is generally nonconstant, and at each point $x \in J^{s-1}Y$,

$$(5.6) \qquad \operatorname{corank} \mathcal{D}_\alpha(x) = \operatorname{rank} \alpha(x).$$

We shall call the vector fields satisfying (5.3) *Hamilton vector fields* related to α. Notice that since we suppose α be smooth, *Hamilton vector fields need not be continuous*.

The characteristic distribution is a subdistribution of the dynamical distribution, $\mathcal{D}_\alpha \subset \Delta_\alpha$, since

$$(5.7) \qquad \Delta^0_\alpha = \operatorname{span}\left\{\eta_\sigma, \eta^i_\sigma, 1 \leq i \leq s-1, 1 \leq \sigma \leq m\right\}.$$

By Theorem 4.2 and Proposition 4.3 we can see the geometric meaning of the distributions $\mathcal{D}_\alpha$ and Δ_α:

5.1. Proposition.
(1) *A section δ of π_{s-1} is an integral section of Δ_α if and only if it is an integral section of $\mathcal{D}_\alpha$.*
(2) *Integral sections of each of the distributions Δ_α and $\mathcal{D}_\alpha$ coincide with Hamilton paths. Holonomic integral sections of Δ_α as well as of $\mathcal{D}_\alpha$ coincide with the $(s-1)$-th prolongations of paths.*

Roughly we can say that, though generally different, the distributions $\mathcal{D}_\alpha$ and Δ_α “intersect on Hamilton paths”. The following theorem shows that both the distributions $\mathcal{D}_\alpha$ and Δ_α are useful for a geometric classification of higher order ODE, and that especially the difference between them can be helpful in understanding “singular” systems.

To this end, consider the matrices

$$(5.8)\qquad \mathrm{B} = (B_{\sigma\nu}), \qquad (\mathrm{B}|\mathrm{A}) = (B_{\sigma\nu}, A_\sigma),$$

where the left (resp. right) index numbers rows (resp. columns), and

$$(5.9)\qquad \mathrm{F} = (2F^{ik}_{\sigma\nu}), \qquad (\mathrm{F}|\mathrm{A}) = \begin{pmatrix} 2F^{0k}_{\sigma\nu} & A_\sigma \\ 2F^{ik}_{\sigma\nu} & 0 \end{pmatrix}$$

where the left indices σ, j, $1 \leq \sigma \leq m, 0 \leq j \leq s-1$, number rows, and the right indices ν, k label columns, $1 \leq \nu \leq m, 0 \leq k \leq s-1$.

Now, we can prove a theorem fundamental for learning the structure of Hamilton paths.

5.2. Theorem. *Let α be a Hamiltonian system defined on an open set $W \subset J^{s-1}Y$, let $x \in W$ be a point. The following five conditions are equivalent:*

(1) *The characteristic distribution $\mathcal{D}_\alpha$ of α is weakly horizontal at x.*

(2) *The dynamical distribution Δ_α is weakly horizontal at x.*

(3) $\Delta_\alpha(x) = \mathcal{D}_\alpha(x)$.

(4) *At x, the equation $i_\zeta\, \alpha = 0$ has a nonvertical solution ζ_x.*

(5) $\operatorname{rank}\mathrm{F}(x) = \operatorname{rank}(\mathrm{F}|\mathrm{A})(x)$.

If the condition $\operatorname{rank}\mathrm{B}(x) = \operatorname{rank}(\mathrm{B}|\mathrm{A})(x)$ *holds then any of the above assertions is satisfied.*

Proof. (1) $\Rightarrow$ (2), since $\mathcal{D}_\alpha \subset \Delta_\alpha$. Next, (1) and (4) are obviously equivalent.

(2) $\Leftrightarrow$ (5): Let ξ be a (not necessarily continuous) vector field belonging to Δ_α, defined in a neighborhood U of x, and such that $T\pi_{s-1}.\xi \neq 0$ at x, let

$$\xi = \xi^0 \frac{\partial}{\partial t} + \sum_{j=0}^{s-1} \xi^\sigma_j \frac{\partial}{\partial q^\sigma_j}$$

be a fiber-chart expression of ξ. By assumption, ξ satisfies the equations

$$(5.10)\qquad \begin{aligned} &\left(A_\sigma - \sum_{k=0}^{s-2} 2F^{0k}_{\sigma\nu}\, q^\nu_{k+1}\right) \xi^0 + \sum_{k=0}^{s-2} 2F^{0k}_{\sigma\nu}\, \xi^\nu_k + B_{\sigma\nu}\, \xi^\nu_{s-1} = 0, \\ &-\sum_{k=0}^{s-2} 2F^{ik}_{\sigma\nu}\, q^\nu_{k+1}\, \xi^0 + \sum_{k=0}^{s-2} 2F^{ik}_{\sigma\nu}\, \xi^\nu_k = 0, \qquad 1 \leq \sigma \leq m, 1 \leq i \leq s-2, \\ &-B_{\sigma\nu} q^\nu_1 \xi^0 + B_{\sigma\nu} \xi^\nu = 0. \end{aligned}$$

Without loss of generality we can assume $\xi^0(x) = -1$. Hence, at x we obtain a system of ms linear non-homogeneous algebraic equations for ms unknowns $\xi^\sigma_j(x)$. Since the matrix of the system is equivalent to the matrix $(\mathrm{F}|\mathrm{A})(x)$, we get from the Frobenius theorem on the existence of solutions of algebraic equations that $\operatorname{rank}\mathrm{F} = \operatorname{rank}(\mathrm{F}|\mathrm{A})$ at x. Conversely, if (5) holds, then by Frobenius theorem at the point x there exists a solution $\xi(x)$ to (5.10) satisfying $\xi^0(x) = -1$. Hence, $\xi(x) \in \Delta_\alpha(x)$ proving (2).

(3) $\Leftrightarrow$ (5): Suppose (3). Then $\eta_0(x)$ is a linear combination of the 1-forms $\eta^i_\sigma(x), 0 \leq i \leq s-1, 1 \leq \sigma \leq m$. Since the matrix of the generators of $\mathcal{D}_\alpha$ is equivalent to the matrix

$$\begin{pmatrix} \mathrm{F} & \mathrm{A} \\ \mathrm{A} & 0 \end{pmatrix} \tag{5.11}$$

where the submatrix (F|A) is the matrix of generators of Δ_α, we get that the last row has to be a linear combination of the other rows (at x). Consequently, considering the condition $F^{ik}_{\sigma\nu} = -F^{ki}_{\nu\sigma}$ which holds for F, we get that (at x) the last column of the matrix (5.11) is a linear combination of its other columns. This means that at x

$$\operatorname{rank}\begin{pmatrix} \mathrm{F} & \mathrm{A} \\ \mathrm{A} & 0 \end{pmatrix} = \operatorname{rank}(\mathrm{F}|\mathrm{A}) = \operatorname{rank}\begin{pmatrix} \mathrm{F} \\ \mathrm{A} \end{pmatrix} = \operatorname{rank}\mathrm{F}.$$

Conversely, if the condition $\operatorname{rank}\mathrm{F}(x) = \operatorname{rank}(\mathrm{F}|\mathrm{A})(x)$ holds then, by the same arguments as above, $\mathcal{D}_\alpha(x)$ is spanned by the forms $\eta^i_\sigma(x)$, $0 \le i \le s-1$, $1 \le \sigma \le m$. Hence $\mathcal{D}_\alpha(x) = \Delta_\alpha(x)$.

It remains to show (2) $\Rightarrow$ (1), which, however, is now trivial.

Finally, looking at the matrix F we can see that $\operatorname{rank}\mathrm{B}(x) = \operatorname{rank}(\mathrm{B}|\mathrm{A})(x)$ means that $\operatorname{rank}\mathrm{F}(x) = \operatorname{rank}(\mathrm{F}|\mathrm{A})(x)$. $\square$

5.3. Remarks. (i) Theorem 5.2 provides a geometric characterization of the weak horizontality conditions of the characteristic and dynamical distribution. Equivalently, it provides us with an explicit algebraic condition (condition (5)) which can be easily applied in every concrete situation to *exclude the points of the phase space where the generalized Hamilton equations* a priori *have no solution*.

(ii) The distribution $\mathcal{D}_\alpha$ (as the characteristic distribution of α), compared with Δ_α, seems at a first sight to be a more natural object describing the Hamiltonian system associated with α. However, according to the above results, both the distributions $\mathcal{D}_\alpha$ and Δ_α are useful from the theoretical and practical point of view. Indeed, if the Hamiltonian system is weakly horizontal then (5.5) leads to consider superfluous (not independent) 1-forms. In such a situation, working with the distribution Δ_α is more simple. Also, there is a theoretical argument for taking into account both the distributions $\mathcal{D}_\alpha$ and Δ_α, namely the condition (3) of the above theorems, which leads to a geometric interpretation of the set of points of the phase space where Hamilton extremals a priori are not allowed to pass through as the set of points where the distributions $\mathcal{D}_\alpha$ and Δ_α are different.

To study proper dynamics, one has to find additional conditions excluding the non-holonomic Hamilton paths. Clearly, while Hamilton paths are connected with non-vertical vector fields, holonomic Hamilton paths are connected with semisprays. We have the following assertions which will help us to learn the *structure of holonomic Hamilton paths* (= prolonged paths).

5.4. Theorem. *Let α be a Hamiltonian system defined on an open subset $W \subset J^{s-1}Y$, $s > 1$, let Δ_α be its dynamical distribution. Let $x \in W$ be a point. The following conditions are equivalent:*

(1) *In a neighborhood of x there exists a (possibly non-continuous) semispray ξ such that $\xi(x) \in \Delta_\alpha(x)$.*

(2) $\operatorname{rank}\mathrm{B}(x) = \operatorname{rank}(\mathrm{B}|\mathrm{A})(x)$.

Proof. Let ξ be a semispray such that $\xi(x) \in \Delta_\alpha(x)$. Then $\xi(x)$ satisfies the equations (5.10), and since $\xi^0 = 1$, $\xi_i^\sigma = q_{i+1}^\sigma$, $1 \le \sigma \le m$, $0 \le i \le s-2$, the equations (5.10) simplify to

(5.12) $$(A_\sigma + B_{\sigma\nu}\xi_{s-1}^\nu)(x) = 0.$$

This system is solvable with respect to the $\xi_{s-1}^\nu(x)$'s, i.e., the condition (2) is satisfied.

Conversely, suppose $\operatorname{rank} \mathrm{B}(x) = \operatorname{rank}(\mathrm{B}|\mathrm{A})(x)$, and let β_x^ν, $1 \le \nu \le m$ be a solution of the equation $A_\sigma(x) + B_{\sigma\nu}(x)\beta_x^\nu = 0$. Putting

$$\xi = \frac{\partial}{\partial t} + \sum_{i=0}^{s-2} q_{i+1}^\sigma \frac{\partial}{\partial q_i^\sigma} + \xi_{s-1}^\sigma \frac{\partial}{\partial q_{s-1}^\sigma}$$

where $\xi_{s-1}^\sigma(x) = \beta_x^\sigma$, $1 \le \sigma \le m$, and $\xi_{s-1}^\sigma(y) = 0$ for all $y \ne x$ we get a semispray satisfying the condition (1). $\square$

Taking into account the equations (5.12) we get

5.5. Theorem. *Let α be a Hamiltonian system defined on an open subset $W \subset J^{s-1}Y$, $s > 1$, let Δ_α be its dynamical distribution. Let r be an integer, $1 \le r \le m$, let $x \in W$ be a point. The following two conditions are equivalent:*

(1) *In a neighborhood of x in U there exist semisprays $\xi_1, \dots, \xi_r$ (not necessarily continuous) possessing the following properties:*

(1_a) *$\xi_1, \dots, \xi_r$ are linearly independent at x,*

(1_b) *$\xi_j(x) \in \Delta_\alpha(x)$, $1 \le j \le r$,*

(1_c) *if ξ is another semispray in a neighborhood of x such that $\xi(x) \in \Delta_\alpha(x)$ then the vectors $\xi_1(x), \dots, \xi_r(x), \xi(x)$ are linearly dependent.*

(2) $\operatorname{rank} \mathrm{B}(x) = \operatorname{rank}(\mathrm{B}|\mathrm{A})(x) = m + 1 - r$.

Proof. Suppose (1) and denote for $i = 1, \dots, r$ by $\beta_i = (\beta_i^\sigma)$ the components of ξ_i at $\partial/\partial q_{s-1}^\sigma$, $1 \le \sigma \le m$, at the point x. The β_i's satisfy the equations $\mathrm{A}(x) + \mathrm{B}(x)\beta_i = 0$, which means that $\operatorname{rank} \mathrm{B}(x) = \operatorname{rank}(\mathrm{B}|\mathrm{A})(x)$, and it holds $\beta_2 = \beta_1 + \bar\beta_2, \dots, \beta_r = \beta_1 + \bar\beta_r$ where $\bar\beta_2, \dots, \bar\beta_r$ are solutions of the homogeneous equations $\mathrm{B}(x)\bar\beta = 0$. If $\sum_{l=2}^r a_l\bar\beta_l = 0$ then $\sum_{l=2}^r a_l(\beta_l - \beta_1) = -(\sum_l a_l)\beta_1 + \sum_l a_l\beta_l = 0$, i.e., by assumption, $a_2 = \dots = a_r = 0$, proving the linear independence of the $\bar\beta_i$'s. Let ξ be another semispray such that $\xi(x) \in \Delta(x)$. Then $\xi(x) = \sum_{j=1}^r b_j\xi_j(x)$ for some constants $b_1, \dots, b_r$ such that $b_1 + \dots + b_r = 1$. Hence $\beta = b_1\beta_1 + \sum_{l=2}^r b_l(\beta_1 + \bar\beta_l) = \beta_1 + \sum_{l=2}^r \bar\beta_l$, proving that $\bar\beta_2, \dots, \bar\beta_r$ form a basis of solutions of the system of the homogeneous equations with the matrix $\mathrm{B}(x)$. This means that $r - 1 = m - \operatorname{rank} \mathrm{B}(x)$.

Conversely, taking a basis $\bar\beta_2, \dots, \bar\beta_r$ of solutions of $\mathrm{B}(x)\bar\beta = 0$ and a particular solution β_1 of $\mathrm{A}(x) + \mathrm{B}(x)\beta = 0$, and putting $\beta_2 = \beta_1 + \bar\beta_2, \dots, \beta_r = \beta_1 + \bar\beta_r$ we get similarly as in the proof of the preceding theorem a system of semisprays $\xi_1, \dots, \xi_r$ satisfying (1_b). Now, if $\sum b_j\xi_j(x) = 0$ for some constants b_j, $1 \le j \le r$, then $\sum b_j = 0$ and $\sum b_j\beta_j = 0$, i.e., $(\sum b_j)\beta_1 + \sum_{l=2}^r b_l\bar\beta_l = \sum_{l=2}^r b_l\bar\beta_l = 0$, which means that $b_l = 0$ for $2 \le l \le r$, and consequently also $b_1 = 0$, proving (1_a). If ξ is another semispray such that $\xi(x) \in \Delta(x)$ then $\beta = \beta_1 + \sum b_l\bar\beta_l = \beta_1 + \sum b_l(\beta_l - \beta_1)$; now, $\xi(x) = (1 - \sum_{l=2}^r b_l)\xi_1(x) + \sum_{l=2}^r b_l\xi_l(x)$, proving (1_c). $\square$

We shall use the following terminology suggested by Theorems 5.2 and 5.4.

If α is a Hamiltonian system defined on an open set $W \subset J^{s-1}Y$, then

$$\tilde{\mathcal{P}} = \left\{x \in J^{s-1}Y \mid \operatorname{rank} \mathrm{F} = \operatorname{rank}(\mathrm{F}|\mathrm{A})\right\} \subset W \tag{5.13}$$

will be called *primary constraint set*, and the set $\mathcal{P} \subset \tilde{\mathcal{P}}$ defined by

$$\mathcal{P} = \left\{x \in J^{s-1}Y \mid \operatorname{rank} \mathrm{B} = \operatorname{rank}(\mathrm{B}|\mathrm{A})\right\} \tag{5.14}$$

will be called *primary semispray-constraint set* of α. Notice that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ need not be submanifolds of $W \subset J^{s-1}Y$. Clearly $\tilde{\mathcal{P}}$ has the meaning of a "set of admissible initial conditions for the Hamilton equations", while the meaning of $\mathcal{P}$ is "a set of admissible initial conditions" for the equations of paths of $E = p_1\alpha$ in W.

If $J^{s-1}Y - \tilde{\mathcal{P}} \neq \emptyset$ we say that the Hamiltonian system α carries *primary dynamical constraints*. Similarly if $J^{s-1}Y - \mathcal{P} \neq \emptyset$ we say that α carries *primary semispray constraints*.

A Hamiltonian system α will be called *semiregular* if Δ_α is weakly horizontal and of a locally constant rank. Apparently semiregular Hamiltonian systems carry no primary dynamical constraints. If, moreover, Δ_α is completely integrable then we have a regular foliation $\mathcal{F}_\alpha$ of W with $(\dim J^{s-1}Y - \operatorname{rank}\alpha)$-dimensional leaves; every Hamilton path is then an embedding of an open subset of the base X into a leaf of $\mathcal{F}_\alpha$ and, conversely, every section of π_{s-1} which ends in a leaf of $\mathcal{F}_\alpha$ is a Hamilton path. Every point in the phase space W is an initial point for a non-uniquely determined extended motion which develops within a leaf of $\mathcal{F}$ passing through the initial point.

Let $s \geq 2$. If α has no primary semispray constraints, and $\operatorname{rank} \mathrm{B}$ is locally constant we say that α is *weakly regular*. Notice that by Theorem 5.2, Δ_α is weakly horizontal and coincides with the characteristic distribution $\mathcal{D}_\alpha$ of α. By Theorem 5.5, Δ_α has a subdistribution $\tilde{\Delta}$ of rank $r = m + 1 - \operatorname{rank} \mathrm{B}$, which can be locally spanned by *semisprays*. Unfortunately, this subdistribution is *completely integrable if and only if* $\operatorname{rank} \mathrm{B} = m$. If this is the case, i.e., if $\det \mathrm{B} \neq 0$, then $\tilde{\Delta}$ is a rank 1 subdistribution of Δ_α spanned by the following semispray, respectively, annihilated by the following 1-forms,

$$\begin{gathered}\frac{\partial}{\partial t} + \sum_{i=0}^{s-2} q^\sigma_{i+1} \frac{\partial}{\partial q^\sigma_i} - B^{\sigma\rho} A_\rho \frac{\partial}{\partial q^\sigma_{s-1}}, \\ A_\sigma dt + B_{\sigma\nu}\, dq^\nu_{s-1}, \omega^\sigma, \dots, \omega^\sigma_{s-2}, \quad 1 \leq \sigma \leq m,\end{gathered} \tag{5.15}$$

where $(B^{\sigma\rho})$ is the inverse matrix to B. A Hamiltonian system α on $W \subset J^{s-1}Y$, $s \geq 2$, such that $\operatorname{rank} \tilde{\Delta} = 1$ will be called *regular*. In the case $s = 1$ we shall speak about a regular Hamiltonian system whenever it is semiregular and $\operatorname{rank} \Delta_\alpha = 1$. Summarizing the "dynamical picture" of a regular Hamiltonian system α on W we get that through every point in W there passes exactly one maximal Hamilton path of α, and the set of Hamilton paths coincides with the set of paths of the dynamical form $E = p_1\alpha$.

Notice that a weakly regular Hamiltonian system need not be semiregular, and conversely, a semiregular Hamiltonian system need not be weakly regular. If α is both semiregular and weakly regular, we call it *strongly semiregular*. The "dynamical picture" for completely integrable strongly semiregular systems shows that there is no subfoliation of $\mathcal{F}$ corresponding to the "semispray subdistribution", i.e., the proper motion itself cannot be characterized by leaves; however, choosing initial conditions, the proper motion proceeds within a leaf of $\mathcal{F}$ passing through the initial point (cannot leave the "enveloping" leaf).

6. Geometric classification of ODE

For a geometric classification of higher-order ODE we can use the above properties and classification of Hamiltonian extensions of ODE, as follows:

A dynamical form E (respectively, a mechanical system $[\alpha]$) will be called *regular* if in a neighborhood of every point $x \in J^{s-1}Y$ it has a regular Hamiltonian extension. A dynamical form (respectively, a mechanical system) which is not regular will be called *singular*.

Obviously, *regular dynamical forms are representable by semisprays*. More precisely, they are characterized as follows:

6.1. Theorem. *Let ms be even. Let $E \in \Lambda^2_{af}(J^sY)$ be a dynamical form, Consider the corresponding class $[\Delta_\alpha]$ of dynamical distributions. In a fiber chart (V, ψ) on Y, $\psi = (t, q^\sigma)$, denote $E = (A_\sigma + B_{\sigma\nu} q^\nu_s)\,\omega^\sigma \wedge dt$. The following conditions are equivalent:*

(1) *The mechanical system $[\alpha]$ is regular.*

(2) *The matrix $(B_{\sigma\nu})$ is everywhere regular.*

(3) *There is a unique dynamical distribution of rank 1 on V_{s-1} belonging to $[\Delta_\alpha]$, and it is of the form*

$$\tilde{\Delta} = \operatorname{span}\left\{\frac{\partial}{\partial t} + \sum_{i=0}^{s-2} q^\sigma_{i+1}\frac{\partial}{\partial q^\sigma_i} - B^{\sigma\rho}A_\rho \frac{\partial}{\partial q^\sigma_{s-1}}\right\}, \tag{6.1}$$

$$\tilde{\Delta}^0 = \operatorname{span}\left\{A_\sigma dt + B_{\sigma\nu} dq^\nu_{s-1}, \omega^\sigma, \dots, \omega^\sigma_{s-2}, 1 \le \sigma \le m\right\},$$

where $(B^{\sigma\nu})$ is the inverse matrix to $(B_{\sigma\nu})$.
The equations for paths of E have an equivalent form

$$q^\sigma_s = -B^{\sigma\rho}A_\rho, \quad 1 \le \sigma \le m. \tag{6.2}$$

6.2. Remark. If $s = 1$ then the formula (6.1) in the condition (3) of the above theorem takes the form

$$\tilde{\Delta} = \operatorname{span}\left\{\frac{\partial}{\partial t} - B^{\sigma\rho}A_\rho \frac{\partial}{\partial q^\sigma}\right\},$$

$$\tilde{\Delta}^0 = \operatorname{span}\left\{A_\sigma dt + B_{\sigma\nu} dq^\nu, 1 \le \sigma \le m\right\}.$$

Proof. We shall prove the theorem for $s \ge 2$; for $s = 1$ the proof is analogous and simple.

Suppose (1), and consider the related dynamical distributions defined on V_{s-1}. There is a distribution Δ_α of rank 1, i.e., such that the generators (3.7) are linearly independent at each point of V_{s-1}. This means that both the matrices $(B_{\sigma\nu})$ and $(F^{ij}_{\sigma\nu})$ where $1 \le i$, $j \le s-2$, are regular on V_{s-1}.

Let (2) hold. Then every dynamical codistribution of $[\alpha]$ on V_{s-1} is spanned by the 1-forms

$$(6.3)\qquad A_\sigma\, dt + \sum_{j=1}^{s-2} 2F^{0j}_{\sigma\nu}\,\omega^\nu_j + B_{\sigma\nu}\, dq^\nu_{s-1},\ \ \omega^\sigma,\ \ \sum_{j=1}^{s-2} F^{ij}_{\sigma\nu}\,\omega^\nu_j,\quad 1 \le i \le s-2.$$

Since we suppose ms even, the antisymmetric square matrix $(F^{ij}_{\sigma\nu})$ has an even number of rows (columns), i.e., it is possible to make a choice such that $\det(F^{ij}_{\sigma\nu}) \neq 0$. For the corresponding 2-form α we then get $\Delta_\alpha = \tilde{\Delta}$. We can see that under assumption that B is regular, the condition $\det(F^{ij}_{\sigma\nu}) \neq 0$ is necessary and sufficient for the corresponding dynamical distribution be of rank 1. This implies uniqueness of the arising rank 1 distribution.

If (3) is satisfied, then (1) follows trivially.

The assertion (4) follows immediately from (3). Finally, if (4) holds, then the matrix B is everywhere regular, and we are done. $\square$

Notice that for a regular dynamical form on J^sY such that sm is even, the distribution $\tilde{\Delta}$ coincides with the *characteristic distribution* of any of the equivalent 2-forms of the maximal rank belonging to the Lepage class $[\alpha]$ of E. This means that for $\Gamma \in \tilde{\Delta}$ and every $\alpha' \in [\alpha]$ such that $\operatorname{dom}\alpha = \operatorname{dom}\alpha'$ and $\operatorname{rank}\alpha' = \dim J^{s-1}Y - 1$,

$$(6.4)\qquad i_\Gamma \alpha' = 0.$$

Next, notice that for $s \ge 2$, $\tilde{\Delta}$ is a *semispray distribution* (i.e., the associated connection is a *semispray connection*).

Taking into account the generators of the form (6.3) we can see that if the number sm is odd then the matrix $(F^{ij}_{\sigma\nu})$ where $1 \le i, j \le s-2$, is *singular* (since it is an antisymmetric $(m(s-2) \times m(s-2))$-matrix). Consequently, the generators (6.3) are linearly dependent. Similarly, for $s = 1$ the corresponding generators (6.3) take the form

$$A_\sigma\, dt + B_{\sigma\nu}\, dq^\nu,$$

where the matrix B is antisymmetric (Theorem 3.2). Summarizing, we get that if E is a regular dynamical on J^sY and ms is odd, $s \ge 2$, then for every Hamiltonian extension α of E, $\tilde{\Delta} \neq \Delta_\alpha$.

More precisely,

6.3. Theorem. *Let ms be odd, $s > 2$. Let $E \in \Lambda^2_{af}(J^sY)$ be a dynamical form, and let $[\Delta_\alpha]$ be the corresponding class of dynamical distributions. In a fiber chart (V, ψ) on Y, $\psi = (t, q^\sigma)$, denote $E = (A_\sigma + B_{\sigma\nu}q^\nu_s)\,\omega^\sigma \wedge dt$. The following conditions are equivalent:*

(1) *The matrix $(B_{\sigma\nu})$ is everywhere regular.*

(2) *On V_{s-1}, every dynamical distribution in $[\Delta_\alpha]$ has a unique rank* 1 *semispray subdistribution,* (6.1).

(3) *The equations for paths of E have an equivalent form* (6.2).

A dynamical form for which the set $J^{s-1}Y - \mathcal{P}$ is nonempty will be called a mechanical system *with primary semispray constraints*. We can also say that systems with primary semispray constraints do not admit a “global” proper dynamics (in the sense that there are *a priori* restrictions on initial points of the proper motion).

A dynamical form will be called *weakly regular* if it carries no primary semispray constraints and rank B is locally constant on the phase space. We have seen that prolongations of paths of a weakly regular dynamical form can be interpreted as integral sections of a distribution of constant rank which is representable by *semisprays*: namely, it is locally spanned by $m + 1 - \operatorname{rank} \mathrm{B}$ semisprays, and is *not completely integrable* (if $\operatorname{rank} \mathrm{B} \neq m$). Prolonged extremals coincide with integral sections of this distribution. Consequently, the dynamical picture for weakly regular systems is the following. Every point in the phase space is an initial point for a non-uniquely determined proper motion (it is "indeterministic" in the sense that the motion cannot be uniquely determined by initial conditions). Since the "semispray distribution" does not give rise to a foliation, the proper motion starting from a fixed initial point cannot be represented as proceeding within a leaf. Consequently, the dynamical picture cannot be obtained by standard techniques, and one must apply the *constraint algorithm* described below.

7. Structure of solutions of ODE

Within the standard theory of distributions of a constant rank, the integration problem has two steps:

(1) to clarify the *structure of solutions* of the distribution in question (namely, if it is completely integrable then the maximal integral manifolds form a *foliation* of the given manifold),

(2) to *find the maximal integral manifolds*, which, in case of involutive distributions, practically means to find adapted charts to the foliation (= complete sets of independent first integrals).

For involutive distributions, the first problem is completely solved by the Frobenius theorem; the second problem is rather complicated and means to apply some of the known integration methods based on symmetries of the distribution and relations with first integrals. If the distribution happens not to be involutive then no general theory, clarifying the structure of solutions and providing methods for finding them explicitly, is known.

We have shown that every system of (generally higher order) ordinary differential equations can be locally interpreted as a system of equations for *holonomic integral sections of a dynamical distribution*. Consequently, the problem of integration of such equations identifies with the problem of *integration of a distribution*. However, it is highly nontrivial, since only in the case of *regular* mechanical systems it is reduced to the known case of finding *maximal* integral manifolds of an *involutive* distribution of a *constant* rank.

To be able to deal with all mechanical systems, we have to clarify aspects of the integration problem in a general situation. Notice that if the order of the equations is ≥ 2, one can distinguish (at least locally) *two levels* of the integration problem:

(1) finding extended dynamics, i.e., Hamilton paths of an associated Hamiltonian system,

(2) finding proper dynamics, i.e., prolonged paths.

The first level of the problem precisely means to find *integral sections* of a distribution. On the other hand, the second level means to pick up only those solutions which are *holonomic*. Obviously, every path defines a one-dimensional (immersed) submanifold

M of the phase space and a *semispray ξ along M* such that for every $y \in M$, $\xi(y)$ belongs to a dynamical distribution Δ_α at y. Conversely, every vector field ξ (even not continuous!) belonging to a dynamical distribution Δ_α, i.e., satisfying the Hamilton equation $i_\xi \alpha = 0$, and such that ξ is a semispray along an at least one-dimensional immersed submanifold M of the phase space, defines a solution of the equations for paths. It should be stressed that ξ need not be a semispray *everywhere* on the phase space. This means that for finding proper dynamics it is generally *not sufficient* to find the subdistribution of a dynamical distribution, spanned by semisprays. We shall call the problem of distinguishing *holonomic* solutions of a distribution the *(higher-order) semispray problem*.

Let us turn to describe a general procedure which enables one to find explicitly *integral sections* of *non-integrable* dynamical distributions of generally *non-constant rank*. We shall call this procedure the *constraint algorithm*. As mentioned above, this algorithm has two levels—including and non-including the higher-order semispray problem.

7.1. Extended dynamics. First, we shall describe an algorithm for finding explicitly the dynamics of a Hamiltonian system α, associated with a given $(s-1)$-th order mechanical system. In general it is not possible to characterize the dynamics by a system of continuous vector fields. On the other hand, as mentioned above, Hamilton paths define one-dimensional (immersed) submanifolds of the phase space such that the vector fields along these submanifolds belong to dynamical distributions. Clearly, such submanifolds can have nonempty intersections. Hence, the problem to be solved is to find *at each point x* of the phase space the bunch of submanifolds where an extended motion "is allowed to develop", i.e., such that every section of each of these submanifolds is a Hamilton path, and conversely, every Hamilton path passing through x is locally embedded in a manifold of this bunch.

Step 1. Find the primary constraint set set $\tilde{\mathcal{P}}$ for α (recall that $\tilde{\mathcal{P}}$ need not be a submanifold of the phase space). If $\tilde{\mathcal{P}} = \emptyset$, there is no extended dynamics, hence no dynamics at all. If $\tilde{\mathcal{P}} \neq \emptyset$, choose a point $x \in \tilde{\mathcal{P}}$, and proceed to the next step.

Step 2. Denote $S_{(1)} = \tilde{\mathcal{P}}$, and $M_{(1)} = \cup_\iota M_{(1)\iota}$ the union of all connected maximal submanifolds $M_{(1)\iota} \subset U$, where U is a neighborhood of x, lying in $S_{(1)}$ and passing through x (here "maximal" means that if N is a connected submanifold passing through x, lying in $S_{(1)} \cap U$, and $M_{(1)\iota} \subset N$ then $M_{(1)\iota} = N$).

Suppose that $M_{(1)} \neq \{x\}$. For each ι consider the restriction of the dynamical distribution Δ_α to $M_{(1)\iota}$, i.e., the distribution $\Delta_{(1)\iota} = \Delta_\alpha \cap TM_{(1)\iota}$ (called the *constrained to $M_{(1)\iota}$* dynamical distribution). If $\Delta_{(1)\iota}$ is weakly horizontal at each point of $M_{(1)\iota}$, we call the manifold $M_{(1)\iota}$ *a final constraint submanifold at x*; the problem now is reduced to the problem of integration of the distribution $\Delta_{(1)\iota}$. Otherwise, one of the following two possibilities occurs: (i) $\Delta_{(1)\iota}$ is trivial at x, or is not weakly horizontal at x; then exclude the manifold $M_{(1)\iota}$ from the bunch $M_{(1)}$. (ii) $\Delta_{(1)\iota}$ is weakly horizontal at x but is not weakly horizontal on $M_{(1)\iota}$; then proceed to the next step.

Step 3. Exclude the points where $\Delta_{(1)\iota}$ is not weakly horizontal from $M_{(1)\iota}$, and denote the resulting set by $S_{(2)\iota}$; clearly, $x \in S_{(2)\iota}$. Repeat the procedure described in Step 2 with $S_{(2)} = \cap_\iota S_{(2)\iota}$ instead of $S_{(1)}$.

After sufficiently many steps we obtain either a bunch of final constraint submanifolds at x, or we find that there is no final constraint submanifold passing through x. If there exists a point such that there is no final constraint submanifold passing through x, we say that the system possesses *secondary dynamical constraints*.

Collecting the points where there exists a bunch of final constraint submanifolds we obtain a *subset* of the phase space where the extended motion proceeds, we call it the set of *admissible initial conditions for the Hamilton equations*. The structure of this set can be complicated; in particular, it need not be a submanifold of the phase space. Considering then the collection of final constraint submanifolds together with to them constrained dynamical distributions, we get the dynamical picture corresponding to the solutions of the Hamilton equations. Typically, this dynamical picture will be rather complicated, measuring the "rate of singularity" of a given mechanical system.

7.2. Proper dynamics: the higher-order semispray problem. Now, we shall be interested in an explicit description of the proper dynamics of the mechanical system. The procedure is as follows:

Find the set $\mathcal{P}$ (recall that $\mathcal{P}$ need not be a submanifold of the phase space). If $\mathcal{P} = \emptyset$, there is no motion. If $\mathcal{P} \neq \emptyset$, choose a point $x \in \mathcal{P}$. Denote $R_{(1)} = \mathcal{P}$. Proceed in the same way as described in 7.1, replacing $S_{(1)}$ by $R_{(1)}$, etc.

As a result of the procedure one gets at each admissible initial point $x \in \mathcal{P}$ (i) the bunch M_x of final constraint submanifolds, and (ii) a system $\mathcal{V}_x$ of vector fields along the manifolds of the bunch M_x which belong to the dynamical distribution Δ_α (at the points of M_x); by construction, all these vector fields are nonvertical.

Let $N \in M_x$, and let ξ be the vector field in $\mathcal{V}_x$ tangent to N. We shall say that ξ can be *identified with a semispray* along a submanifold $N_0 \subset N$ if there exists a submanifold N_0 of N, a neighborhood U of N_0 and a semispray ζ on U such that at each point $y \in N_0$, $\xi(y) = \zeta(y)$. Let $x \in \mathcal{P}$ be a point. We shall call the point x *admissible* if there exists a neighborhood U_x of x and a vector field $\xi \in \mathcal{V}_x$ such that ξ can be identified with a semispray along a submanifold of an element of the bunch M_x. If every point in $\mathcal{P}$ is admissible we say that the mechanical system possesses *no secondary semispray constraints*. Otherwise, we shall say that the mechanical system possesses *secondary semispray constraints*.

Collecting the admissible points of $\mathcal{P}$ we obtain a *subset* of the phase space where the proper motion proceeds; we call this set the *set of admissible initial conditions for the equations for paths*. As expected, this set need not be a submanifold of the phase space. Now, consider the family $\{\mathcal{V}_x\}$ where x runs over the set of admissible points. To get the dynamical picture corresponding to the solutions of the equations for paths, it is sufficient to pick up those elements of $\{\mathcal{V}_x\}$ which (in the above sense) can be identified with semisprays along at least one-dimensional submanifolds of elements of the $\{M_x\}$.

7.3. Remarks. The algorithm described above (proposed in [32]) can be viewed as a generalization and re-interpretation of the *constraint algorithm* developed within the range of Dirac's theory of constrained systems [16], [17], and later generalized to time-dependent and higher-order Lagrangians [9], [40]. In this context, the (higher-order) semispray problem (which in our understanding means a problem of finding *holonomic sections of a distribution of a non-constant rank which cannot be spanned by continuous vector fields*) is parallel to the problem of distinguishing solutions of the Euler–Lagrange equations in the set of solutions of the Hamilton equations in presymplectic mechanics, called the *SODE* and *HODE problem* (= "second-order" and "higher-order differential equation problem"), respectively (cf. [4], [5], [6], [16], [17], [40], and others). Similarly, our concepts of primary and secondary dynamical constraints, resp. of primary and secondary semispray constraints are based on the geometric understanding of the dynamics by means of dynamical distributions (with not a direct correspondence with the more-or-less heuristic concepts of "primary" and "secondary constraints" of Dirac's theory of constrained systems [12]).

We can see that while solutions of regular systems of ODE define a one-dimensional foliation of the phase space (i.e., each point of the phase space represents admissible initial conditions for a uniquely determined motion), solutions of systems of ODE which are *not regular* are *usually* (but not always!) restricted to a certain *subset* (not generally a submanifold) of the phase space. Therefore non-regular systems are often called *constrained systems*. Since the "constraints" are a property of the system itself (they have nothing to do with the underlying structure) they should be better referred to as *internal constraints*, to distinguish them from *external constraints* which are put on the underlying fibered manifold (e.g., holonomic and nonholonomic constraints of classical mechanics); the latter case is studied in Sections 10, 11.

8. Lagrangian systems

Now, we shall discuss the particular but important case of mechanical systems defined by dynamical forms which arise as Euler–Lagrange forms in the calculus of variations.

Recall that a local *Lagrangian* of order r for a fibered manifold $\pi : Y \to X$ is a π_r-horizontal 1-form defined on an open subset of J^rY. A dynamical form $E \in \Lambda^2_{\mathrm{af}}(J^sY)$ is called *(globally) variational* if there exists an integer $r \geq 1$ and a Lagrangian λ on J^rY such that E coincides (possibly up to a projection) with the Euler–Lagrange form of λ. E is called *locally variational* if J^sY can be covered by open sets U_ι in such a way that, for every ι, $E|_{U_\iota}$ is variational. Note that a locally variational form need not be globally variational.

Paths of a locally variational form are called *extremals*, and the equations for extremals are called *the Euler–Lagrange equations*.

The Lepage class (of order $s - 1$) associated to a locally variational form will be called a *Lagrangian system (of order $s - 1$)*. By the following theorem (cf. e.g. [35]) every Lagrangian system can be *uniquely* represented by a *global closed* two-form.

8.1. Theorem. *Let $E \in \Lambda^2_{af}(J^sY)$ be locally variational. Then the Lepage class $[\alpha]$ of E contains a unique closed 2-form, defined on $J^{s-1}Y$.*

Conversely, if in a neighborhood of every point $x \in J^{s-1}Y$, the Lepage class of E contains a closed 2-form α then α is unique, is defined on $J^{s-1}Y$, and the form E is locally variational.

The above theorem directly follows from the following Lemmas (for complete proofs we refer to [36]).

8.2 Lemma [1], [24], [60]. *A dynamical form E on J^sY, $s \geq 1$, is locally variational if and only if in each fiber chart the components E_σ, $1 \leq \sigma \leq m$, of E satisfy the following identities:*

$$\frac{\partial E_\sigma}{\partial q_l^\nu} - (-1)^l \frac{\partial E_\nu}{\partial q_l^\sigma} - \sum_{k=l+1}^{s} (-1)^k \binom{k}{l} \frac{d^{k-l}}{dt^{k-l}} \frac{\partial E_\nu}{\partial q_k^\sigma} = 0, \tag{8.1}$$

for all $0 \leq l \leq s$ *and* $1 \leq \sigma,\ \nu \leq m$.

8.3. Lemma [21], [30]. *Let α be a two-form on $J^{s-1}Y$, $s \geq 1$. The following two conditions are equivalent:*

(1) *$p_1\alpha$ is a dynamical form and $d\alpha = 0$,*

(2) *$p_1\alpha$ is locally variational, and in each fiber chart,*

$$\pi_{s,s-1}^* \alpha = E_\sigma\, \omega^\sigma \wedge dt + \sum_{j,k=0}^{s-1} F_{\sigma\nu}^{jk}\, \omega_j^\sigma \wedge \omega_k^\nu, \qquad F_{\sigma\nu}^{jk} = -F_{\nu\sigma}^{kj}, \tag{8.2}$$

where

$$\begin{aligned} F_{\sigma\nu}^{jk} &= \frac{1}{2} \sum_{l=0}^{s-j-k-1} (-1)^{j+l} \binom{j+l}{l} \frac{d^l}{dt^l} \frac{\partial E_\sigma}{\partial q_{j+k+l+1}^\nu}, \quad 0 \leq j+k \leq s-1, \\ F_{\sigma\nu}^{jk} &= 0, \quad s \leq j+k \leq 2s-2. \end{aligned} \tag{8.3}$$

8.4. Lemma [30]. *Let E be a locally variational form on J^sY, $s \geq 1$. Then there exists a unique closed two-form α such that $p_1\alpha = E$; this two-form is projectable onto $J^{s-1}Y$.*

If E is a locally variational form then the closed two-form belonging to the Lepage class of E will be called a *Lepagean equivalent* of E, and denoted by α_E.

8.5. Remark. The identities (8.1) represent necessary and sufficient conditions for a system of higher-order ordinary differential equations to come from a Lagrangian as a system of its Euler–Lagrange equations. This problem, known as the *local inverse problem of the calculus of variations*, was first studied by H. Helmholtz [18] for second order ordinary differential equations. The solution for higher-order ODE is due to A.L. Vanderbauwhede [60]; the general case (higher-order PDE) has been solved independently by D. Krupka [24], and I. Anderson and T. Duchamp [1]. In what follows we shall call (8.1) *Anderson–Duchamp–Krupka conditions*.

If the conditions (8.1) are satisfied then the E_σ are the Euler–Lagrange expressions of the Lagrange function

$$L = q^\sigma \int_0^1 E_\sigma(t, uq^\nu, \dots, uq_s^\nu)\, du, \tag{8.4}$$

which can be constructed in a neighborhood of every point in J^sY. This formula has been discovered in 1913 by Volterra for the second order case, and later subsequently generalized by M. M. Vainberg [57] and E. Tonti [56]. It is called *Vainberg–Tonti Lagrangian*. Notice that the function L is of the same order as the E_σ's.

Taking into account Lemma 8.3 we easily obtain

8.6. Corollary. *Every locally variational form E on J^sY, $s \geq 1$, (which is not projectable onto J^kY for some $k < s$) is affine in the derivatives of order s, i.e., in every fiber chart,*

$$E_\sigma = A_\sigma + B_{\sigma\nu} q_s^\nu,$$

where $A_\sigma = A_\sigma(t, q^\rho, \ldots, q_{s-1}^\rho)$, $B_{\sigma\nu} = B_{\sigma\nu}(t, q^\rho, \ldots, q_{s-1}^\rho)$. The matrix $(B_{\sigma\nu})$ is symmetric if s is even and antisymmetric if s is odd. Moreover, for the Lepagean equivalent α_E of E (8.2) one has

$$F_{\sigma\nu}^{0,s-1} = B_{\sigma\nu},\ F_{\sigma\nu}^{1,s-2} = -B_{\sigma\nu},\ F_{\sigma\nu}^{2,s-3} = B_{\sigma\nu}, \ldots, F_{\sigma\nu}^{s-1,0} = (-1)^{s-1} B_{\sigma\nu}.$$

Denote by $\mathfrak{Lep}$ the mapping, assigning to a locally variational form its Lepagean equivalent.

8.7. Corollary [36]. *The mapping $\mathfrak{Lep}$ of the set of locally variational forms to the set of closed Lepagean two-forms is bijective and inverse to the mapping p_1.*

8.8. Remark. Since the form α_E is closed, Poincaré Lemma gives us that in a neighborhood of every point there is a (non-unique) 1-form theta such that $\alpha_E = d\theta$. Such 1-forms are called *Cartan forms*, or *Lepagean* 1*-forms* related with E. Notice that the 1-form $\lambda = h\theta$ is a Lagrangian for E, and that θ is uniquely determined by λ (therefore it is denoted by θ_λ). If λ is of order r then θ_λ generally is of order $2r - 1$. In fibered coordinates where $\lambda = L\,dt$ we have

$$\theta_\lambda = L\,dt + \sum_{i=0}^{r-1} \left(\sum_{k=0}^{r-i-1} (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial q_{i+k+1}^\sigma} \right) \omega_i^\sigma.$$

Since E is uniquely determined by λ, we denote $E = E_\lambda$, and call E_λ the *Euler–Lagrange form* of λ. Recall that two Lagrangians, λ_1, λ_2, with the same domain of definition are called *equivalent* if $E_{\lambda_1} = E_{\lambda_2}$. For more details on geometric foundations of the calculus of variations on fibered manifolds, and to the general theory of Lepagean equivalents of Lagrangians we refer to the work of Krupka [22–26].

All important specific features and properties (both mathematical and physical) of Lagrangian systems come from the fact that the form α_E is *closed*. This leads to many fundamental results; let us mention very briefly at least some of them below (for a comprehensive exposition we refer the reader to [36]).

By Theorem 8.1 to every locally variational dynamical form E one has a *distinguished global Hamiltonian extension* α_E which is *completely determined* by the form E itself. Accordingly, the corresponding (global) dynamical distribution is uniquely determined by E. It is denoted by Δ_E and called the *Euler–Lagrange distribution* of the locally variational form E. Therefore, working with *Lagrangian systems* namely the form α_E and the corresponding characteristic distribution $\mathcal{D}_E$ and dynamical distribution Δ_E are used to study extremals and Hamilton extremals. Naturally, all structure and classification properties of mechanical systems introduced and studied in the previous sections apply to the particular case of Lagrangian systems.

Moreover, taking into account Corollary 8.6 and Theorems 6.1 and 6.3 we can see that *if ms is odd then no Lagrangian system is regular*. Consequently, we have

8.9. Theorem. *For Lagrangian systems the conditions*
(1) E *is regular*
(2) *the Euler–Lagrange distribution* Δ_E *has rank one*
are equivalent.

Accordingly, one has the following *geometric definition of a regular Lagrangian*. A Lagrangian λ of order $r \geq 1$ is called *regular* if the associated Euler–Lagrange distribution has rank one [30]. Expressing this condition in fibered coordinates we get

$$\det\left(\frac{\partial E_\sigma}{\partial q_s^\nu}\right) \neq 0,$$

where $E = E_\sigma \omega^\sigma \wedge dt$ is the dynamical form corresponding to the Lagrangian system defined by the Lagrangian λ (notice that $s \leq 2r$ is the "true order" of E, i.e., the lowest order where E_λ is projectable; equivalently, $s-1$ is the *order of the Lagrangian system* defined by the Lagrangian λ).

8.10. Remark. It should be stressed that compared with the standard definition of a regular Lagrangian of order r, namely

$$\det\left(\frac{\partial^2 L}{\partial q_r^\sigma \partial q_r^\nu}\right) \neq 0,$$

(cf. e.g. [3], [12], [17], [43], [54] and many others), our concept of regularity *redefines* regularity to be a *property of the class of equivalent Lagrangians*, and covers many Lagrangians usually considered singular (and processed as singular, e.g., within Dirac's theory of constrained systems). If explicitly rewritten for particular Lagrangians the new definition gives many concrete *regularity conditions* for Lagrangians. For example, a *first-order* Lagrangian $L(t, q^\rho, q_1^\rho)$ is regular if and only if one of the following two conditions is satisfied:

$$\text{(i)} \quad \det\left(\frac{\partial^2 L}{\partial q_1^\sigma \partial q_1^\nu}\right) \neq 0,$$

$$\text{(ii)} \quad \frac{\partial^2 L}{\partial q_1^\sigma \partial q_1^\nu} = 0 \quad \forall \sigma, \nu, \quad \text{and} \quad \det\left(\frac{\partial^2 L}{\partial q^\sigma \partial q_1^\nu} - \frac{\partial^2 L}{\partial q_1^\sigma \partial q^\nu}\right) \neq 0.$$

Apparently, the first condition describes all regular first-order Lagrangians the Euler–Lagrange equations of which are nontrivially of *second* order, i.e., define regular *first-order Lagrangean systems*, while the second condition refers to all regular first-order Lagrangians which are affine in the velocities, i.e., of the form $L = f(t, q^\rho) + g_\nu(t, q^\rho) q_1^\nu$; their Euler–Lagrange equations are first-order ODE, hence these Lagrangians define *regular zero-order Lagrangean systems*.

For more details and other examples we refer to [36].

Also *semiregular* Lagrangian systems turn to be much more simple than general mechanical systems. Due to Cartan Theorem for *closed* 2*-forms* of constant rank, the corank of the characteristic distribution $\mathcal{D}_E$ is a constant equal to an *even* number, and the distribution $\mathcal{D}_E$ is *completely integrable*. Hence, the same holds for the Euler–Lagrange distribution.

8.11. Theorem. *Let α_E be a semiregular Lagrangian system. Then the Euler–Lagrange distribution Δ_E is completely integrable and corank of Δ_E is even.*

By the above theorem it is clear that for semiregular Lagrangian systems one can effectively apply *integration methods based on relations between symmetries of closed 2-forms and complete sets of independent first integrals of their characteristic distributions* (Liouville or Hamilton–Jacobi integration methods). For more details on these integration methods, their applications and different generalizations we refer e.g. to [20], [31], [33], [34], [36], [39], [62-64]. In the regular case these integration methods give a complete solution of the Euler–Lagrange equations. For non-regular semiregular Lagrangian systems they provide us with a complete solution of the *Hamilton equations*. Extremals are then obtained by an additional application of the constraint algorithm (Sec. 7).

Finally, the property $d\alpha_E = 0$ leads to a local representation of Lagrangian systems in a certain "normal form" as follows. As we shall see below, this is closely related to the possibility of lowering the order of the corresponding Lagrangians.

Let us denote

$$s = 2c \qquad \text{if } s \text{ is even,}$$

$$s = 2c+1 \qquad \text{if } s \text{ is odd.}$$

8.12. Theorem (Canonical form of a closed Lepagean two-form) [30]**.** *Let E be a locally variational form of order s, $s \geq 1$, let α_E be its Lepagean equivalent. Then there is an open covering $\mathcal{O}$ of $J^{s-1}Y$ such that*

(1) *for each $W \in \mathcal{O}$ there is a fiber chart (V, ψ) on Y such that $W \subset V_{s-1}$,*

(2) *on each $W \in \mathcal{O}$ there are defined functions H, p^k_ν, $1 \leq \nu \leq m$, $0 \leq k \leq s-c-1$, such that the restriction of α_E to W is expressed in the form*

$$\alpha_E = -dH \wedge dt + \sum_{k=0}^{s-c-1} dp^k_\nu \wedge dq^\nu_k. \tag{8.5}$$

Proof. Since the form α_E is closed, there exists a covering $\mathcal{O}$ satisfying (1) such that on each $W \in \mathcal{O}$ it holds $\alpha_E = d\rho$ for a one-form ρ defined on W. Using Lemma 8.3 and the Poincaré Lemma we obtain (up to a projection)

$$\rho = \left(q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\, du \right) dt + \sum_{k=0}^{s-1} \left(\sum_{j=0}^{s-1} 2q^\sigma_j \int_0^1 (F^{jk}_{\sigma\nu} \circ \chi_{s-1})\, u\, du \right) \omega^\nu_k. \tag{8.6}$$

We shall show that there are functions f, H, p^k_ν, $1 \leq \nu \leq m$, $0 \leq k \leq s-c-1$, on W such that (8.6) can be equivalently expressed in the form

$$\rho = -H\, dt + \sum_{k=0}^{s-c-1} p^k_\nu\, dq^\nu_k + df. \tag{8.7}$$

Consider the mapping $\chi_{s-1,s-c} : [0,1] \times W \to W$,

$$\chi_{s-1,s-c}\big(v, (t, q^\sigma, \dots, q^\sigma_{s-1})\big) = (t, q^\sigma, \dots, q^\sigma_{s-c-1}, vq^\sigma_{s-c}, \dots, vq^\sigma_{s-1}). \tag{8.8}$$

Put

$$f = \sum_{k=s-c}^{s-1} \sum_{j=0}^{s-1-k} 2q_k^\nu q_j^\sigma \int_0^1 \left(\int_0^1 (F_{\sigma\nu}^{jk} \circ \chi_{s-1})\, u\, du \right) \circ \chi_{s-1,s-c}\, dv + \phi(t, q^\rho, \dots, q_{s-c-1}^\rho), \tag{8.9}$$

where ϕ is an arbitrary but fixed function, and define

$$p_\nu^k = \sum_{j=0}^{s-k-1} 2q_j^\sigma \int_0^1 (F_{\sigma\nu}^{jk} \circ \chi_{s-1})\, u\, du - \frac{\partial f}{\partial q_k^\nu}, \tag{8.10}$$

$1 \le \nu \le m, 0 \le k \le s-c-1,$

$$H = -q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\, du + \sum_{k=0}^{s-1} \sum_{j=0}^{s-k-1} 2q_j^\sigma q_{k+1}^\nu \times \int_0^1 (F_{\sigma\nu}^{jk} \circ \chi_{s-1})\, u\, du + \frac{\partial f}{\partial t}. \tag{8.11}$$

Using the identity

$$\frac{\partial f}{\partial q_k^\nu} = \sum_{j=0}^{s-k-1} 2q_j^\sigma \int_0^1 \left(F_{\sigma\nu}^{jk} \circ \chi_{s-1}\right) u\, du, \quad s-c \le k \le s-1, \tag{8.12}$$

we get

$$H = -q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\, du + \sum_{k=0}^{s-c-1} p_\nu^k q_{k+1}^\nu + \frac{df}{dt}. \tag{8.13}$$

Substituting into (8.7) we obtain (8.6). This completes the proof. □

For any fixed function $\phi(t, q^\rho, \dots, q_{s-c-1}^\rho)$, the functions H and p_ν^k, $1 \le \nu \le m$, $0 \le k \le s-c-1$, defined by (8.11) and (8.10), are called the *Hamiltonian* and *momenta* of the locally variational form E. The function ϕ itself is called a *gauge function*. Notice that Hamiltonian and momenta are functions related *directly* to a given dynamical form E, i.e., they refer to the *whole class of equivalent Lagrangians*. We stress that different choices of ϕ's in (8.9) lead to different sets of Hamiltonian and momenta for a given Lagrangian system, but all of them are of order $s-1$.

Let λ_1 and λ_2 be two equivalent Lagrangians, λ_1 of order k and λ_2 of order r. If $k > r$, we shall say that the order of the Lagrangian λ_1 *can be reduced* to r. Lagrangians of the lowest possible order are called *minimal-order Lagrangians*.

8.13. Corollary (Order-reduction) [60], [30]. *Every Lagrangian can be locally reduced to the lowest possible order, i.e., to the order $s/2$ if the order s of the Euler–Lagrange equations is even, and to $(s+1)/2$ if the order s of the Euler–Lagrange equations is odd.*

Proof. The result is obtained by putting

$$\lambda_{\min} = \lambda - h\,df \tag{8.14}$$

where λ is the Vainberg–Tonti Lagrangian of E, and f is defined by (8.9). It is an easy exercise to show that this Lagrangian is of order $c = s/2$ (respectively, $c+1 = (s+1)/2$) if s is even (respectively, odd). □

The formulas (8.11) and (8.10) can be interpreted in a way which relates them to (local) *minimal-order Lagrangians*. Then different sets of a Hamiltonian and momenta correspond to different *equivalent minimal-order* Lagrangians. Namely, it holds

$$\begin{aligned} p^k_\nu &= \sum_{j=0}^{s-c-k-1} (-1)^j \frac{d^j}{dt^j} \frac{\partial L_{\min}}{\partial q^\nu_{k+1+j}}, \qquad 1 \le \nu \le m, 0 \le k \le s-c-1, \\ H &= -L_{\min} + \sum_{k=0}^{s-c-1} p^k_\nu\, q^\nu_{k+1}, \end{aligned} \tag{8.15}$$

where $\lambda_{\min} = L_{\min} dt$ is a minimal-order Lagrangian for α_E.

8.14. Remark. Let us discuss an important particular family of Lagrangian systems which are frequently considered in higher-order mechanics, namely, the *time-independent Lagrangian systems*. To this end, consider a fibered manifold $\pi : R \times M \to R$. We say that a Lagrangian system α_E is *autonomous*, or *time-independent* if in a neighborhood of every point it possesses local *time-independent minimal-order Lagrangians* (i.e., such that $\partial L_{\min}/\partial t = 0$, where t is the global coordinate on R). It is easy to see that Hamiltonians corresponding to different equivalent time-independent minimal-order Lagrangians differ only by a constant. Consequently, *a time-independent Lagrangean system on $\pi : R \times M \to R$ possesses a unique up to a constant time-independent Hamiltonian*. In fibered coordinates we have this Hamiltonian represented in the form

$$\begin{aligned} H = {} & -q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\, du \\ & + \sum_{k=0}^{s-1} \sum_{j=0}^{s-k-1} 2q^\sigma_j\, q^\nu_{k+1} \int_0^1 \left(F^{jk}_{\sigma\nu} \circ \chi_{s-1}\right) u\, du + c. \end{aligned}$$

It is natural to call this Hamiltonian the *total energy* of the time-independent Lagrangian system α_E.

While time-independent Lagrangian systems on $R \times T^{s-1}M$ are characterized by a unique up to a constant Hamiltonian, *time-dependent* Lagrangian systems (even on $\pi : R \times M \to R$), possess *no* distinguished Hamiltonians.

9. Examples of dynamics of highly singular systems

In this section we denote $\omega^i = dq^i - \dot{q}^i\, dt$. The examples are taken from [32], [36].

9.1. Consider the Lagrangian

$$L = \tfrac{1}{2}\big((\dot q^1)^2 + (q^1)^2 q^2\big) \tag{9.1}$$

on $R \times R^2 \times R^2$ [3]. This Lagrangian defines a *singular first-order Lagrangian system* α. Since

$$p_1 = \dot q^1, \qquad p_2 = 0, \qquad H = \tfrac{1}{2}\big((\dot q^1)^2 - (q^1)^2 q^2\big),$$

we get

$$\begin{aligned}\alpha &= -dH \wedge dt + dp_1 \wedge dq^1\\ &= -\big(\dot q^1\, d\dot q^1 - \tfrac{1}{2}(q^1)^2\, dq^2 - q^1 q^2\, dq^1\big) \wedge dt + d\dot q^1 \wedge dq^1.\end{aligned}$$

The annihilators of the Euler–Lagrange and the characteristic distribution are, respectively,

$$\begin{aligned}\Delta_\alpha^0 &= \operatorname{span}\big\{\tfrac{1}{2}(q^1)^2\, dt, \omega^1, d\dot q^1 - q^1 q^2\, dt\big\},\\ \mathcal{D}_\alpha^0 &= \operatorname{span}\big\{\tfrac{1}{2}(q^1)^2\, dt, \omega^1, d\dot q^1 - q^1 q^2\, dt, \dot q^1\, d\dot q^1\\ &\quad - \tfrac{1}{2}(q^1)^2\, dq^2 - q^1 q^2\, dq^1\big\}.\end{aligned}$$

(A) Extended dynamics. Obviously, Δ_α is weakly horizontal at the points $q^1 = 0$, i.e., the primary constraint set is

$$\tilde{\mathcal{P}} = \big\{x \in J^1Y \mid q^1(x) = 0\big\}.$$

In other words, this Lagrangian system possesses *primary dynamical constraints*.

The set $\tilde{\mathcal{P}}$ is a submanifold of the phase space. At each point of $\tilde{\mathcal{P}}$,

$$\Delta_\alpha = \mathcal{D}_\alpha = \operatorname{span}\left\{\frac{\partial}{\partial t} + \dot q^1 \frac{\partial}{\partial q^1}, \ \frac{\partial}{\partial q^2}, \ \frac{\partial}{\partial \dot q^2}\right\} = \operatorname{annih}\{\omega^1, d\dot q^1\}.$$

Let us compute the constrained to $\tilde{\mathcal{P}}$ Euler–Lagrange distribution. At the points of $\tilde{\mathcal{P}}$ where $\dot q^1 \neq 0$ we have

$$\Delta_\alpha|_{\tilde{\mathcal{P}}} = \operatorname{span}\left\{\frac{\partial}{\partial q^2}, \ \frac{\partial}{\partial \dot q^2}\right\},$$

which is a vertical distribution, and at the points of $\tilde{\mathcal{P}}$ where $\dot q^1 = 0$ we get a weakly horizontal distribution

$$\Delta_\alpha|_{\tilde{\mathcal{P}}} = \operatorname{span}\left\{\frac{\partial}{\partial t}, \ \frac{\partial}{\partial q^2}, \ \frac{\partial}{\partial \dot q^2}\right\}.$$

This means that the Lagrangian system possesses *secondary dynamical constraints*. The (final) constraint set is the manifold $M = \{x \in J^1Y \mid q^1(x) = 0, \ \dot q^1(x) = 0\} \subset J^1Y$.

Summarizing, we get the following dynamical picture for the Hamiltonian system associated with our Lagrangian system α: the extended dynamics is *constrained to a submanifold* $M = \{x \in J^1Y \mid q^1(x) = \dot q^1(x) = 0\}$ of the phase space. On this submanifold, the extended motion is indeterministic (being not uniquely determined by the initial conditions), and is given by a weakly horizontal distribution of rank 3, spanned by the vector fields

$$\frac{\partial}{\partial t} + f\,\frac{\partial}{\partial q^2} + g\,\frac{\partial}{\partial \dot{q}^2},$$

where f, g are arbitrary functions on M.

(B) Proper dynamics. Computing the semispray-constraint set we get

$$\mathcal{P} = \left\{x \in J^1Y \mid q^1(x) = 0\right\},$$

i.e., $\mathcal{P} = \tilde{\mathcal{P}}$. This means that the Lagrangian system possesses *primary semispray constraints*, which identify with the primary dynamical constraints. Similarly as above we get

$$\Delta_\alpha|_{\mathcal{P}} = \operatorname{span}\left\{\frac{\partial}{\partial q^2},\ \frac{\partial}{\partial \dot{q}^2}\right\}$$

at the points of $\mathcal{P}$ where $\dot{q}^1 \neq 0$, and

$$\Delta_\alpha|_{\mathcal{P}} = \operatorname{span}\left\{\frac{\partial}{\partial t},\ \frac{\partial}{\partial q^2},\ \frac{\partial}{\partial \dot{q}^2}\right\}$$

at the points of $\mathcal{P}$ where $\dot{q}^1 = 0$, i.e., the Lagrangian system possesses *secondary semispray constraints*. As above, denote $M = \{x \in J^1Y \mid q^1(x) = \dot{q}^1(x) = 0\}$. The preceding weakly horizontal distribution along M which is of rank 3 has a subdistribution of rank 2, spanned by the following semisprays along M

$$\frac{\partial}{\partial t} + \dot{q}^2\frac{\partial}{\partial q^2} + g\,\frac{\partial}{\partial \dot{q}^2}, \tag{9.2}$$

where g is a function on M. Hence, *the proper dynamics is constrained to* M and is *indeterministic* there: prolongations of extremals coincide with the integral sections of the distribution spanned by the vector fields (9.2).

The above dynamical picture gives us also information about properties of *extremals* on the configuration space. Namely, we can see that *all* solutions of the Euler–Lagrange equations are embedded in the submanifold $Q = \{x \in Y \mid q^1(x) = 0\}$, and that *every* section lying in this submanifold is an extremal.

9.2. Consider Cawley's Lagrangian

$$L = \dot{q}^1\dot{q}^3 + \tfrac{1}{2}(q^2)^2q^3 \tag{9.3}$$

on $R \times R^3 \times R^3$ [7], [6]. This Lagrangian defines a first order Lagrangian system

$$\alpha = \left(q^2q^3\omega^2 + \tfrac{1}{2}(q^2)^2\omega^3\right) \wedge dt - \omega^1 \wedge d\dot{q}^3 - \omega^3 \wedge d\dot{q}^1.$$

Computing the distributions $\mathcal{D}_\alpha$ and Δ_α we get

$$\mathcal{D}_\alpha = \operatorname{annih}\left\{q^2q^3\,dt, \omega^1, q^2q^3\,\omega^2, \omega^3, d\dot{q}^1 - \tfrac{1}{2}(q^2)^2\,dt, d\dot{q}^3\right\},$$
$$\Delta_\alpha = \operatorname{annih}\left\{q^2q^3\,dt, \omega^1, \omega^3, d\dot{q}^1 - \tfrac{1}{2}(q^2)^2\,dt, d\dot{q}^3\right\}.$$

We can see that $\mathcal{D}_\alpha \subset \Delta_\alpha$ and $\mathcal{D}_\alpha \neq \Delta_\alpha$. The function $f = \dot{q}^3$ is a *first integral* of the distributions $\mathcal{D}_\alpha$ and Δ_α.

(A) Extended dynamics. It holds $\operatorname{rank} \mathrm{F} = \operatorname{rank}(\mathrm{F}|\mathrm{A})$ if and only if $q^2q^3 = 0$, i.e., the Lagrangian system possesses *primary dynamical constraints*, and

$$\tilde{\mathcal{P}} = \left\{x \in J^1Y \mid q^2q^3 = 0\right\},$$

which is not a submanifold of the phase space.

(i) Let $x \notin \tilde{\mathcal{P}}$. The Euler–Lagrange distribution is at x spanned by vertical vectors, hence, there is no dynamics at this point.

(ii) Let $x \in \tilde{\mathcal{P}}$. Then we have

$$\begin{aligned}\mathcal{D}_\alpha = \Delta_\alpha &= \operatorname{span}\left\{\frac{\partial}{\partial t} + \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^3\frac{\partial}{\partial q^3} + \frac{1}{2}(q^2)^2\frac{\partial}{\partial \dot{q}^1},\ \frac{\partial}{\partial q^2},\ \frac{\partial}{\partial \dot{q}^2}\right\}\\ &= \operatorname{annih}\left\{\omega^1, \omega^3, d\dot{q}^1 - \frac{1}{2}(q^2)^2\,dt, d\dot{q}^3\right\},\end{aligned}$$

i.e., $\operatorname{rank} \mathcal{D}_\alpha = \operatorname{rank} \Delta_\alpha = 3$ at x.

For $x \in \tilde{\mathcal{P}}, q^2 \neq 0$ we get that the bunch of submanifolds in $\tilde{\mathcal{P}}$ passing through x consists from a single submanifold $M_x = \{q^3 = 0\}$. The constrained to M_x Euler–Lagrange distribution is

$$\Delta_\alpha|_{M_x} = \operatorname{span}\left\{\frac{\partial}{\partial q^2},\ \frac{\partial}{\partial \dot{q}^2}\right\},$$

at the points of M_x where $\dot{q}^3 \neq 0$, which is nowhere weakly horizontal, and

$$\Delta_\alpha|_{M_x} = \operatorname{span}\left\{\frac{\partial}{\partial t} + \dot{q}^1\frac{\partial}{\partial q^1} + \frac{1}{2}(q^2)^2\frac{\partial}{\partial \dot{q}^1},\ \frac{\partial}{\partial q^2},\ \frac{\partial}{\partial \dot{q}^2}\right\},$$

at the points of M_x where $\dot{q}^3 = 0$, which is weakly horizontal. This means that the Lagrangian system possesses *secondary dynamical constraints*, and we must exclude from $\tilde{\mathcal{P}}$ the points such that $q^2 \neq 0$, $q^3 = 0, \dot{q}^3 \neq 0$. At each of the remaining points we have a (unique) final constraint submanifold $M_1 = \{y \in M_x \mid \dot{q}^3 = 0\}$. Along M_1, the Euler–Lagrange distribution is reduced to the *completely integrable* distribution of rank 3, spanned by

$$\frac{\partial}{\partial t} + \dot{q}^1\frac{\partial}{\partial q^1} + \frac{1}{2}(q^2)^2\frac{\partial}{\partial \dot{q}^1},\ \frac{\partial}{\partial q^2},\ \frac{\partial}{\partial \dot{q}^2}.$$

Suppose that $x \in \tilde{\mathcal{P}}$, $q^3 \neq 0$. We get $M_x = \{q^2 = 0\}$, and the constrained to M_x Euler–Lagrange distribution is

$$\Delta_\alpha|_{M_x} = \operatorname{span}\left\{\frac{\partial}{\partial t} + \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^3\frac{\partial}{\partial q^3},\ \frac{\partial}{\partial \dot{q}^2}\right\};$$

it is weakly horizontal. M_x is a (unique) *final constraint submanifold at the point* x and $\Delta_\alpha|_{M_x}$ is a *completely integrable* distribution of rank 2.

The only remaining points in the phase space to be considered are $x \in \tilde{\mathcal{P}}$, $q^2 = 0$, $q^3 = 0$. The bunch of submanifolds at x now consists of two manifolds, $M_x^1 = \{q^3 = 0\}$ and $M_x^2 = \{q^2 = 0\}$. The Euler–Lagrange distribution Δ_α is weakly horizontal in the points of M_x^1 where $\dot{q}^3 = 0$. Along M_x^2, Δ_α is weakly horizontal and of rank 2, hence M_x^2 is a final constraint submanifold at x.

Summarizing the results we get the following picture of the extended dynamics: extended motion is *constrained to the subset*

$$(9.4) \qquad \{q^2 = 0\} \cup \{q^3 = \dot{q}^3 = 0\}$$

of the phase space, which is a union of two intersecting closed submanifolds. Along the submanifold $\{q^2 = 0\}$, Hamilton extremals are integral sections of the completely integrable distribution of rank 2, spanned by the vector fields

$$(9.5) \qquad \frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} + \dot{q}^3 \frac{\partial}{\partial q^3}, \ \frac{\partial}{\partial \dot{q}^2},$$

i.e., the extended motion there is *semiregular* and proceeds within *two-dimensional leaves*. Along the submanifold $\{q^3 = \dot{q}^3 = 0\}$ we get Hamilton extremals as integral sections of the completely integrable distribution

$$(9.6) \qquad \frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} + \frac{1}{2}(q^2)^2 \frac{\partial}{\partial \dot{q}^1}, \ \frac{\partial}{\partial q^2}, \ \frac{\partial}{\partial \dot{q}^2}$$

of rank 3, i.e., the extended motion there is *semiregular* and proceeds within *three-dimensional leaves* of the corresponding foliation.

(B) Proper dynamics. The system possesses *primary semispray constraints* which coincide with the primary dynamical constraints, since

$$\operatorname{rank} \mathrm{B} = \operatorname{rank} (\mathrm{B}|\mathrm{A}) \quad \text{if and only if} \quad q^2 q^3 = 0.$$

This means that we are lead to consider the semispray problem for the subset (9.4) and the distributions (9.5) and (9.6). We can see that the system possesses *secondary semispray constraints*: to a point $x \in \{q^2 = 0\}$ there is a semispray ζ such that $\zeta(x)$ belongs to the distribution (9.5) at x if and only if $\dot{q}^2(x) = 0$. The Euler–Lagrange distribution constrained to the closed submanifold $\{q^2 = \dot{q}^2 = 0\}$ is of rank 1, and it is spanned by the vector field

$$(9.7) \qquad \frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} + \dot{q}^3 \frac{\partial}{\partial q^3}$$

(which is a semispray along $\{q^2 = \dot{q}^2 = 0\}$). The distribution (9.6) on $\{q^3 = \dot{q}^3 = 0\}$ has a semispray subdistribution of rank 2, spanned by the vector fields

$$(9.8) \qquad \frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} + \dot{q}^2 \frac{\partial}{\partial q^2} + \frac{1}{2}(q^2)^2 \frac{\partial}{\partial \dot{q}^1} + g \frac{\partial}{\partial \dot{q}^2},$$

where g is a function on $\{q^3 = \dot{q}^3 = 0\}$.

In other words, the proper motion is *constrained to the subset*

$$(9.9) \qquad \{q^2 = \dot{q}^2 = 0\} \cup \{q^3 = \dot{q}^3 = 0\}$$

of the phase space. On the submanifold $\{q^2 = \dot{q}^2 = 0\}$ it is *regular*, described by the vector field (9.7) (through each point there passes exactly one maximal integral section = 1-jet prolongation of an extremal). Consequently, on this subset the motion is uniquely determined by the initial conditions. On the submanifold $\{q^3 = \dot{q}^3 = 0\}$ it is *weakly regular*, described by the vector fields (9.8). Since the distribution (9.8) *is not completely integrable*, we do not have a foliation defined by this distribution. However, since (9.8) is a subdistribution of the semiregular completely integrable distribution (9.6), the Lagrangian system is in fact *strongly semiregular* on $\{q^3 = \dot{q}^3 = 0\}$, which means that the prolonged extremals are embedded in the 3-dimensional leaves of the foliation of (9.6).

Notice that *on the configuration space*, the solutions of the Euler–Lagrange equations (*extremals*) are constrained to the subset $\{q^2 = 0\} \cup \{q^3 = 0\}$.

10. Constraint structure on a fibered manifold

In what follows, $r \geq 1$, $1 \leq k \leq m - 1$.

By a *constraint* in $J^r Y$ we shall mean a fibered submanifold of the fibered manifold $\pi_{r,r-1}$. If $\mathcal{Q}$ is a constraint in $J^r Y$, $\operatorname{codim}\mathcal{Q} = k$, locally given by the equations

$$f^i = 0, \quad 1 \leq i \leq k, \tag{10.1}$$

then by definition

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial q_r^\sigma}\right) = k. \tag{10.2}$$

Notice that (10.2) means that the equations (10.1) can be "locally solved" with respect to k of the functions q_r^σ. Without loss of generality one may consider the $q_r^{m-k+1}, \ldots, q_r^m$ as functions of the $(t, q^\sigma, \ldots, q_{r-1}^\sigma, q_r^1, \ldots, q_r^{m-k})$. Thus, from the definition one immediately gets that every constraint of codimension k can be covered by a family of *adapted fiber charts* (U, χ), $\chi = (t, q^\sigma, \ldots, q_{r-1}^\sigma, q_r^1, \ldots, q_r^{m-k}, f^1, \ldots, f^k)$, where (t, q^σ) are fibered coordinates on $\pi_r(U)$. In particular one has *normal charts* where $\mathcal{Q}$ is described by the equations

$$f^i \equiv q_r^{m-k+i} - g^i\big(t, q^\sigma, \ldots, q_r^1, \ldots, q_r^{m-k}\big) = 0. \tag{10.3}$$

A section γ of π defined on an open set $I \subset X$ will be called a *holonomic path in $\mathcal{Q}$* if $J^r\gamma(x) \in \mathcal{Q}$ for every $x \in I$.

Consider on $\mathcal{Q}$ an atlas $\mathcal{A}$ of adapted fiber charts. For every $(U, \chi) \in \mathcal{A}$ put

$$\mathcal{C}^0_{U,\varphi} = \operatorname{span}\big\{\varphi^i, df^i, 1 \leq i \leq k\big\},$$

where φ^i are linearly independent 1-forms such that

$$h\varphi^i = f^i\, dt, \quad 1 \leq i \leq k. \tag{10.4}$$

$\mathcal{C}_{U,\varphi}$ is a distribution of rank $rm+m+1-2k$ on U, in general not completely integrable. Holonomic integral sections of the distribution $\mathcal{C}_{U,\varphi}$, i.e. solutions of the "constraint equations" $f^i \circ J^r\gamma = 0$ and their prolongations $(df^i/dt) \circ J^{r+1}\gamma = 0$, coincide with the holonomic paths in $\mathcal{Q} \cap U$.

The above construction provides us, for a fixed U, with a *family* of distributions $\mathcal{C}_{U,\varphi}$ with the same holonomic integral sections. However, there is a unique and natural choice for the φ^i's, namely $\varphi^i = f^i\, dt + \varphi^i_\sigma \omega^\sigma$, where

$$\varphi^i_\sigma = \frac{\partial f^i}{\partial q^\sigma_r}, \quad 1 \le i \le k. \tag{10.5}$$

Such forms will be called *constraint* 1*-forms*. The meaning of these forms comes from the following lemma which is proved by an easy computation.

10.1. Lemma [38]. *Let F be the canonical morphism of the module $\mathcal{V}_U(\pi)$ of π-vertical vector fields on $\pi_{r,0}(U)$ over the ring of functions on $\pi_{r,0}(U)$ to the module $\mathcal{V}_U(\pi_{r,0})$ of $\pi_{r,0}$-vertical vector fields over the ring of $\pi_{r,0}$-projectable functions on U, defined by*

$$F(\partial/\partial q^\sigma) = \partial/\partial q^\sigma_r, \qquad F(g\xi) = (g \circ \pi_{r,0})F(\xi).$$

One-forms φ^i, $1 \le i \le k$, on U are constraint 1*-forms if and only if for $1 \le i \le k$,*
(1) *$h\varphi^i = f^i\, dt$,*
(2) *$p\varphi^i$ are $\pi_{r,0}$-horizontal, and*
(3) *for every π-vertical vector field ξ on $\pi_{r,0}(U)$,*

$$i_{J^r\xi}\varphi^i = i_{F(\xi)} df^i. \tag{10.6}$$

We shall call (10.5) *generalized Chetaev expressions*, and the corresponding distribution *constraint distribution* associated with U; it will be denoted by $\mathcal{C}_U$.

In this way, in a neighborhood of a constraint $\mathcal{Q}$ there is a canonical system of local constraint distributions, subordinate to a cover of $\mathcal{Q}$ by adapted fiber charts. Along the constraint $\mathcal{Q}$ each of these distributions is tangent to $\mathcal{Q}$, and holonomic paths in $\mathcal{Q}$ piecewise coincide with holonomic integral sections of these distributions. Taking another cover of $\mathcal{Q}$ by adapted fiber charts one gets in a neighborhood of the constraint a different system of local constraint distributions. However, by the following theorem, all these local distributions give rise to a *unique* (global) *distribution on the constraint*.

10.2. Theorem [35], [38]. *Let $\mathcal{Q} \subset J^rY$ be a constraint. Let $\{U_\iota\}_{\iota\in I}$ be a cover of $\mathcal{Q}$ by adapted fiber charts, and $\{\mathcal{C}_{U_\iota}\}$ the subordinate system of local constraint distributions. Then for every ι, $\kappa \in I$,*

$$\mathcal{C}_{U_\iota} = \mathcal{C}_{U_\kappa} \quad \text{on} \quad U_\iota \cap U_\kappa \cap \mathcal{Q}.$$

Proof. One has to show that if $\iota : \mathcal{Q} \to J^rY$ is the canonical embedding, and $\{\eta^p_\iota\}$ and $\{\eta^p_\kappa\}$, $1 \le p \le 2k$, are linearly independent annihilators of $\mathcal{C}_{U_\iota}$ and $\mathcal{C}_{U_\kappa}$, respectively, then $\eta^p_\iota = \sum_s b^p_s \eta^s_\kappa$ for some regular matrix (b^p_s) on $U_\iota \cap U_\kappa \cap \mathcal{Q}$. Suppose that $\mathcal{Q}$ is defined by $f^i = 0$ and $\bar{f}^i = 0$ on U_ι and U_κ, respectively. Then on $U_\iota \cap U_\kappa$, $\mathcal{C}^0_{U_\iota}$ is spanned by the 1-forms df^i and the constraint 1-forms φ^i, and $\mathcal{C}^0_{U_\kappa}$ is spanned by $d\bar{f}^i$ and $\bar{\varphi}^i$, where

$$\varphi^i = f^i dt + \frac{\partial f^i}{\partial q^\sigma_r}\,\omega^\sigma, \quad \text{respectively,} \quad \bar{\varphi}^i = \bar{f}^i d\bar{t} + \frac{\partial \bar{f}^i}{\partial \bar{q}^\sigma_r}\,\bar{\omega}^\sigma,$$

$1 \leq i \leq k$. Since at each point $x \in \mathcal{Q}$ (belonging to $U_\iota \cap U_\kappa$) both df^i and $d\bar{f}^i$ define the tangent distribution $T_x\mathcal{Q}$, we must have $d\bar{f}^i(x) = a^i_j(x)df^j$, where (a^i_j) is a regular matrix. Consequently, since $\bar{\omega}^\sigma = (\partial\bar{q}^\sigma/\partial q^\nu)\omega^\nu$, we get

$$\left(\frac{\partial\bar{f}^i}{\partial\bar{q}^\sigma_r}\right)_x \left(\frac{\partial\bar{q}^\sigma}{\partial q^\nu}\right)_x = c(x)\, a^i_j(x) \left(\frac{\partial f^j}{\partial q^\nu_r}\right)_x$$

where $c(x) \neq 0$. Hence, on the constraint submanifold,

$$\iota^*\bar{\varphi}^i = \frac{\partial\bar{f}^i}{\partial\bar{q}^\sigma_r}\frac{\partial\bar{q}^\sigma}{\partial q^\nu}\,\iota^*\omega^\nu = c\,a^i_j\,\frac{\partial f^j}{\partial q^\nu_r}\,\iota^*\omega^\nu = (c\,a^i_j)\,\iota^*\varphi^j,$$

and we are done. $\square$

The obtained distribution on $\mathcal{Q}$ is called the *canonical distribution* of the constraint, or *Chetaev bundle* over $\mathcal{Q}$ [38]. Note that corank $\mathcal{C}$ with respect to $\mathcal{Q}$ is k, and that by definition

$$\mathcal{C} = \text{annih}\left\{\iota^*\varphi^i, 1 \leq i \leq k\right\}.$$

In keeping with the first-order case, horizontal rank 1 subdistributions of $\mathcal{C}$ are called *constraint connections*. They are locally spanned by *constraint semisprays* which in *normal coordinates* take the form

$$\Gamma = \frac{\partial}{\partial t} + \sum_{p=0}^{r-2} q^\sigma_{p+1}\frac{\partial}{\partial q^\sigma_p} + \sum_{l=1}^{m-k} q^l_r\frac{\partial}{\partial q^l_{r-1}}$$
$$+ \sum_{i=1}^{k} g^i \frac{\partial}{\partial q^{m-k+i}_{r-1}} + \sum_{l=1}^{m-k}\Gamma^l\frac{\partial}{\partial q^l_r}.$$

In accordance with [35], [38], the pair $(\mathcal{Q}, \mathcal{C})$ where $\mathcal{Q}$ is a constraint in J^rY and $\mathcal{C}$ is the canonical distribution of the constraint is called a *constraint structure of order r*. The ideal on $\mathcal{Q}$ generated by the 1-forms annihilating the canonical distribution is called the *constraint ideal* and is denoted by $\mathcal{I}(\mathcal{C}^0)$. Every p-form on $\mathcal{Q}$, $p \geq 1$, belonging to the constraint ideal is then called a *constraint p-form*.

10.3. Remark (Geometrical models for classical constraints)**.** Let us show relations of the above model of constraints with some frequently used geometric representations of classical (i.e., holonomic and linear non-holonomic) constraints.

(i) Constraints which are realized as *a fibered submanifold $\mathcal{Q}_0$ of Y* of codimension $k < m$ are called *holonomic*. Locally they are given by equations $u^i(t, q^\sigma) = 0$, $1 \leq i \leq k$, such that rank $(\partial u^i/\partial q^\sigma) = k$. Within our scheme this means that $f^i = du^i/dt$, and

$$\varphi^i = \frac{du^i}{dt}dt + \frac{\partial u^i}{\partial q^\sigma}\omega^\sigma = \pi^*_{1,0}du^i,$$

i.e., all the constraint 1-forms are $\pi_{1,0}$-projectable and closed. Consequently, the corresponding constraint ideal is a *differential ideal*, the canonical distribution $\mathcal{C}$ is *completely integrable*, and along $J^1\mathcal{Q}_0$ it is nothing but the tangent distribution $TJ^1\mathcal{Q}_0$. Its projection is the tangent distribution to $\mathcal{Q}_0$, giving sense to the classical concepts of *degrees of freedom*, *possible displacements*, and *virtual displacements* defined to be the dimension of the manifold $\mathcal{Q}_0$, the tangent distribution to $\mathcal{Q}_0$, and its π-vertical subdistribution, respectively. The constraint structure for holonomic systems becomes $(J^1\mathcal{Q}_0, TJ^1\mathcal{Q}_0)$.

(ii) Consider a *completely integrable distribution $\mathcal{D}$ of corank $k < m$ on Y*. This means that in a neighborhood of every point in Y there are k linearly independent 1-forms du^i spanning the annihilator $\mathcal{D}^0$. Such a distribution represents constraints usually called *semiholonomic* or *linear integrable* (referring to the equations of the constraints which are affine in velocities). In our notations, $f^i = du^i/dt$, $\varphi^i = \pi^*_{1,0}du^i$. Thus, the constraint 1-forms are $\pi_{1,0}$-projectable and closed, i.e., the corresponding constraint ideal is a *differential ideal*, and the corresponding constraint distribution is *completely integrable* and projects onto $\mathcal{D}$. The constraint structure on J^1Y is $(\mathcal{Q}, \mathcal{C})$, where $\mathcal{Q}$ is locally defined by

$$\frac{du^i}{dt} = \frac{\partial u^i}{\partial t} + \frac{\partial u^i}{\partial q^\sigma}\dot{q}^\sigma = 0, \quad 1 \le i \le k,$$

and $\mathcal{C}^0 = \mathrm{span}\{\iota^*\pi^*_{1,0}du^i,\ 1 \le i \le k\}$.

(iii) The constraints are given by *a distribution $\mathcal{D}$ of corank $k < m$ on Y* which is not supposed to be completely integrable. They are called *simple non-holonomic* [35] or *linear nonintegrable*. Denote $\mathcal{D}^0 = \mathrm{span}\{\eta^i = a^i\,dt + b^i_\sigma dq^\sigma, 1 \le i \le k\}$. Then in our notations,

$$\varphi^i = \pi^*_{1,0}\eta^i = \left(a^i + b^i_\sigma\dot{q}^\sigma\right)dt + b^i_\sigma\omega^\sigma, \qquad f^i = a^i + b^i_\sigma\dot{q}^\sigma,$$

i.e., the constraint 1-forms are $\pi_{1,0}$-projectable (the corresponding constraint distribution projects onto $\mathcal{D}$). The constraint structure on J^1Y is $(\mathcal{Q}, \mathcal{C})$, where $\mathcal{Q}$ is locally defined by

$$a^i + b^i_\sigma\dot{q}^\sigma = 0, \quad 1 \le i \le k,$$

and $\mathcal{C}^0 = \mathrm{span}\{\iota^*\pi^*_{1,0}\eta^i, 1 \le i \le k\}$.

11. Constrained systems

With help of the canonical distribution, mechanical systems subject to constraints can be intrinsically characterized as *mechanical systems on constraint submanifolds* [38].

If $[\alpha]$ is a mechanical system on $J^{s-1}Y$, $s \ge 2$, corresponding to a dynamical form E of order s, one can consider a constraint submanifold of J^rY with $r \ge 1$, possibly *different* from $s-1$. First, let us suppose that constraints are given on the *phase space* $J^{s-1}Y$ where the unconstrained dynamics proceeds.

Let $[\alpha]$ be a mechanical system on $J^{s-1}Y$, and let $(\mathcal{Q}, \mathcal{C})$ be a constraint structure on $J^{s-1}Y$. Denote by ι the embedding of $\mathcal{Q}$ into $J^{s-1}Y$, and by $\mathcal{I}(\mathcal{C}^0)$ the constraint ideal on $\mathcal{Q}$. For $\alpha \in [\alpha]$ put

$$\alpha_{\mathcal{Q}} = \iota^*\alpha \mod \mathcal{I}(\mathcal{C}^0), \tag{11.1}$$

and denote by $[\alpha_{\mathcal{Q}}]$ the mechanical system generated by the class $\alpha_{\mathcal{Q}}$. Hence, by definition, every element of the class $[\alpha_{\mathcal{Q}}]$ is a 2-form on $\mathcal{Q}$ equal to

$$\iota^*\alpha + \text{constraint 2-form} + \text{2-contact 2-form}.$$

It is easy to see that if $\alpha_1, \alpha_2 \in [\alpha]$ then $[(\alpha_1)_{\mathcal{Q}}] = [(\alpha_2)_{\mathcal{Q}}]$. This means that by (11.1) we have assigned to a mechanical system on $J^{s-1}Y$ a mechanical system on the constraint $\mathcal{Q}$. Naturally, we shall call the class $[\alpha_{\mathcal{Q}}]$ the *constrained system* related to the mechanical system $[\alpha]$ and the constraint structure $(\mathcal{Q}, \mathcal{C})$.

The following proposition brings an intrinsic form of the *equations of motion* of the constrained system.

11.1. Proposition [38]. *Let $[\alpha]$ be a mechanical system on $J^{s-1}Y$, $(\mathcal{Q}, \mathcal{C})$ a constraint structure on $J^{s-1}Y$, and $[\alpha_{\mathcal{Q}}]$ the corresponding constrained system. A section γ of π is a path of the constrained system $[\alpha_{\mathcal{Q}}]$ if and only if $J^{s-1}\gamma$ is an integral section of the canonical distribution $\mathcal{C}$, and for every π_{s-1}-vertical vector field $\xi \in \mathcal{C}$ it satisfies the equation*

$$J^{s-1}\gamma^* i_\xi \alpha_{\mathcal{Q}} = 0, \tag{11.2}$$

where $\alpha_{\mathcal{Q}}$ is (any) 2-form belonging to $[\alpha_{\mathcal{Q}}]$.

We shall find a *coordinate expression* for the constrained system $[\alpha_{\mathcal{Q}}]$ [38]. Let $x \in \mathcal{Q}$ be a point and consider fibered coordinates $(t, q^\sigma, q_1^\sigma, \dots, q_{s-1}^\sigma)$ in a neighborhood U of x. For simplicity, suppose that the constraint $\mathcal{Q}$ is in U given by equations in normal form,

$$f^i \equiv q_{s-1}^{m-k+i} - g^i\big(t, q^\sigma, \dots, q_{s-1}^1, \dots, q_{s-1}^{m-k}\big) = 0, \quad 1 \le i \le k.$$

First of all, notice that on U we have the following basis of 1-forms:

$$\begin{aligned}&\big(dt, \omega^1, \dots, \omega^{m-k}, \varphi^1, \dots, \varphi^k, \omega_1^\sigma, \dots, \omega_{s-2}^\sigma,\\ &\quad dq_{s-1}^1, \dots, dq_{s-1}^{m-k}, df^1, \dots df^k\big),\end{aligned} \tag{11.3}$$

where (φ^i, df^i), $1 \le i \le k$, are generators of the constraint codistribution $\mathcal{C}_U^0$ on U,

$$\varphi^i = \big(q_{s-1}^{m-k+i} - g^i\big)\, dt - \sum_{l=1}^{m-k} \frac{\partial g^i}{\partial q_{s-1}^l}\, \omega^l + \omega^{m-k+i}. \tag{11.4}$$

Expressing a representative α of the class $[\alpha]$ in the basis (11.3) using

$$\begin{aligned}\omega^{m-k+i} &= \varphi^i - \big(q_{s-1}^{m-k+i} - g^i\big)\, dt + \sum_{l=1}^{m-k} \frac{\partial g^i}{\partial q_{s-1}^l}\, \omega^l,\\ dq_{s-1}^{m-k+i} &= df^i + dg^i,\end{aligned}$$

computing $\iota^*\alpha$ and omitting constraint forms and 2-contact forms we get a representative of the constrained system $[\alpha_{\mathcal{Q}}]$ in the form

$$\alpha_{\mathcal{Q}} = \sum_{l=1}^{m-k} A'_l \omega^l \wedge dt + \sum_{l,p=1}^{m-k} B'_{lp} \omega^l \wedge dq_{s-1}^p, \tag{11.5}$$

where

$$
(11.6)\qquad
\begin{aligned}
A'_l &= (A_l \circ \iota) + \sum_{i=1}^{k} (A_{m-k+i} \circ \iota)\, \frac{\partial g^i}{\partial q^l_{s-1}} \\
&\quad + \sum_{j=1}^{k} (B_{l,m-k+j} \circ \iota) \frac{\hat{d} g^j}{dt} + \sum_{i,j=1}^{k} (B_{m-k+i,m-k+j} \circ \iota)\, \frac{\partial g^i}{\partial q^l_{s-1}}\, \frac{\hat{d} g^j}{dt}, \\
B'_{lp} &= (B_{lp} \circ \iota) + \sum_{j=1}^{k} (B_{l,m-k+j} \circ \iota) \frac{\partial g^j}{\partial q^p_{s-1}} \\
&\quad + \sum_{i=1}^{k} (B_{m-k+i,p} \circ \iota)\, \frac{\partial g^i}{\partial q^l_{s-1}} + \sum_{i,j=1}^{k} (B_{m-k+i,m-k+j} \circ \iota)\, \frac{\partial g^i}{\partial q^l_{s-1}}\, \frac{\partial g^j}{\partial q^p_{s-1}},
\end{aligned}
$$

and

$$
\frac{\hat{d} g^i}{dt} = \frac{\partial g^i}{\partial t} + \sum_{r=0}^{s-2} \frac{\partial g^i}{\partial q^\sigma_r}\, q^\sigma_{r+1} = \frac{d g^i}{dt} - \sum_{p=1}^{m-k} q^p_s \frac{\partial g^i}{\partial q^p_{s-1}}.
$$

Now, the *equations of motion of the constrained system* $[\alpha_{\mathcal{Q}}]$ have the following form of a mixed system of $m-k$ ODE of order s

$$
(11.7)\qquad A'_l + \sum_{j=1}^{m-k} B'_{lj} q^j_s = 0 \quad \text{along } J^s\gamma,
$$

and k ODE of order $s-1$, $f^i \circ J^{s-1}\gamma = 0$, for the components $\gamma^1, \dots \gamma^m$ of sections γ of π.

Another family of non-holonomic systems is characterized by the property that *the constraints depend on higher derivatives than those corresponding to the phase space*. In particular, a problem of this kind has been originally studied within classical mechanics ($s = 2$) when constraints on accelerations have been considered ($r = 2$) [11], [13], [19], [48], [58], [59], and others. Within our approach this case, however, can be viewed as a particular case of constrained systems studied above.

If $E \in \Lambda^2_{af}(J^sY)$ is a dynamical form, $[\alpha]$ its Lepage class, and $\mathcal{Q} \subset J^rY$, $r \geq s$, a non-holonomic constraint, we put

$$
\alpha_{\mathcal{Q}} = \iota^* \pi^*_{r,s-1} \alpha \ \operatorname{mod} \mathcal{I}(\mathcal{C}^0),
$$

and we call the arising class $[\alpha_{\mathcal{Q}}]$ the *constrained system*, related to the mechanical system $[\alpha]$ on $J^{s-1}Y$ and the constraint structure $(\mathcal{Q}, \mathcal{C})$ on J^rY. Now, equations of motion in both the intrinsic and coordinate form are obtained analogously to the preceding case.

Finally, it remains to investigate the case $r < s-1$. Consider a non-holonomic constraint $\mathcal{Q}_r \subset J^rY$, and suppose that locally it is given by the equations

$$
u^i_r(t, q^\sigma, \dots, q^\sigma_r) = 0.
$$

$\mathcal{Q}_r$ naturally prolongs to a submanifold $\mathcal{P}$ of $J^{s-1}Y$, given by the equations

$$
(11.8)\qquad u^i_r = 0, \quad u^i_{r+1} \equiv \frac{du^i_r}{dt} = 0, \quad \dots, \quad u^i_{s-1} \equiv \frac{d^{s-1-r} u^i_r}{dt^{s-1-r}} = 0.
$$

Put $f^i = u^i_{s-1}$. The submanifold $\mathcal{Q}$ of $J^{s-1}Y$ given by the equations $f^i = 0$ is a non-holonomic constraint of order $s-1$, and $\mathcal{P} \subset \mathcal{Q}$. Denote by $\iota_{\mathcal{P}}$ and ι the canonical embedding $\mathcal{P} \to \mathcal{Q}$ and $\mathcal{Q} \to J^{s-1}Y$, respectively. The canonical distribution $\mathcal{C}$ on $\mathcal{Q}$ is spanned by the 1-forms

$$(11.9) \qquad \iota^*\varphi^i = \left(\frac{\partial u^i_r}{\partial q^\sigma_r} \circ \iota\right) \omega^\sigma = \iota^*\pi^*_{s-1,r}\,\varphi^i_r, \quad 1 \le i \le k,$$

where φ^i_r are the constraint 1-forms referring to $\mathcal{Q}_r$. Denote by $\mathcal{C}_{\mathcal{P}}$ the subdistribution of the canonical distribution $\mathcal{C}$, restricted to $\mathcal{P}$, which is tangent to $\mathcal{P}$. In other words, for every $x \in \mathcal{P}$, put

$$(11.10) \quad \mathcal{C}_{\mathcal{P}}(x) = \mathcal{C}(x) \cap T_x\mathcal{P}.$$

$\mathcal{C}_{\mathcal{P}}$ will be called the *induced canonical distribution*.

11.2. Proposition [38]. *At every $x \in \mathcal{P}$,*

$$(11.11) \quad \mathcal{C}^0_{\mathcal{P}} = \iota^*_{\mathcal{P}}\mathcal{C}^0 = \operatorname{span}\{\iota^*_{\mathcal{P}}\iota^*\varphi^i\}.$$

Proof. Let $\xi_x \in \mathcal{C}_{\mathcal{P}}(x)$. Then $i_{\xi_x}\iota^*_{\mathcal{P}}\iota^*\varphi^i(x) = 0$, since $\xi_x \in \mathcal{C}$. Conversely, let $\xi_x \in T_x\mathcal{P}$ annihilate $\mathcal{C}^0_{\mathcal{P}}$ at x. Then $i_{\xi_x}\iota^*\varphi^i(x) = \iota^*\varphi^i(x)(\xi_x) = \iota^*_{\mathcal{P}}\iota^*\varphi^i(x)(\xi_x) = 0$, i.e., $\xi_x \in \mathcal{C}(x)$. Consequently, $\xi_x \in \mathcal{C}(x) \cap T_x\mathcal{P}$. $\square$

Let $[\alpha]$ be a mechanical system on $J^{s-1}Y$. If $\alpha_{\mathcal{Q}}$ is a representative of the constrained system $[\alpha_{\mathcal{Q}}]$ on $\mathcal{Q}$ related to $[\alpha]$ and the constraint structure $(\mathcal{Q}, \mathcal{C})$, put

$$(11.12) \quad \alpha_{\mathcal{P}} = \iota^*_{\mathcal{P}}\,\alpha_{\mathcal{Q}} \bmod \mathcal{I}(\mathcal{C}^0_{\mathcal{P}})$$

Due to Proposition 11.2, the *class* $[\alpha_{\mathcal{Q}}]$ is pulled back to the *class* $[\alpha_{\mathcal{P}}]$, hence, this procedure gives us a *constrained system on* $\mathcal{P}$, related to the mechanical system $[\alpha]$ on $J^{s-1}Y$ and the non-holonomic constraint $\mathcal{Q}_r \subset J^rY$. The equations of motion are now easily obtained in the following form:

11.3. Proposition. *Let $[\alpha]$ be a mechanical system on $J^{s-1}Y$, $\mathcal{Q}_r$ a non-holonomic constraint in J^rY, $1 \le r < s-1$. Let as above, $(\mathcal{P}, \mathcal{C}_{\mathcal{P}})$ be the induced constraint structure on $J^{s-1}Y$. A section $\gamma : I \to Y$ of π is a path of the constrained system $[\alpha_{\mathcal{P}}]$ if and only if $J^{s-1}\gamma(I) \subset \mathcal{P}$, and for every π_{s-1}-vertical vector field $\xi \in \mathcal{C}_{\mathcal{P}}$ and every $\alpha_{\mathcal{P}} \in [\alpha_{\mathcal{P}}]$ such that $J^{s-1}\gamma(I) \cap \operatorname{dom}\alpha_{\mathcal{P}} \neq \emptyset$, it satisfies the equation*

$$(11.13) \quad J^{s-1}\gamma^* i_\xi \alpha_{\mathcal{P}} = 0.$$

Propositions 11.1 and 11.3 mean that dynamics of constrained systems can be represented by means of *distributions* on the constraints. Namely, by a *constraint dynamical distribution* $\Delta_{\alpha_{\mathcal{Q}}}$ we understand a subdistribution of the canonical distribution $\mathcal{C}$, annihilated by means of the 1-forms $i_\xi\alpha_{\mathcal{Q}}$, where ξ runs over all vertical vector fields on $\mathcal{Q}$ belonging to $\mathcal{C}$. Now, *Hamiltonian extensions* of constrained systems can be studied. In analogy with the unconstrained case, by *constraint Hamilton paths* we understand integral sections of constraint dynamical distributions.

The concept of *regularity*, resp., *semiregularity* for constrained systems is introduced and studied in the same way as in the unconstrained case, providing analogous results.

Notice that *a constrained system related with a regular mechanical system need not be regular*, and conversely, *a singular mechanical system may become regular under a suitable constraint*. In particular, notice that all constrained systems which arise from mechanical systems of order $s-1$ and are subject to constraints of order *greater* than $s-1$ are *singular* (as mechanical systems on $\mathcal{Q} \subset J^r Y$).

Applying the theory of Lagrangian systems to constrained systems we immediately get the following results:

11.4. Theorem. *A constrained system* $[\alpha]$ *is Lagrangian if and only if the class* $[\alpha_Q]$ *contains a closed representative.*

11.5. Corollary. *A constrained system arising from a Lagrangian system is Lagrangian.*

11.6. Mechanical systems with holonomic constraints. Within the presented setting, higher-order holonomic systems can be easily treated as a particular case of non-holonomic systems (cf. [38]).

Recall that a *holonomic constraint* (or *a system of* k *(*$k < m$*) independent holonomic constraints*) is a fibered submanifold of codimension k in the fibered manifold $\pi : Y \to X$.

Let $\iota_0 : \mathcal{Q}_0 \to Y$ be a a holonomic constraint of codimension k. Then at each point $x \in Y$ there is a chart (U, χ), $\chi = (t, q^1, \ldots, q^{m-k}, u^1, \ldots, u^k)$ such that $\mathcal{Q}_0$ is on U defined by the equations

$$u^i = 0, \quad 1 \le i \le k,$$

and the functions u^i satisfy the condition

$$\text{rank}\left(\frac{\partial u^i}{\partial q^\sigma}\right) = k. \tag{11.14}$$

The submanifold $\mathcal{Q}_0$ of Y prolongs to a submanifold $\mathcal{P} \equiv J^{s-1}\mathcal{Q}_0 \subset J^{s-1}Y$ of the codimension sk. $J^{s-1}\mathcal{Q}_0$ is locally defined by the equations

$$u^i = 0, \quad \frac{du^i}{dt} = 0, \quad \ldots, \quad \frac{d^{s-1}u^i}{dt^{s-1}} = 0, \quad 1 \le i \le k,$$

i.e., it can be covered by adapted fiber coordinates $(t, q^j, u^i_0, q^j_1, u^i_1, \ldots, q^j_{s-1}, f^i)$, where $1 \le j \le m-k$, $1 \le i \le k$, and

$$u^i_0 = u^i, \quad u^i_1 = \frac{du^i}{dt}, \quad \ldots, \quad f^i \equiv u^i_{s-1} = \frac{d^{s-1}u^i}{dt^{s-1}}.$$

In this way, $J^{s-1}\mathcal{Q}_0$ is a submanifold of the non-holonomic constraint $\mathcal{Q} \subset J^{s-1}Y$ of codimension k, locally defined by the equations $f^i = 0$, $1 \le i \le k$. Using similar notations as above, we have $\iota : \mathcal{Q} \to J^{s-1}Y$, $\iota_{\mathcal{P}} : J^{s-1}\mathcal{Q}_0 \to \mathcal{Q}$, and $J^{s-1}\iota_0 = \iota \circ \iota_{\mathcal{P}}$.

Theorem.

$$\mathcal{C}_{\mathcal{P}} = T J^{s-1}\mathcal{Q}_0.$$

Proof. If $\mathcal{Q}_0$ is a holonomic constraint, we get for the constraint 1-forms on the associated non-holonomic constraint $\mathcal{Q} \subset J^{s-1}Y$,

$$\iota^* \varphi^i = \frac{\partial u^i}{\partial q^\sigma}\, \omega^\sigma .$$

Hence, for the canonical distribution $\mathcal{C}$ on $\mathcal{Q}$ we have $\mathcal{C}^0 = \mathrm{span}\{\iota^* \pi^*_{s-1,0}\, p\, du^i\}$. Now, the induced canonical distribution $\mathcal{C}_\mathcal{P}$ on $\mathcal{P} = J^{s-1}\mathcal{Q}_0$ is annihilated by the 1-forms $J^{s-1}\iota_0^* \varphi^i = J^{s-1}\iota_0^*\, p\, du^i = J^{s-1}\iota_0^* du^i - J^{s-1}\iota_0^* h\, du^i = 0$, since along $J^{s-1}\iota_0$ the equations $u^i = 0$ and $du^i/dt = 0$ hold. Thus $\mathcal{C}^0_\mathcal{P} = \{0\}$, and we are done. □

The above theorem means that holonomic constraints are not "true" constraints, since they induce *no constraints in the tangent bundle to the constraint submanifold*. As a consequence we get the well-known result saying that holonomic constrained systems are nothing but *pull-backs of unconstrained systems* to the corresponding fibered submanifolds. More precisely, for a holonomic constrained system on $J^{s-1}\mathcal{Q}_0$ we have

$$\alpha_\mathcal{P} = J^{s-1}\iota_0^* \alpha,$$

where α represents the unconstrained system on $J^{s-1}Y$.

Now it is easy to see that if the unconstrained mechanical system is Lagrangian, and λ is its (possibly local) Lagrangian of order r (i.e., if the class $[\alpha]$ on $J^{s-1}Y$ has a unique closed representative locally equal to $d\theta_\lambda$), we have

Corollary.

$$\alpha_\mathcal{P} = J^{s-1}\iota_0^* d\theta_\lambda = d\, J^{s-1}\iota_0^* \theta_\lambda = d\theta_{J^r \iota_0^* \lambda}.$$

11.7. Semiholonomic constraints. A nonholonomic constraint in J^rY is called *semiholonomic of degree* p if

$$f^i = \frac{d^p u^i}{dt^p}$$

for some functions u^i, $1 \le i \le k$. Since *semiholonomic constraints* represent a particular case of non-holonomic constraints, the geometric setting for mechanical systems subject to semiholonomic constraints (of any order and degree) is quite obvious and means no difficulties at all.

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