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The variational sequence: Local and global properties[1]

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Abstract. The aim of this paper is to discuss some aspects of local and global properties of the Euler–Lagrange and Helmholtz–Sonin mappings of the calculus of variations in the r-th order field theory, i.e. on r-jet prolongations of fibered manifolds over a n-dimensional base with $n > 1$, within the framework of the variational sequence, i.e. the quotient of the De Rham sequence with respect to its subsequence of contact differential forms. Such a discussion is, in general, based on the concept of sheaves of differential forms. In the paper a globally defined representation of the variational sequence by forms is constructed which is closely related to the standard concepts in the calculus of variations. There is a close relationship between elements of the quotient sheaves (*classes* of forms) and the quotient mappings on one hand and the standard objects of the calculus of variations, as lagrangians, Euler–Lagrange and Helmholtz–Sonin forms, and Euler–Lagrange and Helmholtz–Sonin mappings on the other hand.

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1. Introduction

One of the most important questions in the calculus of variations is the characterization of local and global properties of the Euler–Lagrange and Helmholtz–Sonin mappings, especially their kernels and images. The general solution of this problem

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on an r-jet prolongation of a given fibered manifold can give the answers pertaining to the problems of variationally trivial lagrangians and variational equations of motion in the r-th order field theory or mechanics. The close relationship between the exterior derivative of a differential form and the Euler–Lagrange mapping in the classical sense, formulated by Lepage and Dedecker, has been developed during the last two decades by many authors (Anderson, Betounes, Duchamp, Gotay, Krupka, Krupková, Kuperschmidt, Olver, Pommaret, Saunders, Takens, Tulczyjew, Vinogradov etc.) and it then led to the concept of the variational sequence on finite jet prolongations of fibered manifolds, introduced and systematically studied by Krupka [8–10]. The variational sequence is constructed as the quotient of the well-known De Rham exact sequence of spaces of differential forms with respect to its subsequence of certain spaces of contact forms. This subsequence is chosen in such a way that the Euler–Lagrange and Helmholtz–Sonin mappings, considered in the generalized concept, are contained in the corresponding quotient sequence of mappings. The theoretical background for the study of the variational sequence is, among others, the theory of sheaves which was presented in details and elaborated for the purposes of the variational sequence calculus by Krupka [11]. Some aspects of the variational sequence were studied by several authors belonging to Krupka's school: Štefánek [17] found a "non-physical" local representation of the r-th order variational sequence in mechanics. Musilová [15] and Musilová and Krbek [16] described the (global) "physical" representation of the physically relevant part of the r-th order variational sequence in mechanics, including the reconstruction of classes of forms from their representatives. Kašparová [5, 6] has been studying the first order variational sequence in field theory and found the global representatives of physically relevant classes of forms. The problem of variationally trivial lagrangians was solved by Krupka and Musilová [12]. Some results in the theory of representations of the variational sequence in field theory, including studies of the trivial variational problem, were presented by Grigore in [2] and [3]. In the first of these papers the representation of q-forms is given for $1 \leq q \leq n+1$, for $q = n+2$ it is only proved that the class of $\pi^{r,0}$-horizontal forms can be represented by the Helmholtz–Sonin form. The representatives of general classes of $(n+2)$-forms have not been sought. Some problems concerning the variational sequence in field theory were recently discussed also by Vitolo in [18] and by Francaviglia, Palese and Vitolo in [1].

In this paper we discuss some properties of the r-th order variational sequence on fibered manifolds over n-dimensional base. We construct its representation for classes of q-forms, $1 \leq q \leq n+2$, especially for the physically relevant part, i.e. for classes of n-forms, $(n+1)$-forms and $(n+2)$-forms. Following the ideas of Krupka [10] for mechanics, we present the representation of the variational sequence for $1 \leq q \leq n+2$. We give the generalized definition of the Euler–Lagrange and Helmholtz–Sonin form as well as the Euler–Lagrange and Helmholtz–Sonin mapping. We show that our representatives are global for $1 \leq q \leq n+2$.

2. Underlying structures and basic notations

Throughout the paper we use the following standard notation, used by Krupka (see e.g. [10, 13]): Y is a $(n+m)$-dimensional *fibered manifold* with the *n-dimensional base* X and *projection* π. For an arbitrary integer $r \geq 0$, J^rY is the *r-jet prolongation* of Y, π^r and $\pi^{r,s}$ for $r \geq s \geq 0$ being the *canonical projections* of J^rY on X and J^sY, respectively, $N_r = \dim J^rY = n + \sum_{j=0}^{r} M_j = n + m\binom{n+r}{n}$, where $M_j = m\binom{n+j-1}{j}$. Moreover, we denote $P_r = \sum_{j=0}^{r-1} M_j + 2n - 1$. By γ and $J^r_x\gamma$ we denote a *section of the fibered manifold* Y (or *section of* π) and its r*-jet at* x, respectively. The mapping $J^r\gamma : x \to J^r\gamma(x) = J^r_x\gamma$ is the r*-jet prolongation* of γ. $\Gamma_\Omega(\pi)$ is the set of all sections of π defined on $\Omega \subset X$. Let (V, ψ), $\psi = (x^i, y^\sigma)$, $1 \leq i \leq n$ and $1 \leq \sigma \leq m$ be a *fibered chart* on Y. Then we denote (U, φ) and (V^r, ψ^r) the *associated chart* on X and *associated fibered chart* on J^rY, respectively. Here $U = \pi(V)$, $\varphi = (x^i)$, $1 \leq i \leq n$, $V^r = (\pi^{r,0})^{-1}(V)$, $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 \ldots j_r})$, $1 \leq j_1, \ldots, j_r \leq n$. The variables $y^\sigma_{j_1 \ldots j_k}$ are completely symmetrical in all indices contained in the multiindex $J = (j_1 \cdots j_k)$. The integer $k = |J|$ is the *length of the multiindex* J. (For y^σ the corresponding multiindex is considered to be of zero length.) Other kinds of multiindices used in the paper are of the form $\binom{\sigma}{J} = \binom{\sigma}{j_1 \ldots j_k}$, $0 \leq |J| \leq r$.

Let $\Omega^r_0 V$ be the ring of smooth functions on V^r. Denote by $\Omega^r_q V$ the $\Omega^r_0 V$*-module of smooth differential* q*-forms* on V^r, $\Omega^r_{q,c} V \subset \Omega^r_q V$, the submodule of *contact* q*-forms* (for $1 \leq q \leq n$) and *strongly contact* q*-forms* (for $n + 1 \leq q \leq N_r$), and $\mathrm{d}\Omega^r_{q-1,c} V \subset \Omega^r_q V$ the subset of exterior derivatives of contact (strongly contact) $(q-1)$-forms. Let $\Theta^r_q V = \mathrm{d}\Omega^r_{q-1,c} V + \Omega^r_{q,c} V$. For $2 \leq q \leq n$ it holds $\mathrm{d}\Omega^r_{q-1,c} V \subset \Omega^r_{q,c} V$, i.e. $\Theta^r_q V = \Omega^r_{q,c} V$, and of course, $\Theta^r_1 V = \Omega^r_1 V$. $\Theta^r_q V$ is trivial for $q > P_r$. In addition we denote by $\omega^\sigma_J = \mathrm{d}y^\sigma_J - y^\sigma_{Ji}\mathrm{d}x^i$, $0 \leq |J| \leq r - 1$, contact 1-forms, and by $\omega_i = (-1)^{i-1}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n$, $\omega_0 = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ the most frequently used horizontal forms. It holds $\mathrm{d}x^i \wedge \omega_i = \omega_0$ (without summation over i) and $\mathrm{d}\omega^\sigma_{j_1 \ldots j_{k-1}} \wedge \omega_{j_k} = -\omega^\sigma_{j_1 \ldots j_k} \wedge \omega_0$.

Any q-form $\varrho \in \Omega^r_q V$ is generated by forms $(\mathrm{d}x^i, \omega^\sigma_J, \mathrm{d}y^\sigma_I)$, $1 \leq i \leq n$, $0 \leq |J| \leq r - 1$, $|I| = r$. The notation ω^σ_J and $\mathrm{d}y^\sigma_I$ means that $\omega^\sigma_J = \omega^\sigma_{j_1 \ldots j_k}$ for $|J| = k$ and $\mathrm{d}y^\sigma_I = \mathrm{d}y^\sigma_{j_1 \ldots j_r}$.

3. Contact forms

In this section we review the definitions and basic properties of the contact and strongly contact forms on the r-jet prolongation of a fibered manifold. For a more detailed description and proofs the reader is referred to the fundamental papers of Krupka [13, 14].

Let us denote $\dim X = n$ and let $r \geq 0$ be an integer. We can assign to every vector $\xi \in TJ^{r+1}Y$ at a point $J^{r+1}_x\gamma \in J^{r+1}Y$ a tangent vector $h\xi \in TJ^rY$ at the point $J^r_x\gamma = \pi^{r+1,r}(J^{r+1}_x\gamma) \in J^rY$ by

$$h\xi = T_xJ^r\gamma \circ T\pi^{r+1} \cdot \xi.$$

The mapping $h : TJ^{r+1}Y \to TJ^rY$ defined by this formula is a vector bundle morphism over the jet projection $\pi^{r+1,r}$; we call h the *horizontalization*. The tangent vector $h\xi$ is π^r*-horizontal* and it is called the *horizontal component* of ξ. A tangent vector ξ is π^{r+1}*-vertical*, if and only if $h\xi = 0$. Using complementary construction, one can assign to every tangent vector $\xi \in TJ^{r+1}Y$ at a point $J^{r+1}_x\gamma \in J^{r+1}Y$ a tangent vector $p\xi \in TJ^rY$ at $J^r_x\gamma \in J^rY$ by

$$T\pi^{r+1,r} \cdot \xi = h\xi + p\xi,$$

$p\xi$ is a π^r-vertical vector, and ξ is $\pi^{r+1,r}$*-vertical* if and only if $h\xi = 0$, $p\xi = 0$.

Let $\xi \in TJ^{r+1}Y$ be a tangent vector at a point $J^{r+1}_x\gamma \in J^{r+1}Y$, and let (V, ψ), $\psi = (x^i, y^\sigma)$ be a fibered chart at the point $y = \gamma(x) \in V$. If ξ has an expression

$$\xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma_I \frac{\partial}{\partial y^\sigma_I},$$

with summation through all multiindices $I = (i_1, i_2, \dots, i_k)$ such that $0 \le i_1 \le i_2 \le \dots \le i_k$, $0 \le k \le r$, then

$$h\xi = \xi^i \left(\frac{\partial}{\partial x^i} + y^\sigma_{Ii} \frac{\partial}{\partial y^\sigma_I} \right), \qquad p\xi = (\Xi^\sigma_I - y^\sigma_{Ii}\xi^i)\frac{\partial}{\partial y^\sigma_I},$$

where Ii is the unique permutation $(j_1, j_2, \dots, j_{k+1})$ of the set $(i_1, i_2, \dots, i_k, i)$ such that $j_1 \le j_2 \le \dots \le j_{k+1}$.

For any open subset V of Y we denote, as in Section 2, $\Omega^r_0 V$ the ring of smooth functions on $V^r = (\pi^{r,0})^{-1}(V) \subset J^rY$. The $\Omega^r_0 V$-module of smooth differential q-forms on V^r is denoted by $\Omega^r_q V$. Let $V \subset Y$ be an open set. The horizontalization $h : TJ^{r+1}Y \to TJ^rY$ induces a decomposition of any q-form $\varrho \in \Omega^r_q V$, where $q \ge 1$, in the following sense: Let $\xi_1, \xi_2, \dots, \xi_q$ be tangent vectors to $J^{r+1}Y$ at a point $J^{r+1}_x\gamma \in V^{r+1}$. Let us decompose each of these vectors as above. The horizontal components all belong to a n-dimensional vector subspace of the vector space tangent to J^rY at $J^r_x\gamma$. Thus the unique decomposition exists

$$(\pi^{r+1,r})^*\varrho = \sum_{k=0}^{q} p_k\varrho,$$

$$\begin{aligned} &p_k\varrho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \dots, \xi_q) \\ &\quad = \frac{1}{(q-k)!k!}\,\varepsilon^{i_1 i_2 \dots i_q}\varrho(J^r_x\gamma)(h\xi_{i_1}, \dots, h\xi_{i_{q-k}}, p\xi_{i_{q-k+1}}, \dots, p\xi_{i_q}) \end{aligned}$$

$\varepsilon^{i_1,\dots,i_q}$ being the generalized Levi-Civitta symbol. Especially, $p_q\varrho(J^{r+1}_x\gamma)(\xi_1, \dots, \xi_q) = \varrho(J^r_x\gamma)(p\xi_1, \dots, p\xi_q)$.

It is, of course, evident, that for $q > n$ it holds $p_k\varrho = 0$ for $0 \le k \le q - n - 1$. Obviously, for any function $f \in \Omega^r_0 V$,

$$p_k(f\varrho) = (\pi^{r+1,r})^* f \cdot p_k\varrho, \quad 0 \le k \le q.$$

The form $p_k\varrho$ is called the *k-contact component* of the form ϱ. If $(\pi^{r+1,r})^* = \sum_{s=k}^{q} p_s\varrho$ for some k, $0 \le k \le q$, i.e. $p_0\varrho = \dots = p_{k-1}\varrho = 0$, the form ϱ is called *k-contact*. The number k is the *degree of contactness*. The 0-contact component of the form is called its *horizontal component* and it is denoted by $h\varrho$. The form $p\varrho = \sum_{k=1}^{q} p_k\varrho$ is called the *contact component* of the form. It holds

$$(\pi^{r+1,r})^*\varrho = h\varrho + p\varrho.$$

The form is called π^r*-horizontal* or *contact* if $(\pi^{r+1,r})^*\varrho = h\varrho$ (i.e. $p\varrho = 0$), or $(\pi^{r+1,r})^*\varrho = p\varrho$ (i.e. $h\varrho = 0$). For $q > n$ every q-form is contact. Let $q > n$. The form ϱ for which $p_{q-n}\varrho = 0$ is called *strongly contact*. Let $f : V^r \to \mathbf{R}$ be a function. Then we define

$$hf = (\pi^{r+1,r})^* f.$$

For any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$ it holds

$$hdx^i = dx^i,\ pdx^i = 0,\ hdy^\sigma_I = y^\sigma_{Ii} dx^i,\ pdy^\sigma_{Ii} = (\pi^{r+1,r})^* dy^\sigma_I - y^\sigma_{Ii} dx^i.$$

For the 1-dimensional base the decomposition of a k-form ϱ into its contact components is extremely simple:

$$(\pi^{r+1,r})^* \varrho = p_{k-1}\varrho + p_k \varrho.$$

Let us now present a brief review of basic properties of contact and strongly contact forms on $J^r Y$ in the coordinate form, adapted for practical purposes of our calculations. For a more detailed description and proofs the reader is referred to the fundamental papers of Krupka [13, 14]. The forms

$$\big(\mathrm{d}x^i, \omega^\sigma_{j_1}, \dots, \omega^\sigma_{j_1 \dots j_{r-1}}, \mathrm{d}y^\sigma_{j_1 \dots j_r}\big), \quad \text{where } \omega^\sigma_{j_1 \dots j_k} = \mathrm{d}y^\sigma_{j_1 \dots j_k} - y^\sigma_{j_1 \dots j_k i}\, \mathrm{d}x^i, \tag{1}$$

define the *contact base* of 1-forms on V^r. For a function $f \in \Omega^r_0 V$ we denote by $\mathrm{d}_i f$ its total derivative with respect to the variable x^i,

$$\mathrm{d}_i f = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^\sigma_J} y^\sigma_{Ji} = \mathrm{d}'_i f + \frac{\partial f}{\partial y^\sigma_I}\, y^\sigma_{Ii}, \quad 0 \le |J| \le r,\ |I| = r.$$

Lemma 1. *Let $W \subset Y$ be an open set, $q \ge$ an integer, and $\varrho \in \Omega^r_q V$ a q-form. Let (V, ψ) be a fibered chart on Y for which $W \subset V$. Let ϱ have the chart expression*

$$\varrho = \sum_{s=0}^{q} A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_s}_{\sigma_s,\, i_{s+1} i_{s+2} \dots i_q}\, \mathrm{d}y^{\sigma_1}_{I_1} \wedge \mathrm{d}y^{\sigma_2}_{I_2} \wedge \cdots \wedge \mathrm{d}y^{\sigma_s}_{I_s} \wedge \mathrm{d}x^{i_{s+1}} \wedge \mathrm{d}x^{i_{s+2}} \wedge \cdots \wedge \mathrm{d}x^{i_q} \tag{2}$$

with coefficients antisymmetrical in all multiindices $\big(\binom{I_1}{\sigma_1}, \dots, \binom{I_s}{\sigma_s}\big)$, $0 \le |I_p| \le r$, $1 \le p \le s$, antisymmetrical in all indices $(i_{s+1}, \dots, i_q)$ and symmetrical in all indices within each multiindex I_p. Then there exists the unique decomposition

$$(\pi^{r+1,r})^* \varrho = h\varrho + p\varrho = h\varrho + p_1\varrho + \cdots + p_q\varrho, \tag{3}$$

in which for every $1 \le k \le q$ it holds

$$\begin{aligned} p_k\varrho &= C^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_k}_{\sigma_k, i_{k+1} i_{k+2} \dots i_q}\, \omega^{\sigma_1}_{I_1} \wedge \omega^{\sigma_2}_{I_2} \wedge \cdots \wedge \omega^{\sigma_k}_{I_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \mathrm{d}x^{i_{k+2}} \wedge \cdots \wedge \mathrm{d}x^{i_q}, \\ C^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_k}_{\sigma_k,\, i_{k+1} i_{k+2} \dots i_q} &= \sum_{s=k}^{q} \binom{s}{k} A^{I_1 I_2}_{\sigma_1 \sigma_2} \cdots {}^{I_k}_{\sigma_k} \cdots {}^{I_s}_{\sigma_s,\, i_{s+1} i_{s+2} \dots i_q}\, y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} \cdots y^{\sigma_s}_{I_s i_s}, \\ &\mathrm{alt}\,(i_{k+1} i_{k+2} \cdots i_q). \end{aligned} \tag{4}$$

(Note, that the summations are taken over all independent choices of indices in each multiindex, e.g. $(i_1 \cdots i_p) = I$, $|I| = p$). The proof can be found in [13].

In our calculations we frequently use the $(q-n)$-contact component of a q^- form ϱ for $n < q \le N_r$. For $k = q - n$ the equation (4) gives

$$\begin{aligned} p_{q-n}\varrho &= C^{I_1}_{\sigma_1} \cdots {}^{I_{q-n}}_{\sigma_{q-n}, i_{q-n+1} \dots i_q}\, \varepsilon^{i_{q-n+1} \dots i_q}\, \omega^{\sigma_1}_{I_1} \wedge \cdots \wedge \omega^{\sigma_{q-n}}_{I_{q-n}} \wedge \omega_0 \\ &= B^{I_1}_{\sigma_1} \cdots {}^{I_{q-n}}_{\sigma_{q-n}}\, \omega^{\sigma_1}_{I_1} \wedge \cdots \wedge \omega^{\sigma_{q-n}}_{I_{q-n}} \wedge \omega_0. \end{aligned} \tag{5}$$

The following lemma describes the local structure of contact forms. (For the proof see [13, 14].)

Lemma 2. *Let* $W \subset Y$ *be an open set and* $\varrho \in \Omega^r_q W$ *a* q*-form. Let* (V, ψ) *be any fibered chart on* Y *for which* $V \subset W$*. Then*

(a) *for* $1 \le q \le n$ *the form* ϱ *is contact if and only if it can be expressed as*

$$\varrho = \Phi^J_\sigma\, \omega^\sigma_J \quad \textit{for } q = 1, \quad \textit{and} \quad \varrho = \omega^\sigma_J \wedge \Psi^J_\sigma + d\Psi \quad \textit{for } 2 \le q \le n, \tag{6}$$

where $\Phi^J_\sigma \in \Omega^r_0 V$ *are some functions,* $\Psi^J_\sigma \in \Omega^r_{q-1} V$ *some* $(q-1)$*-forms, and* $\Psi \in \Omega^r_{q-1} V$ *is a contact* $(q-1)$*-form which can be expressed as* $\omega^\sigma_I \wedge \chi^I_\sigma$ *for some* $(q-2)$*-forms* $\chi^I_\sigma \in \Omega^r_{q-2} V$, $0 \le |J| \le r-1$, $|I| = r-1$.

(b) *for* $n < q \le N_r$ *the form* ϱ *is strongly contact if and only if it can be expressed as*

$$\varrho = \omega^{\sigma_1}_{J_1} \wedge \cdots \wedge \omega^{\sigma_p}_{J_p} \wedge \mathrm{d}\omega^{\sigma_{p+1}}_{I_{p+1}} \wedge \cdots \wedge \mathrm{d}\omega^{\sigma_{p+s}}_{I_{p+s}} \wedge \Phi^{J_1 \dots J_p I_{p+1} \dots I_{p+s}}_{\sigma_1 \dots \sigma_p \sigma_{p+1} \dots \sigma_{p+s}}, \tag{7}$$

where $\Phi^{J_1 \dots J_p I_{p+1} \dots I_{p+s}}_{\sigma_1 \dots \sigma_p \sigma_{p+1} \dots \sigma_{p+s}} \in \Omega^r_{q-p-2s} V$, $0 \le |J_l| \le r-1$, $1 \le l \le p$, $|I_j| = r-1$, $p+1 \le j \le p+s$, *and summation is made over such all* p *and* s *for which* $p+s \ge q-n+1$, $p+2s \le q$.

4. Variational sequence

For the case of field theory we follow in this section the general ideas of Krupka [8] and basic concepts presented in [9, 10] for mechanics. Let Ω^r_q, $q \ge 0$, be the *direct image* of the sheaf of smooth q-forms over $J^r Y$ by the jet projection $\pi^{r,0}$ (functions are considered as 0-forms). Denote

$$\begin{aligned} &\Omega^r_{q,c} = \ker p_0 \quad \text{for } 1 \le q \le n, \quad \Omega^r_{q,c} = \ker p_{q-n} \quad \text{for } q > n \quad \text{and} \\ &\Theta^r_q = \Omega^r_{q,c} + \mathrm{d}\Omega^r_{q-1,c}, \end{aligned} \tag{8}$$

where p_0 and p_{q-n} are morphisms of sheaves induced by mappings p_0 and p_{q-n}, assigning to a form ϱ its horizontal and p_{q-n} contact component, respectively. $\mathrm{d}\Omega^r_{q-1,c}$ is the *image sheaf* of $\Omega^r_{q-1,c}$ by d. For every open set $W \subset Y$, $\Omega^r_q W$ is the Abelian group of q-forms on $W^r = (\pi^{r,0})^{-1}(W)$ and $\Omega^r_{q,c} W$ is the Abelian group of contact and strongly contact q-forms for $1 \le q \le n$ and $q > n$, respectively, expressed locally by Lemma 2. $\mathrm{d}\Omega^r_{q-1,c} W$ is the subgroup of $\Omega^r_q W$ given as $\{\varrho \in \Omega^r_q W | \varrho = \mathrm{d}\eta,\ \eta \in \Omega^r_{q,c} W\}$. Let us consider the sequence

$$\{0\} \to \Theta^r_1 \to \cdots \to \Theta^r_n \to \Theta^r_{n+1} \to \Theta^r_{n+2} \to \cdots \to \Theta^r_{P_r} \to \{0\}, \tag{9}$$

with arrows (except the first one) given by exterior derivatives d. The following lemma describes a basic property of this sequence.

Lemma 3. *Let* $W \subset Y$ *be an open set, and let* $\varrho \in \Theta^r_q W$ *be a form,* $1 \le q \le N_r$*. Then there exists the unique decomposition* $\varrho = \varrho_c + \mathrm{d}\bar{\varrho}_c$*, where* $\varrho_c \in \Omega^r_{q,c} W$ *and* $\bar{\varrho}_c \in \Omega^r_{q-1,c} W$.

Proof. For $1 \le q \le n$ it holds $\mathrm{d}\Omega^r_{q-1,c} V \subset \Omega^r_{q,c} V$, and thus only the case $q \ge n+1$ needs proof. Let $\varrho \in \Theta^r_q V$. Then it is evident that there exist forms $\varrho_c \in \Omega^r_{q,c} V$ and $\bar{\varrho}_c \in \Omega^r_{q-1,c} V$ such that $\varrho = \varrho_c + \mathrm{d}\bar{\varrho}_c$. Let $\varrho = 0$, i.e. $\varrho_c = -\mathrm{d}\bar{\varrho}_c$, i.e. $\mathrm{d}\varrho_c = 0$. We shall prove that both forms ϱ_c and $\bar{\varrho}_c$ vanish. Because ϱ_c is a strongly contact q-form, it holds $p_{q-n}\varrho_c = 0$.Then the chart expression of ϱ_c is of the form

$$(\pi^{r+1,r})^*\varrho_c = \sum_{k=q-n+1}^{q} A_{\sigma_1}^{J_1}\cdots{}^{J_k}_{\sigma_k,i_{k+1}\dots i_q}\, \omega^{\sigma_1}_{J_1} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{i_q},$$

where coefficients $A_{\sigma_1}^{J_1}\cdots{}^{J_k}_{\sigma_k,i_{k+1}\dots i_q} \in \Omega_0^{r+1}V$, $q-n+1 \leq k \leq q$ are antisymmetrical in multiindices $\left(\binom{J_1}{\sigma_1}, \dots, \binom{J_k}{\sigma_k}\right)$ and in indices $(i_{k+1}, \dots, i_q)$, and symmetrical in all indices within each multiindex J_p, $1 \leq p \leq k$. By the exterior derivative we obtain

$$\begin{aligned}
0 &= (\pi^{r+1,r})^*\mathrm{d}\varrho_c \\
&= \sum_{k=q-n+1}^{q} \mathrm{d}A_{\sigma_1}^{J_1}\cdots{}^{J_k}_{\sigma_k,i_{k+1}\dots i_q} \wedge \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{i_q} \\
&+ A_{\sigma_1}^{J_1}\cdots{}^{J_k}_{\sigma_k,i_{k+1}\dots i_q} \big(\mathrm{d}\omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} - \omega^{\sigma_1}_{J_1} \wedge \mathrm{d}\omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \\
&+ \cdots + (-1)^{k+1}\omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \mathrm{d}\omega^{\sigma_k}_{J_k}\big) \wedge \mathrm{d}x^{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{i_q}.
\end{aligned}$$

Taking into account that $\mathrm{d}\omega^{\sigma}_{J} = -\omega^{\sigma}_{Ji} \wedge \mathrm{d}x^i$ and rearranging the summations, we have for the k-contact component of ϱ_c the following expression:

$$\begin{aligned}
&(\pi^{r+2,r+1})^* p_k \mathrm{d}\varrho_c \\
&= (-1)^q \big(\mathrm{d}_{i_{q+1}} A_{\sigma_1}^{J_1}\cdots{}^{J_k}_{\sigma_k,i_{k+1}\dots i_q}\, \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{i_{q+1}} \\
&+ (-1)^q k A_{\sigma_1}^{J_1}\cdots{}^{J_k}_{\sigma_k,i_{k+1}\dots i_q}\, \omega^{\sigma_1}_{J_1 i_{q+1}} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{i_{q+1}}\big) \\
&+ (-1)^{k-1} \left(\frac{\partial A_{\sigma_1}^{J_1}\cdots{}^{J_{k-1}}_{\sigma_{k-1},i_{k+1}\dots i_{q+1}}}{\partial y^{\sigma_k}_{J_k}} \right)_{\mathrm{alt}\left(\binom{J_1}{\sigma_1}\cdots\binom{J_k}{\sigma_k}\right)} \omega^{\sigma_1}_{J_1} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \\
&\wedge \mathrm{d}x^{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{i_{q+1}}
\end{aligned}$$

for $q-n+2 \leq |J_k| \leq q+1$. For $k = q-n+1$ the last term is missing. All summations range over $0 \leq |J_p| \leq r$,$1 \leq p \leq k$, with the exception of the last term, in which $|J_k| = r+1$. Especially, for $k = q-n+1$ we obtain

$$\begin{aligned}
(\pi^{r+2,r+1})^* p_{q-n+1}\varrho_c &= (-1)^q \big(\mathrm{d}_{i_{q+1}} A_{\sigma_1}^{J_1}\cdots{}^{J_{q-n+1}}_{\sigma_{q-n+1},i_{q-n+2}\dots i_q}\, \omega^{\sigma_1}_{J_1} \wedge \cdots \wedge \omega^{\sigma_{q-n+1}}_{J_{q-n+1}} \\
&\wedge \mathrm{d}x^{i_{q-n+2}} \wedge \cdots \wedge \mathrm{d}x^{i_{q+1}} + (-1)^q (q-n+1) A_{\sigma_1}^{J_1}\cdots{}^{J_{q-n+1}}_{\sigma_{q-n+1},i_{q-n+2}\dots i_q}\, \omega^{\sigma_1}_{J_1 i_{q+1}} \\
&\wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_{q-n+1}}_{J_{q-n+1}} \wedge \mathrm{d}x^{i_{q-n+1}} \wedge \cdots \wedge \mathrm{d}x^{i_{q+1}}\big).
\end{aligned}$$

For $|J_1| = r$ the form $\omega^{\sigma_1}_{J_1 i_{q+1}}$ should be an element of $\Omega_1^{r+2}V$. Thus, taking into account the antisymmetry of coefficients, the condition $p_{q-n+1}\mathrm{d}\varrho_c = 0$ leads to the relation

$$A_{\sigma_1}^{J_1}\cdots{}^{J_{q-n+1}}_{\sigma_{q-n+1},i_{q-n+2}\dots i_q} = 0$$

as soon as any one of the multiindices $|J_p|$, $1 \leq p \leq k$ is of length r. We obtain the chart expression of ϱ_c as follows

$$\varrho_c = \sum_{k=q-n+1}^{q} A_{\sigma_1}^{J_1}\cdots{}^{J_k}_{\sigma_k,i_{k+1}\dots i_q}\, \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{i_q},$$

with $0 \leq |J_p| \leq r-1$. We can see that the form ϱ_c is ω-generated. Thus, in the expression for $(\pi^{r+2,r+1})^* p_k \mathrm{d}\varrho_c$ the summation is made over $0 \leq |J_p| \leq r-1$ the only exception being the term

$$(-1)^{k-1}\left(\frac{\partial A^{J_1}_{\sigma_1} \cdots {}^{J_{k-1}}_{\sigma_{k-1}, i_{k+1} \ldots i_{q+1}}}{\partial y^{\sigma_k}_{J_k}}\right)_{\mathrm{alt}\left(\binom{J_1}{\sigma_1} \cdots \binom{J_k}{\sigma_k}\right)} \omega^{\sigma_1}_{J_1} \wedge \cdots \wedge \omega^{\sigma_k}_{J_k}$$
$$\wedge\, \mathrm{d}x^{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{i_{q+1}}$$

in which it could be $|J_k| = r$. For $k = q-n+2$ it holds

$$\left(\frac{A^{J_1}_{\sigma_1} \cdots {}^{J_{q-n+1}}_{\sigma_{q-n+1}, i_{q-n+3} \cdots i_{q+1}}}{\partial y^{\sigma_{q-n+2}}_{J_{q-n+2}}}\right)_{\mathrm{alt}\left(\binom{J_1}{\sigma_1} \cdots \binom{J_{q-n+2}}{\sigma_{q-n+2}}\right)} = 0,$$

as soon as any one of the multiindices $J_1, \ldots, J_{q-n+2}$ is of length r. This is caused by the coefficients $A^{J_1}_{\sigma_1} \cdots {}^{J_{q-n+1}}_{\sigma_{q-n+1}, i_{q-n+3} \cdots i_{q+1}}$ being zero for some $|J_p| = r$. Expressing $(\pi^{r+2,r+1})^* p_{q-n+2} \mathrm{d}\varrho_c$ we can repeat the procedure and finally obtain $\varrho_c = 0$ (all coefficients are zeros). Then $\mathrm{d}\bar{\varrho}_c = 0$ and the same argumentation as for ϱ_c leads to the conclusion that $\bar{\varrho}_c = 0$. This finishes the proof.

Thus, the sequence (9) is an *exact subsequence* of the de Rham sequence

$$\{0\} \to \Omega^r_1 \to \cdots \to \Omega^r_n \to \Omega^r_{n+1} \to \Omega^r_{n+2} \to \cdots \to \Omega^r_{N_r} \to \{0\}.$$

The quotient sequence

$$\begin{aligned} &\{0\} \to \mathbf{R}_Y \to \Omega^r_0 \to \Omega^r_1/\Theta^r_1 \to \cdots \\ &\to \Omega^r_n/\Theta^r_n \to \Omega^r_{n+1}/\Theta^r_{n+1} \to \Omega^r_{n+2}/\Theta^r_{n+2} \to \cdots \\ &\to \Omega^r_{P_r}/\Theta^r_{P_r} \to \Omega^r_{P_r+1} \to \cdots \to \Omega^r_{N_r} \to \{0\} \end{aligned} \tag{10}$$

is called the *variational sequence of the r-th order*. It is, of course, also exact. We denote quotient mappings as follows

$$E^r_q : \Omega^r_q/\Theta^r_q \ni [\varrho] \longrightarrow E^r_q([\varrho]) = [\mathrm{d}\varrho] \in \Omega^r_{q+1}/\Theta^r_{q+1}. \tag{11}$$

The mappings E^r_n and E^r_{n+1} generalize the classical concept of Euler–Lagrange mapping and Helmholtz–Sonin mapping of calculus of variations, respectively. They represent "physically relevant" terms of the variational sequence.

Using the chart expressions of forms we can prove the following lemma:

Lemma 4. *Let $W \subset Y$ be an open set, and let $\varrho \in \Theta^{r+1}_q W$ be a form, $1 \leq q \leq N_r$. Let ϱ be $(\pi^{r+1,r})$-projectable, i.e. $\varrho = (\pi^{r+1,r})^* \eta$ for a form $\eta \in \Omega^r_q W$. Then η is an element of $\Theta^r_q W$.*

Proof. By hypothesis assume that $\varrho = (\pi^{r+1,r})^* \eta$. Aided by Lemma 3 we can further write $\varrho_c + \mathrm{d}\bar{\varrho}_c = (\pi^{r+1,r})^* \eta$. Taking the exterior derivative of this equation we obtain $\mathrm{d}\varrho_c = (\pi^{r+1,r})^* \mathrm{d}\eta$. Let us use the decomposition of $\mathrm{d}\eta$:

$$\mathrm{d}\varrho_c = \sum_{k=1}^{q+1} p_k \mathrm{d}\eta$$

and

$$p_k \mathrm{d}\varrho_c = (\pi^{r+2,r+1})^* p_k \mathrm{d}\eta = p_k \mathrm{d}\,(p_{k-1}\eta + p_k\eta)$$

using Lemma 3 of the second chapter in [13]. Applying this identity for $k = q+1, \dots, 1$ and using coordinate expressions (4) for $p_k\eta$ we recover (due to the fact that the expressions are polynomial in the jet coordinates y^ν_K, $|K| = r+1$) the $\pi^{r+1,r}$-projectability of ϱ_c. The complete result follows from linearity by reapplying the procedure to $\mathrm{d}\bar{\rho}_c$.

Let us consider the following scheme:

$$\begin{tikzcd}
\{0\} \arrow[r] & \Theta^{r+1}_q \arrow[r] & \Omega^{r+1}_q \arrow[r] & \Omega^{r+1}_q/\Theta^{r+1}_q \arrow[r] & \{0\} \\
\{0\} \arrow[r] & \Theta^{r}_q \arrow[r] \arrow[u] & \Omega^{r}_q \arrow[r] \arrow[u] & \Omega^{r}_q/\Theta^{r}_q \arrow[r] \arrow[u] & \{0\}
\end{tikzcd}$$

in which the first two “uparrows” represent the immersions by pullbacks and the third one defines the quotient mapping

$$Q^{r+1,r}_q : \Omega^r_q/\Theta^r_q \longrightarrow \Omega^{r+1}_q/\Theta^{r+1}_q.$$

Using Lemma 4 we can immediately see that the mapping $Q^{r+1,r}_q$ is injective. The (injective) mappings

$$Q^{s,r}_q : \Omega^r_q/\Theta^r_q \longrightarrow \Omega^s_q/\Theta^s_q, \quad r < s \tag{12}$$

can be defined in a quite analogous way.

The study of global properties of the variational sequence is based on the following facts proved by Krupka [8, 10]:

1. Each sheaf Ω^r_q is fine.
2. The variational sequence (in the shortened notation denoted by $\{0\} \to \mathbf{R}_Y \to \mathcal{V}$) is an acyclic resolution from the constant sheaf $\mathbf{R}_Y$ over Y.
3. For every $q \geq 0$ it holds $H^q(\Gamma(\mathbf{R}_Y, \mathcal{V})) = H^q(Y, \mathbf{R})$, where

$$\Gamma(Y, \mathcal{V}) : \{0\} \to \Gamma(Y, \mathbf{R}_Y) \to \Gamma(Y, \Omega^r_0) \to \Gamma(Y, \Omega^r_1)$$
$$\to \cdots \to \Gamma(Y, \Omega^r_{N_r}) \to \{0\}$$

is the cochain complex of global sections and $H^q(\Gamma(\mathbf{R}_Y, \mathcal{V}))$ denotes its q-th cohomology group.

5. Representation of the variational sequence

In this section we use the injectivity of mappings $Q^{s,r}_q$ to discuss the problem of the representation of the variational sequence by the appropriately chosen (exact) sequence of mappings of spaces of forms. Let W be an open subset of Y. Two q-forms $\varrho, \eta \in \Omega^r_q W$ belonging to the same class $\Omega^r_q W/\Theta^r_q W$ are called *equivalent*. Two q-forms $\varrho \in \Omega^r_q W$ and $\eta \in \Omega^t_q W$ are called *equivalent in the generalized sense* if there exists an integer $s \geq r, t$ for which $(\pi^{s,r})^*\varrho - (\pi^{s,t})^*\eta \in \Theta^s_q W$. Any mapping

$$\Phi^{s,r}_q : \Omega^r_q W/\Theta^r_q W \ni [\varrho] \longrightarrow \Phi^{s,r}_q([\varrho]) = \varrho_0 \in \Omega^s_q W$$

with $\varrho_0 \in [(\pi^{s,r})^*\varrho]$ (i.e. ϱ is equivalent with ϱ_0 in the generalized sense), is called *representation* of $\Omega^r_q W/\Theta^r_q W$. Because of the injectivity of mappings $Q^{s,r}_q$ (see Definition (12) and Lemma 4) the representation mappings $\Phi^{s,r}_q$ are injective too.

This injectivity enables us to define the *representation of the variational sequence* by forms as the lower row of the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^r_q/\Theta^r_q \arrow[r] \arrow[d] & \Omega^r_{q+1}/\Theta^r_{q+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Omega^s_q \arrow[r] & \Omega^s_{q+1} \arrow[r] & \cdots
\end{tikzcd}$$

in which the upper row is the variational sequence, the “downarrows” represent the mappings $\Phi^{s,r}_q$ and mappings of the lower row are defined by

$$(13) \qquad E^{s,r}_q : \Omega^s_q \longrightarrow \Omega^s_{q+1}, \quad E^{s,r}_q = \Phi^{s,r}_{q+1} \circ E^r_q \circ (\Phi^{s,r}_q)^{-1}, \quad E^{s,r}_0 = \Phi^{s,r}_1 \circ E^r_0.$$

In the following we shall show that there exists such a representation of the variational sequence (i.e. the integer $s \geq r$ and mappings $E^{s,r}_q$) for which $E^{s,r}_n$ assigns to every lagrangian of the r-th order its Euler–Lagrange form and $E^{s,t}_{n+1}$, $r \leq t \leq s$, assigns to every dynamical form on $J^t Y$ its Helmholtz–Sonin form. Such a representation will be called *physical*. It is given by following requirements for mappings $(\Phi^{s,r}_q)$:

$$(14) \qquad \Phi^{s,r}_n([\lambda]) = (\pi^{s,r})^*\lambda, \quad \Phi^{s,r}_{n+1}([\mathrm{d}\lambda]) = \mathcal{E}_\lambda, \quad \Phi^{s,t}(\mathrm{d}\mathcal{E}) = \mathcal{H}_\mathcal{E}.$$

The key Theorem 1 characterizes locally a representation of the r-th order variational sequence up to its physically relevant part, i.e. for $1 \leq q \leq n+2$. The Examples 1 and 2 succeeding this theorem show that the representation presented there is physical, i.e. it fulfills conditions (14).

Now, let us construct the mappings $\Phi^{s,r}_q$.

It is evident that for every q, for which $1 \leq q \leq n$, the q-forms ϱ and $h\varrho$ are equivalent in the generalized sense. Thus, the form $h\varrho$ can be considered as the (global) representative of the class $[\varrho]$ and we can define

$$(15) \qquad \Phi^{s,r}_q : \Omega^r_q/\Theta^r_q \ni [\varrho] \longrightarrow \Phi^{s,r}_q([\varrho]) = (\pi^{s,r+1})^* h\varrho \in \Omega^s_q,$$

for arbitrary $s \geq r+1$. Let $W \subset Y$ be an open set and let $\varrho \in \Omega^r_{n+1}W$. Let (V, ψ) be a fibered chart on Y such that $V \subset W$. We shall find a integer s and a form $\alpha \in \Omega^s_{n+1}V$, such that α belongs to the class $[(\pi^{s,r})^*\varrho]$, and $p_1\alpha$ is $\pi^{s+1,0}$-horizontal. The first mentioned condition means that α is of the form $\alpha - (\pi^{s,r})^*\varrho = \theta_c + \mathrm{d}\bar{\theta}_c$ for some $\theta_c \in \Omega^s_{n+1,c}V$, and some $\bar{\theta}_c \in \Omega^s_{n,c}V$. Then $p_1\alpha - (\pi^{s+1,r+1})^* p_1\varrho = p_1\mathrm{d}\bar{\theta}_c$. Let $(\pi^{s+1,s})^*\bar{\theta}_c$ be expressed in the fibered chart (V, ψ) as

$$(\pi^{s+1,s})^*\bar{\theta}_c = \sum_{k=0}^{s} Q^{j_1\dots j_k,i}_\sigma \omega^\sigma_{j_1\dots j_k} \wedge \omega_i + \sum_{l=2}^{n} p_l\bar{\theta}_c,$$

i.e.

$$p_1\mathrm{d}\bar{\theta}_c = \sum_{k=0}^{s} \big(h\,\mathrm{d}Q^{j_1\dots j_k,i}_\sigma \wedge \omega^\sigma_{j_1\dots j_k} \wedge \omega_i - Q^{(j_1\dots j_k,i)}_\sigma \omega^\sigma_{j_1\dots j_k i} \wedge \omega_0\big).$$

Coefficients $Q_\sigma^{j_1\cdots j_k,i}$ are elements of $\Omega_0^{r+1}V$ and $(j_1\ldots j_k, i)$ denotes the full symmetrization. Suppose $p_1\alpha$ to be of the form $p_1\alpha = \sum_{k=0}^s A_\sigma^{j_1\cdots j_k}\omega^\sigma_{j_1\cdots j_k} \wedge \omega_0$ and $p_1\varrho = \sum_{k=0}^s B_\sigma^{j_1\cdots j_k}\omega^\sigma_{j_1\cdots j_k} \wedge \omega_0$. Then we obtain

$$\begin{aligned}
&\sum_{k=0}^{s} A_\sigma^{j_1\cdots j_k}\omega^\sigma_{j_1\cdots j_k} \wedge \omega_0 - \sum_{k=0}^{r} B_\sigma^{j_1\cdots j_k}\omega^\sigma_{j_1\cdots j_k} \wedge \omega_0 \\
&\quad = -\mathrm{d}_i\, Q_\sigma^{,i}\omega^\sigma \wedge \omega_0 - \sum_{k=1}^{s} \left(\mathrm{d}_i\, Q_\sigma^{j_1\cdots j_k,i} + Q_\sigma^{(j_1\cdots j_k,j_{k+1})}\right)\omega^\sigma_{j_1\cdots j_k} \wedge \omega_0 \\
&\quad - Q_\sigma^{(j_1\cdots j_s,j_{s+1})}\omega_{j_1\cdots j_{s+1}} \wedge \omega_0,
\end{aligned}$$

which gives the following system of equations for coefficients $Q_\sigma^{(j_1\cdots j_k,i)}$:

$$Q_\sigma^{(j_1\cdots j_s,j_{s+1})} = 0 \quad \Rightarrow \quad Q_\sigma^{j_1\cdots j_s,j_{s+1}} = q_\sigma^{j_1\cdots j_s,j_{s+1}},$$

where $q_\sigma^{(j_1\cdots j_s,j_{s+1})} = 0$,

$$\begin{aligned}
&A_\sigma^{j_1\cdots j_k} + \mathrm{d}_i\, Q_\sigma^{j_1\cdots j_k,i} + Q_\sigma^{(j_1\cdots j_{k-1},j_k)} = 0 \quad \text{for } r+1 \le k \le s, \\
&\left(A_\sigma^{j_1\cdots j_k} - B_\sigma^{j_1\cdots j_k}\right) + \mathrm{d}_i\, Q_\sigma^{j_1\cdots j_k,i} + Q_\sigma^{(j_1\cdots j_{k-1},j_k)} = 0 \quad \text{for } 1 \le k \le r, \\
&A_\sigma - B_\sigma + \mathrm{d}_i\, Q_\sigma^{,i} = 0.
\end{aligned}$$

Solving this system we obtain step by step:

$$\begin{aligned}
&Q_\sigma^{(j_1\cdots j_{k-1},j_k)} = -\mathrm{d}_i\, Q_\sigma^{j_1\cdots j_k,i} - A_\sigma^{j_1\cdots j_k} \\
&\Rightarrow Q_\sigma^{j_1\cdots j_{k-1},j_k} = q_\sigma^{j_1\cdots j_{k-1},j_k} - \mathrm{d}_i\, Q_\sigma^{j_1\cdots j_k,i} - A_\sigma^{j_1\cdots j_k} \quad \text{for } r+1 \le k \le s,
\end{aligned}$$

where $q_\sigma^{(j_1\cdots j_k,i)} = 0$. Then

$$\begin{aligned}
&Q_\sigma^{j_1\cdots j_{s-1},j_s} = q_\sigma^{j_1\cdots j_{s-1},j_s} - \mathrm{d}_{j_{s+1}} q_\sigma^{j_1\cdots j_s,j_{s+1}} - A_\sigma^{j_1\cdots j_s}, \\
&Q_\sigma^{j_1\cdots j_{s-2},j_{s-1}} = q_\sigma^{j_1\cdots j_{s-2},j_{s-1}} - \mathrm{d}_{j_s} q_\sigma^{j_1\cdots j_{s-1},j_s} + \mathrm{d}_{j_s}\mathrm{d}_{j_{s+1}} q_\sigma^{j_1\cdots j_s,j_{s+1}} \\
&\quad - A_\sigma^{j_1\cdots j_{s-1}} + \mathrm{d}_{j_s} A_\sigma^{j_1\cdots j_s},
\end{aligned}$$

and recurrently

$$\begin{aligned}
Q_\sigma^{j_1\cdots j_{k-1},j_k} &= \sum_{l=0}^{s-k+1} (-1)^l \mathrm{d}_{j_{k+1}} \cdots \mathrm{d}_{j_{k+l}} q_\sigma^{j_1\cdots j_{k+l-1},j_{k+l}} \\
&- \sum_{l=1}^{s-k+1} (-1)^l \mathrm{d}_{j_{k+1}} \cdots \mathrm{d}_{j_{k+l-1}} A_\sigma^{j_1\cdots j_{k+l-1}}
\end{aligned}$$

for $r+1 \le k \le s$. Putting into this formula the expressions $(A_\sigma^{j_1\cdots j_k} - B_\sigma^{j_1\cdots j_k})$ instead of $A_\sigma^{j_1\cdots j_k}$ we obtain the corresponding relations for $1 \le k \le r$. Finally, for $k = 1$ we have

$$\begin{aligned}
Q_\sigma^{,j_1} &= \sum_{l=0}^{s}(-1)^l \mathrm{d}_{j_2} \cdots \mathrm{d}_{j_{l+1}} q_\sigma^{j_1\cdots j_l,j_{l+1}} \\
&- \sum_{l=1}^{s}(-1)^l \mathrm{d}_{j_2} \cdots \mathrm{d}_{j_l} \left(A_\sigma^{j_1\cdots j_l} - B_\sigma^{j_1\cdots j_l}\right),
\end{aligned}$$

where $B_\sigma^{j_1\cdots j_l}=0$ for $r+1\le k\le s$. Finally

$$A_\sigma - B_\sigma = -\mathrm{d}_{j_1} Q_\sigma^{,j_1}$$

and thus

$$A_\sigma - B_\sigma = \mathrm{d}_{j_1} q_\sigma^{,j_1} + \sum_{l=1}^{s}(-1)^l \mathrm{d}_{j_1}\cdots \mathrm{d}_{j_{l+1}} q_\sigma^{j_1\cdots j_l, j_{l+1}}$$

$$-\sum_{l=1}^{s}(-1)^l\big(A_\sigma^{j_1\cdots j_l} - B_\sigma^{j_1\cdots j_l}\big)=0.$$

Due to the symmetry of the operator $\mathrm{d}_{j_1}\cdots\mathrm{d}_{j_{l+1}}$ and the antisymmetry of $q_\sigma^{j_1\cdots j_l, j_{l+1}}$ it holds

$$\sum_{l=1}^{s}(-1)^l \mathrm{d}_{j_1}\cdots \mathrm{d}_{j_{l+1}} q_\sigma^{j_1\cdots j_l, j_{l+1}}=0.$$

Without any loss of generality we put $q_\sigma^{,j_1}=0$ and we finally obtain

$$\sum_{l=0}^{s}(-1)^l \mathrm{d}_{j_1}\cdots \mathrm{d}_{j_l}\big(A_\sigma^{j_1\cdots j_l} - B_\sigma^{j_1\cdots j_l}\big)=0.$$

The requirement of $\pi^{s+1,0}$-horizontality of the representative gives $A_\sigma^{j_1\cdots j_k}=0$ for $1\le k\le s$ and

$$A_\sigma = \sum_{l=0}^{r}(-1)^l \mathrm{d}_{j_1}\cdots \mathrm{d}_{j_l} B_\sigma^{j_1\cdots j_l}.$$

It is evident that the coefficients A_σ are elements of $\Omega_0^{2r+1}V$. The representative of the class $[\varrho]$ has the form

$$\sum_{l=0}^{r}(-1)^l \mathrm{d}_{j_1}\cdots \mathrm{d}_{j_l} B_\sigma^{j_1\cdots j_l}.$$

Now, let us apply the analogous construction for $q=n+2$. Let $\varrho\in\Omega_{n+2}^r V$. We wish t_0 find an integer s and a form $\alpha\in\Omega_{n+2}^s V$ such that $\alpha\sim[\pi^{s,r})^*\varrho]$, i.e. $\alpha-(\pi^{s,r})^*\varrho=\theta_c+\mathrm{d}\bar\theta_c$ for some forms $\theta_c\in\Omega_{n+2,c}^s V$ and $\bar\theta\in\Omega_{n+1,c}^s V$. This leads to the condition $p_2\alpha-(\pi^{s+1,r+1})^*p_2\varrho=p_2\,\mathrm{d}\bar\theta_c$. Suppose that in the fibered chart (V,ψ) the forms $p_2\alpha$, $p_2\varrho$ and $p_2\bar\theta_c$ have the following chart expressions:

$$p_2\alpha=\sum_{k,l=0}^{s} A_{\sigma\nu}^{(j_1\ldots j_k)(k_1\ldots k_l)}\omega_{j_1\ldots j_k}^\sigma\wedge\omega_{k_1\ldots k_l}^\nu\wedge\omega_0,$$

$$p_2\varrho=\sum_{k,l=0}^{r} B_{\sigma\nu}^{(j_1\ldots j_k)(k_1\ldots k_l)}\omega_{j_1\ldots j_k}^\sigma\wedge\omega_{k_1\ldots k_l}^\nu\wedge\omega_0,$$

$$p_2\bar\theta_c=\sum_{k,l=0}^{s} Q_{\sigma\nu}^{(j_1\ldots j_k)(k_1\ldots k_l),i}\omega_{j_1\ldots j_k}^\sigma\wedge\omega_{k_1\ldots k_l}^\nu\wedge\omega_i\,.$$

Then the requirement $p_2\alpha - (\pi^{s+1,r+1})^* p_2\varrho = p_2\,\mathrm{d}\bar{\theta}_c$, the fact that $p_1\mathrm{d}\bar{\theta}_c = 0$ and thus $(\pi^{s+1,r+1})^* p_2\mathrm{d}\bar{\theta}_c = p_2\mathrm{d}p_1\bar{\theta}_c + p_2\mathrm{d}p_2\bar{\theta}_c = p_2\mathrm{d}p_2\bar{\theta}_c$ gives

$$\begin{aligned}
&\sum_{k,l=0}^{s} \big(A_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l)} - B_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l)}\big)\ \omega^{\sigma}_{j_1\dots j_k} \wedge \omega^{\nu}_{k_1\dots k_l} \wedge \omega_0 \\
&\quad - \sum_{k,l=0}^{s} \big(\mathrm{d}_i\, Q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),i}\,\omega^{\sigma}_{j_1\dots j_k} \wedge \omega^{\nu}_{k_1\dots k_l} \wedge \omega_0 \\
&\quad + Q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),i}\,\omega^{\sigma}_{j_1\dots j_k i} \wedge \omega^{\nu}_{k_1\dots k_l} \wedge \omega_0 \\
&\quad + Q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),i}\,\omega^{\sigma}_{j_1\dots j_k} \wedge \omega^{\nu}_{k_1\dots k_l i} \wedge \omega_0\big) = 0,
\end{aligned}$$

where we consider $B_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l)} = 0$ as soon as any of indices $(j_1, \dots, j_k, k_1, \dots, k_l)$ exceeds r. After some calculations we obtain the following system of equations for coefficients $Q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l)}$:

$$\begin{aligned}
&A_{\sigma\nu} - B_{\sigma\nu} - \mathrm{d}_i\, Q_{\sigma\nu}^{,i} = 0, \\
&\big(A_{\sigma\nu}^{(j_1\dots j_k)(\,)} - B_{\sigma\nu}^{(j_1\dots j_k)(\,)} - \mathrm{d}_i\, Q_{\sigma\nu}^{(j_1\dots j_k)(\,),i} - Q_{\sigma\nu}^{(\underline{j}_1\dots \underline{j}_{k-1})(\,),\underline{j}_k}\big) = 0
\end{aligned}$$

for $1 \le k \le s$,

$$\begin{aligned}
&\big(A_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l)} - B_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l)} - \mathrm{d}_i\, Q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),i} \\
&\quad -\ Q_{\sigma\nu}^{(\underline{j}_1\dots \underline{j}_{k-1})(k_1\dots k_l),\underline{j}_k} - Q_{\sigma\nu}^{(j_1\dots j_k)(\underline{k}_1\dots \underline{k}_{l-1}),\underline{k}_l}\big) = 0
\end{aligned}$$

for $1 \le k, l \le s,\ l \le k$, and

$$Q_{\sigma\nu}^{(\underline{j}_1\dots \underline{j}_s)(\,),\underline{j}_{s+1}} = 0, \qquad Q_{\sigma\nu}^{(\underline{j}_1\dots \underline{j}_s)(k_1\dots k_l),\underline{j}_{s+1}} = 0.$$

The “underlines” under indices denote the symmetrization. So, $Q_{\sigma\nu}^{(\underline{j}_1\dots \underline{j}_{k-1})(k_1\dots k_l),\underline{j}_k}$ denotes that the symmetrization is made over indices $(j_1, \dots, j_{k-1}, j_k)$.

Now, we shall solve the presented equations: We can express the coefficients $Q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),\,j_{k+1}}$ as

$$Q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),\,j_{k+1}} = Q_{\sigma\nu}^{(\underline{j}_1\dots \underline{j}_k)(k_1\dots k_l),\underline{j}_{k+1}} + q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),\,j_{k+1}},$$

where $q_{\sigma\nu}^{(\underline{j}_1\dots \underline{j}_k)(k_1\dots k_l),\underline{j}_{k+1}} = 0$,

$$Q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),k_{l+1}} = Q_{\sigma\nu}^{(j_1\dots j_k)(\underline{k}_1\dots \underline{k}_l),\underline{k}_{l+1}} + \bar{q}_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),k_{l+1}},$$

where $\bar{q}_{\sigma\nu}^{(j_1\dots j_k)(\underline{k}_1\dots \underline{k}_l),\underline{k}_{l+1}} = 0$. Solving the equations for Q's we obtain recurrently

$$\begin{aligned}
&Q_{\sigma\nu}^{(j_1\dots j_s)(k_1\dots k_l),\,j_{s+1}} = q_{\sigma\nu}^{(j_1\dots j_s)(k_1\dots k_l),\,j_{s+1}}, \quad 1 \le l \le s, \\
&Q_{\sigma\nu}^{(j_1\dots j_s)(\,),\,j_{s+1}} = q_{\sigma\nu}^{(j_1\dots j_s)(\,),\,j_{s+1}} \quad \text{for}\ \ l = 0, \\
&Q_{\sigma\nu}^{(j_1\dots j_{s-1})(\,),\,j_s} = q_{\sigma\nu}^{(j_1\dots j_{s-1})(\,),\,j_s} - \mathrm{d}_{j_{s+1}} q_{\sigma\nu}^{(j_1\dots j_s)(\,),\,j_{s+1}} \\
&\quad + A_{\sigma\nu}^{(j_1\dots j_s)(\,)} - B_{\sigma\nu}^{(j_1\dots j_s)(\,)}, \\
&Q_{\sigma\nu}^{(j_1\dots j_{s-2})(\,),\,j_{s-1}} = q_{\sigma\nu}^{(j_1\dots j_{s-2})(\,),\,j_{s-1}} - \mathrm{d}_{j_s} q_{\sigma\nu}^{(j_1\dots j_{s-1})(\,),\,j_s} - \mathrm{d}_{j_{s+1}} q_{\sigma\nu}^{(j_1\dots j_s)(\,),\,j_{s+1}} \\
&\quad + \big(A_{\sigma\nu}^{(j_1\dots j_{s-1})(\,)} - B_{\sigma\nu}^{(j_1\dots j_{s-1})(\,)}\big) - \mathrm{d}_{j_s}\big(A_{\sigma\nu}^{(j_1\dots j_s)(\,)} - B_{\sigma\nu}^{(j_1\dots j_s)(\,)}\big),
\end{aligned}$$

$$\begin{aligned} Q_{\sigma\nu}^{(j_1\dots j_{k-1})(\,),j_k} &= \sum_{l=0}^{s-k+1} (-1)^l \mathrm{d}_{j_{k+1}} \cdots \mathrm{d}_{j_{k+l}} q_{\sigma\nu}^{(j_1\dots j_{k+l-1})(\,),j_{k+l}} \\ &- \sum_{l=1}^{s-k+1} (-1)^l \mathrm{d}_{j_{k+1}} \cdots \mathrm{d}_{j_{k+l-1}} \big(A_{\sigma\nu}^{(j_1\dots j_{k+l-1})(\,)} - B_{\sigma\nu}^{(j_1\dots j_{k+l-1})(\,)}\big), \\ Q_{\sigma\nu}^{(\,)(\,),j_1} &= \sum_{l=0}^{s} (-1)^1 \mathrm{d}_{j_2} \cdots \mathrm{d}_{j_{l+1}} q_{\sigma\nu}^{(j_1\dots j_l)(\,),j_{l+1}} \\ &- \sum_{l=1}^{s} (-1)^l \mathrm{d}_{j_2} \cdots \mathrm{d}_{j_l} \big(A_{\sigma\nu}^{(j_1\dots j_l)(\,)} - B_{\sigma\nu}^{(j_1\dots j_l)(\,)}\big). \end{aligned}$$

The last relation has been obtained for $k = 1$. Moreover, it holds

$$A_{\sigma\nu} - B_{\sigma\nu} - \mathrm{d}_{j_1} Q_{\sigma\nu}^{(\,)(\,),j_1} = 0,$$

and thus

$$\begin{aligned} &\sum_{l=0}^{s} \mathrm{d}_{j_1} \cdots \mathrm{d}_{j_l} \big(A_{\sigma\nu}^{(j_1\dots j_l)(\,)} - B_{\sigma\nu}^{(j_1\dots j_l)(\,)}\big) \\ &\quad - \sum_{l=0}^{s} (-1)^l \mathrm{d}_{j_1} \cdots \mathrm{d}_{j_{l+1}} q_{\sigma\nu}^{(j_1\dots j_l)(\,),j_{l+1}} = 0. \end{aligned}$$

Taking again into account the symmetry of the operator $\mathrm{d}_{j_1} \cdots \mathrm{d}_{j_{l+1}}$ and the antisymmetry of $q_{\sigma\nu}^{(j_1\dots j_l)(\,),j_{l+1}}$, and putting $q_{\sigma\nu}^{(\,)(\,),j_1} = 0$, we obtain

$$\sum_{l=0}^{s} \mathrm{d}_{j_1} \cdots \mathrm{d}_{j_l} \big(A_{\sigma\nu}^{(j_1\dots j_l)(\,)} - B_{\sigma\nu}^{(j_1\dots j_l)(\,)}\big) = 0.$$

Repeating the procedure for $Q_{\sigma\nu}^{(j_1\dots j_{k-1})(k_1),j_k}$ we obtain

$$\begin{aligned} Q_{\sigma\nu}^{(j_1\dots j_{k-1})(k_1),j_k} &= \sum_{l=0}^{s-k+1} (-1)^l \mathrm{d}_{j_{k+1}} \cdots \mathrm{d}_{j_{k+l}} q_{\sigma\nu}^{(j_1\dots j_{k+l-1})(k_1),j_{k+l}} \\ &+ \sum_{l=0}^{s-k+1} (-1)^l \mathrm{d}_{j_{k+1}} \cdots \mathrm{d}_{j_{k+l-1}} Q_{\sigma\nu}^{(j_1\dots j_{k+l-1})(\,),k_1} \\ &- \sum_{l=1}^{s-k+1} (-1)^l \mathrm{d}_{j_{k+1}} \cdots \mathrm{d}_{j_{k+l-1}} \big(A_{\sigma\nu}^{(j_1\dots j_{k+l-1})(k_1)} - B_{\sigma\nu}^{(j_1\dots j_{k+l-1})(k_1)}\big). \end{aligned}$$

$Q_{\sigma\nu}^{(j_1\dots j_{k+l-1})(\,),k_1}$ are determined by the proceeding set of relations. Finally, for $k = 1$ we obtain

$$\begin{aligned} Q_{\sigma\nu}^{(\,)(k_1),j_1} &= \sum_{l=0}^{s} (-1)^l \mathrm{d}_{j_2} \cdots \mathrm{d}_{j_{l+1}} q_{\sigma\nu}^{(j_1\dots j_l)(k_1),j_{l+1}} \\ &- \sum_{l=1}^{s} (-1)^l \mathrm{d}_{j_2} \cdots \mathrm{d}_{j_l} \big(A_{\sigma\nu}^{(j_1\dots j_l)(k_1)} - B_{\sigma\nu}^{(j_1\dots j_l)(k_1)}\big) \end{aligned}$$

$$+\sum_{l=1}^{s}(-1)^l \mathrm{d}_{j_2}\cdots\mathrm{d}_{j_l}\left(\sum_{p=0}^{s-l}(-1)^p \mathrm{d}_{j_{l+2}}\cdots\mathrm{d}_{j_{l+p+1}} q_{\sigma\nu}^{(j_1\dots j_l k_1 j_{l+2}\dots j_{l+p})(\,),\,j_{l+p+1}}\right.$$
$$\left.-\sum_{p=1}^{s-l}\mathrm{d}_{j_{l+2}}\cdots\mathrm{d}_{j_{l+p}}\left(A^{(j_1\dots j_l k_1 j_{l+2}\dots j_{l+p})(\,)}-B^{(j_1\dots j_l k_1 j_{l+2}\dots j_{l+p})(\,)}\right)\right).$$

Completing the procedure we finally obtain functions $Q_{\sigma\nu}^{(j_1\dots j_k)(k_1\dots k_l),i}$. For obtaining such a representative which fulfills the relation (14) for the specially chosen class $[\eta]$, where η is the exterior derivative of a $\pi^{r,0}$-horizontal form, we choose a form $\alpha\in[(\pi^{s,r})^*\varrho]$ with the 2-contact component given by the following chart expression

$$p_2\alpha = C_{\sigma\nu}^{j_1\dots j_k}\,\omega^{\sigma}_{j_1\dots j_k}\wedge\omega^{\nu}\wedge\omega_0.$$

Then the representative of the class $[\varrho]$ is

$$(\pi^{s,2r+1})^*\sum_{j=0}^{2r}\left[\sum_{p=0}^{j}\sum_{l=j-p}^{r}(-1)^l\binom{l}{j-p}\mathrm{d}_{i_{j+1}}\cdots\mathrm{d}_{i_{p+l}}\,B_{\sigma\nu}^{i_1\dots i_p,i_{p+1}\dots i_{p+l}}\right]\omega^{\sigma}_{i_1\dots i_j}$$
$$\wedge\,\omega^{\nu}\wedge\omega_0,$$

sym $(i_1\dots i_j)$, $s\geq 2r+1$, is the representation of $\Omega^r_{n+2}V/\Theta^r_{n+2}V$. So, we can formulate the following theorem:

Theorem 5. *Let $W\subset Y$ be an open set, and let $q\geq$ be an integer. Let (V,ψ) be a fibered chart on Y for which $V\subset W$.*

(a) *Let $1\leq q\leq n$ and let $\varrho\in\Omega^r_q W$ be a form. Then the mapping*

$$\text{(16)}\qquad \Phi^{s,r}_q:\Omega^r_q V/\Theta^r_q V\ni\varrho\longrightarrow\Phi^{s,r}_q([\varrho])=(\pi^{s,r})^*h\varrho\in\Omega^s_q V,\quad s\geq r+1$$

is the representation of $\Omega^r_q V/\Theta^r_q V$.

(b) *Let $q=n+1$ and let $\varrho\in\Omega^r_{n+1}W$ be a form expressed in the fibered chart (V,ψ) by the relation*

$$\text{(17)}\qquad p_1\varrho = B^J_\sigma\,\omega^\sigma_J\wedge\omega_0,$$

in which coefficients $B^J_\sigma\in\Omega^{r+1}_0 V$, $0\leq|J|\leq r$, are given by the chart expression of ϱ following eqs. $(2-5)$. *Then the mapping*

$$\Phi^{s,r}_{n+1}:\Omega^r_{n+1}V/\Theta^r_{n+1}V\ni\varrho\longrightarrow\Phi^{s,r}_{n+1}([\varrho])=\varrho_0\in\Omega^s_{n+1}V,\quad s\geq 2r+1$$

assigning to the class $[\varrho]$ the form

$$\text{(18)}\qquad \varrho_0=(\pi^{s,2r+1})^*\left(\sum_{l=0}^{r}(-1)^l\mathrm{d}_{j_1}\cdots\mathrm{d}_{j_l}B^{j_1\cdots j_l}_\sigma\right)\omega^\sigma\wedge\omega_0$$

is the representation of $\Omega^r_{n+1}V/\Theta^r_{n+1}V$.

(c) *Let $q=n+2$ and let $\varrho\in\Omega^r_{n+2}W$ be a form expressed in the fibered chart (V,ψ) by the relation*

$$\text{(19)}\qquad p_2\varrho = B^{JK}_{\sigma\nu}\,\omega^\sigma_J\wedge\omega^\nu_K\wedge\omega_0,$$

in which coefficients $B^{JK}_{\sigma\nu} \in \Omega^{r+1}_0 V$, $0 \le |J| \le r$, *are given by the chart expression of* ϱ *following eqs.* $(2-5)$. *Then the mapping*

$$\Phi^{s,r}_{n+2} : \Omega^r_{n+2} V/\Theta^r_{n+2} V \ni \varrho \longrightarrow \Phi^{s,r}_{n+2}([\varrho]) = \varrho_0 \in \Omega^s_{n+2} V, \quad s \ge 2r+1$$

assigning to the class $[\varrho]$ *the form*

$$\begin{aligned}\varrho_0 = (\pi^{s,2r+1})^* \sum_{j=0}^{2r} \Bigg[\sum_{p=0}^{j} \sum_{l=j-p}^{r} (-1)^l \binom{l}{j-p} \mathrm{d}_{i_{j+1}} \cdots \mathrm{d}_{i_{p+l}} \\ \times B^{i_1 \ldots i_p, i_{p+1} \ldots i_{p+l}}_{\sigma\nu} \Bigg] \omega^\sigma_{i_1 \cdots i_j} \wedge \omega^\nu \wedge \omega_0,\end{aligned} \tag{20}$$

sym $(i_1 \ldots i_j)$, $s \ge 2r+1$, *is the representation of* $\Omega^r_{n+2} V/\Theta^r_{n+2} V$.

Proof. The proof is constructive and precedes the stated theorem.

The representative (18) of a class $[\varrho]$ of $(n+1)$-forms generated by ϱ is called *Euler–Lagrange form* of the class $[\varrho]$. The representative (20) of a class $[\varrho]$ of $(n+2)$-forms generated by ϱ is called its *Helmholtz–Sonin form*. Following the relation (13) which defines the representation of the variational sequence we can use Theorem 1 for a form $\mathrm{d}\varrho$, $\varrho \in \Omega^r_n W$ or $\varrho \in \Omega^r_{n+1} W$, for obtaining the chart expressions of *Euler–Lagrange* and *Helmholtz–Sonin* mappings $E^{s,r}_n$ and $E^{s,r}_{n+1}$, respectively. These mappings represent the generalization of the well-known “classical” Euler–Lagrange and Helmholtz–Sonin mappings of the calculus of variations.

Example 1. Let $W \subset Y$ be an open set. Let $\lambda \in \Omega^r_n W$ be a lagrangian given in a fibered chart (V, ψ), $V \subset W$, by the expression

$$\lambda = \mathcal{L}\,\omega_0, \qquad \mathcal{L} \in \Omega^r_0 V.$$

Using Theorem 1(b) we obtain immediately

$$\mathcal{E}_\lambda = \Phi^{2r,r}_{n+1}([\mathrm{d}\lambda]) = \left(\sum_{l=0}^{r} (-1)^l \mathrm{d}_{j_1} \cdots \mathrm{d}_{j_l} \frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1 \ldots j_l}} \right) \omega^\sigma \wedge \omega_0 \tag{21}$$

which is evidently the Euler–Lagrange form of the lagrangian λ.

More generally,let $\varrho \in \Omega^r_n W$ be a form and $[\varrho]$ its class represented by the horizontal form $\lambda_\varrho = \Phi^{r+1,r}_n([\varrho])$. λ_ϱ has the chart expression

$$\lambda_\varrho = h\varrho = \mathcal{L}_\varrho\,\omega_0, \qquad \mathcal{L} \in \Omega^{r+1}_0 V,$$

where $\mathcal{L}_\varrho$ is affine in variables y^σ_{r+1}. Using Lemma 4 and Theorem 1(b) we obtain immediately

$$\Phi^{2r+1,r}_{n+1}([\mathrm{d}\varrho]) = \Phi^{2r+1,r+1}_{n+1}([\mathrm{d}\lambda_\varrho]) = \mathcal{E}_{\lambda_\varrho},$$

where $\mathcal{E}_{\lambda_\varrho}$ is determined by the function $\mathcal{L}_\varrho$ following the equation (21) for $s = 2r+1$ instead of $2r$.

Example 2. Now, let $\eta \in \Omega^r_n W$ be a generally chosen n-form, i.e. $[\eta] \in \Omega^r_n W/\Theta^r_n W$. Let (V, ψ) be a fibered chart on Y for which $V \subset W$. We have

$$\Phi^{r+1,r}_n([\eta]) = h\eta = \mathcal{L}\omega_0,$$

where $\mathcal{L} \in \Omega_0^{r+1} V$. We shall find the representative (18) of the class $[\eta]$. We have

$$
\begin{aligned}
&(\pi^{r+1,r})^* \, \mathrm{d}\eta = \mathrm{d}\left(h\eta + p_1\eta + \sum_{k=2}^{n} p_k \eta \right), \\
&(\pi^{r+2,r+1})^* p_1 \, \mathrm{d}\eta = p_1 \, \mathrm{d}(h\eta + p_1\eta).
\end{aligned}
$$

Taking into account the chart expression of $(h\eta + p_1\eta)$ in the form

$$
h\eta + p_1\eta = \mathcal{L}\omega_0 + \sum_{k=0}^{r} P_\sigma^{j_1 \cdots j_k, i} \omega^\sigma_{j_1 \cdots j_k} \wedge \omega_i ,
$$

where $P_\sigma^{j_1 \cdots j_k, i} \in \Omega_0^{r+1} V$, we obtain

$$
\begin{aligned}
p_1 \, \mathrm{d}(h\eta + p_1\eta) = {} & \left(\frac{\partial \mathcal{L}}{\partial y^\sigma} - \mathrm{d}_i P_\sigma^{\ i} \right) \omega^\sigma \wedge \omega_0 \\
& + \sum_{k=1}^{r} \left(\frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1 \cdots j_k}} - \mathrm{d}_i P_\sigma^{j_1 \cdots j_k, i} + P_\sigma^{(j_1 \cdots j_{k-1}, j_k)} \right) \omega^\sigma_{j_1 \cdots j_k} \wedge \omega_0 \\
& + \left(\frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1 \cdots j_{r+1}}} - P_\sigma^{(j_1 \cdots j_r, j_{r+1})} \right) \omega^\sigma_{j_1 \cdots j_{r+1}} \wedge \omega_0 .
\end{aligned}
$$

Then the representative (18) is

$$
\Phi^{2r+1,r}_{n+1}([\mathrm{d}\eta]) = \left(\sum_{l=0}^{r+1} (-1)^l \mathrm{d}_{j_1} \cdots \mathrm{d}_{j_l} B_\sigma^{j_1 \cdots j_l} \right) \omega^\sigma \wedge \omega_0 ,
$$

where

$$
\begin{aligned}
B_\sigma &= \frac{\partial \mathcal{L}}{\partial y^\sigma} - \mathrm{d}_i P_\sigma^{\ ,i} , \\
B_\sigma^{j_1 \cdots j_l} &= \frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1 \cdots j_l}} - \mathrm{d}_i P_\sigma^{j_1 \cdots j_l, i} - P_\sigma^{(j_1 \cdots j_{l-1}, j_l)}, \quad \text{for } 1 \le l \le r, \\
B_\sigma^{j_1 \cdots j_{r+1}} &= \frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1 \cdots j_{r+1}}} - P_\sigma^{(j_1 \cdots j_r, j_{r+1})} .
\end{aligned}
$$

Taking into account that

$$
P_\sigma^{j_1 \cdots j_l, j_{l+1}} = P_\sigma^{(j_1 \cdots j_l, j_{l+1})} + p_\sigma^{j_1 \cdots j_l, j_{l+1}} ,
$$

where $p_\sigma^{(j_1 \cdots j_l, j_{l+1})} = 0$ and calculating the representative we obtain

$$
\begin{aligned}
\Phi^{2r+1,r}_{n+1}([\mathrm{d}\eta]) = {} & \sum_{l=0}^{r+1} (-1)^l \mathrm{d}_{j_1} \cdots \mathrm{d}_{j_l} \left(\frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1 \cdots j_l}} \right) \\
& + \sum_{l=2}^{r+2} (-1)^l \mathrm{d}_{j_1} \cdots \mathrm{d}_{j_l} p_\sigma^{j_1 \cdots j_{l-1}, j_l} .
\end{aligned}
$$

The second sum vanishes because of the symmetry of the operator $\mathrm{d}_{j_1} \cdots \mathrm{d}_{j_l}$ and the antisymmetry of functions $p_\sigma \, j_1 \cdots j_{l-1}, \, j_l$.

Finally

$$\Phi^{2r+1,r}_{n+1}([\mathrm{d}\eta]) = \sum_{l=0}^{r+1}(-1)^l \mathrm{d}_{j_1}\cdots\mathrm{d}_{j_l}\left(\frac{\partial\mathcal{L}}{\partial y^\sigma_{j_1\dots j_l}}\right).$$

On the other hand, it holds $p_1\,\mathrm{d}\Theta_{h\eta} = \mathcal{E}_{h\eta}$, where $\Theta_{h\eta}$ is a Lepagean equivalent of the lagrangian $h\eta = \mathcal{L}\omega_0$, and $\mathcal{E}_{h\eta}$ is its Euler–Lagrange form. Thus

$$\Phi^{2r+1,r}_{n+1}(\mathrm{d}\eta]) = p_1\,\mathrm{d}\Theta_{h\eta} = \mathcal{E}_{h\eta}.$$

This example shows that the representative of $\mathrm{d}\eta$ for an arbitrarily chosen n-form η (not necessarily a lagrangian) is directly obtained as the 1-contact component of the exterior derivative of a Lepagean equivalent of the corresponding lagrangian $h\eta$.

Example 3. Let $W \subset Y$ be an open set. Let $\mathcal{E} \in \Omega^r_{n+1}W$ be a dynamical form given in the fibered chart (V, ψ), $V \subset W$, by the expression

$$\mathcal{E} = \varepsilon_\sigma\,\omega^\sigma \wedge \omega_0, \qquad \varepsilon_\sigma \in \Omega^r_0 V.$$

Then

$$\varrho = \mathrm{d}\mathcal{E} = \sum_{0\le|J|\le r}\frac{\partial\varepsilon_\nu}{\partial y^\sigma_J}\,\omega^\sigma_J \wedge \omega^\nu \wedge \omega_0.$$

On the other hand, in general, we have

$$p_2\varrho = B^{JK}_{\sigma\nu}\omega^\sigma_J \wedge \omega^\nu_K \wedge \omega_0, \qquad B^{JK}_{\sigma\nu} + B^{KJ}_{\nu\sigma} = 0.$$

Thus,

$$B^{0J}_{\sigma\nu} = -B^{J0}_{\nu\sigma} = -\frac{1}{2}\frac{\partial\varepsilon_\sigma}{\partial y^\nu_J}, \qquad J = (j_1\cdots j_k),\ 1 \le k \le r,$$

$$B^{00}_{\sigma\nu} = -B^{00}_{\nu\sigma} = \left(\frac{\partial\varepsilon_\nu}{\partial y^\sigma}\right)_{\mathrm{alt}(\sigma\nu)},$$

other coefficients $B^{JK}_{\sigma\nu}$ being zero. Using Theorem 1(c) we obtain

$$\begin{aligned}\mathcal{H}_{\mathcal{E}} = \Phi^{2r+1,r}_{n+1}([\mathrm{d}\mathcal{E}]) = \frac{1}{2}\Bigg[\sum_{j=0}^{2r}\Bigg(&\frac{\varepsilon_\nu}{\partial y^\sigma_{i_1\dots i_j}} - (-1)^j\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{i_1\dots i_j}}\\ &- \sum_{l=j+1}^{r}(-1)^l\binom{l}{j}\mathrm{d}_{i_{j+1}}\cdots\mathrm{d}_{i_l}\frac{\partial\varepsilon_\sigma}{\partial y^\nu_{i_1\dots i_l}}\Bigg)\Bigg]\omega^\sigma_{i_1\dots i_j}\wedge\omega^\nu\wedge\omega_0,\end{aligned} \tag{22}$$

which is the Helmholtz–Sonin form of the dynamical form $\mathcal{E}$.

More generally, let $\varrho \in \Omega^r_{n+1}W$ be a form and $[\varrho]$ its class represented by the dynamical form

$$\mathcal{E}_\varrho = \Phi^{2r+1,r}_{n+1}([\varrho]) = (\varepsilon_\varrho)_\sigma\omega^\sigma \wedge \omega_0, \qquad (\varepsilon_\varrho)_\sigma \in \Omega^{2r+1}_0 V,$$

given by (18). Using Lemma 4 and Theorem 1(c) we can obtain

$$\Phi^{s,r}_{n+2}([\mathrm{d}\varrho]) = \Phi^{s,2r+1}_{n+2}([\mathrm{d}\mathcal{E}_\varrho]) = \mathcal{H}_{\mathcal{E}_\varrho}, \qquad s \ge 2r+1.$$

These results are in agreement with those of Krupka (see [8]) and Kašparová ([5] for the 1-st order variational sequence). Examples 1 and 2 show that the obtained representation of the variational sequence fulfills the requirement (14), i.e. it is physical.

6. Global properties of the representation

The construction of the representative mappings $\Phi^{s,r}_q$ in the previous section for $1 \le q \le n$ is given by the horizontalization h, and thus, it is global. For $q = n+1$ the globality of the definition of the representatives of the type (18) is mentioned in [1] with the reference to a proof using an integration method. For the 1-st order variational sequence the globality of representatives (18) and (20) was proved in [4, 6], with the use of the integration of appropriately chosen forms. Note that the construction method given for representatives preceding Theorem 1 is manifestly correct since it is given by subtraction of globally defined differential forms. In this section though we follow the idea of the integration method to prove the correctness (globality) of higher order representatives (18) and, as a new result, (20).

Theorem 6. *Let* (V, ψ) *be a fibered chart on* Y. *Let* $1 \le q \le n+2$ *and* $\varrho \in \Omega^r_q Y$ *be a form. Then the class* $[\varrho]$ *is represented by eqs.* (16), (18) *and* (20) *globally, for* $1 \le q \le n$, $q = n+1$ *and* $q = n+2$, *respectively.*

Proof. Because of globality of the horizontalization mapping h only the cases $q = n+1,\ n+2$ need proof. Let Ω be a piece of manifold X.

Let $q = n+1$ and let $\varrho \in \Omega^r_{n+1} W$ be a form with the chart expression given by eqs. (2–5), (17), i.e.

$$(\pi^{r+1,r})^*\varrho = B^J_\sigma\, \omega^\sigma_J \wedge \omega_0 + \sum_{k=2}^{n+1} p_k\varrho, \quad \text{summation over} \quad 0 \le |J| \le r. \tag{23}$$

Let ξ be a π-vertical vector field such that $\operatorname{supp}\xi \subset \pi^{-1}(\Omega)$, and let $\xi = \xi^\sigma(\partial/\partial y^\sigma)$ be its chart expression in (V, ψ). Let us define (for $s \ge r$, in general)

$$\eta_\Omega = \int_\Omega J^s\gamma^* \circ (\pi^{s,r+1})^* h i_{J^r\xi}\varrho$$

Using the fact that ξ is vertical we obtain

$$\eta_\Omega = \int_\Omega J^s\gamma^* \circ (\pi^{s,r+1})^* i_{J^{r+1}\xi} p_1\varrho.$$

Further

$$\int_\Omega J^s\gamma^* \circ (\pi^{s,r+1})^* \left(B^J_\sigma \cdot \mathrm{D}_J\xi^\sigma\right)\omega_0,$$

We have denoted by D_J the symbol $\mathrm{d}_{j_1}\cdots\mathrm{d}_{j_k}$ for $J = (j_1 \cdots j_k)$, $1 \le k \le r$. Due to the properties of total derivative, the operator D_J is symmetrical in all indices contained in multiindex J. By the properties of the pullback mapping it holds

$$\eta_\Omega = \int_\Omega (B^J_\sigma \cdot \mathrm{D}_J\xi^\sigma)(J^{r+1}\gamma)\,\omega_0,$$

Using recursively the relation

$$\begin{aligned}\big((f\mathrm{d}_j g)\circ J^{r+1}\gamma\big)\,\omega_0 &= \big((\mathrm{d}_j(fg) - g\mathrm{d}_j f)\circ J^{r+1}\gamma\big)\,\omega_0\\ &= \big(\mathrm{d}_i(fg)\circ J^{r+1}\gamma\big)\delta^i_j\,\omega_0 - \big((g\mathrm{d}_j f)\circ J^{r+1}\gamma\big)\,\omega_0\\ &= \big(\mathrm{d}_i(fg)\circ J^{r+1}\gamma\big)\,\mathrm{d}x^i\wedge\omega_j - \big((g\mathrm{d}_j f)\circ J^{r+1}\gamma\big)\,\omega_0\\ &= J^{r+1}\gamma^*\big((\pi^{r+1,r})^*\mathrm{d}((fg)\wedge\omega_j) - (g\mathrm{d}_j f)\,\omega_0\big)\end{aligned}$$

for functions f, g, Stokes theorem and the assumption concerning the support of ξ we have

$$\begin{aligned}\eta_\Omega &= \int_\Omega\left(\xi^\sigma\cdot\sum_{l=1}^{r}(-1)^l\mathrm{d}_{j_1}\cdots\mathrm{d}_{j_l}B_\sigma^{j_1\dots j_l}\right)(J^{2r+1}\gamma)\,\omega_0\\ &= \int_\Omega\left(\sum_{l=1}^{r}(-1)^l\mathrm{d}_{j_1}\cdots\mathrm{d}_{j_l}B_\sigma^{j_1\dots j_l}\right)\omega^\sigma(J^{2r+1}\xi)(J^{2r+1}\gamma)\wedge\omega_0,\end{aligned}$$

$$\eta_\Omega = \int_\Omega (J^{2r+1}\gamma)^* i_{J^{2r+1}\xi}\varrho_0. \tag{24}$$

Since this expression was defined in a coordinate-free way the expression inside of the integral defines the representative ϱ_0 of a form ϱ globally and the representation mapping is thus defined correctly.

Let $q = n+2$ and let $\varrho\in\Omega^r_{n+2}Y$ be a form, for which

$$\pi^{r+1,r}\varrho = B^{JK}_{\sigma\nu}\,\omega^\sigma_J\wedge\omega^\nu_K\wedge\omega_0 + \sum_{k=3}^{n+2}p_k\varrho, \tag{25}$$

with coefficients $B^{JK}_{\sigma\nu}$ given by (2–5). Let ζ be another vector field which fulfills the same conditions as ξ. We define

$$\eta_\Omega = \int_\Omega J^s\gamma^*\circ(\pi^{s,r+1})^* h i_{J^r\xi} i_{J^r\zeta}\varrho. \tag{26}$$

Then

$$\begin{aligned}\eta_\Omega &= \int_\Omega J^s\gamma^*\circ(\pi^{s,r+1})^* i_{J^{r+1}\xi} i_{J^{r+1}\zeta} p_2\varrho\\ &= \int_\Omega J^s\gamma^*(\pi^{s,r+1})^*\left(2\xi^\sigma_J\,\zeta^\nu_K\,B^{JK}_{\sigma\nu}\right)\,\omega_0\\ &= \int_\Omega(\mathrm{D}_J\xi^\sigma)\left(2B^{JK}_{\sigma\nu}\cdot\mathrm{D}_K\zeta^\nu\right)(J^{r+1}\gamma)\,\omega_0,\end{aligned}$$

with the operator D_J previously defined as $\mathrm{d}_{j_1}\cdots\mathrm{d}_{j_k}$, $J = (j_1\cdots j_k)$. Applying the procedure used for $q = n+1$ in the first part of the proof to the n-form $(2B^{JK}_{\sigma\nu}D_K\zeta^\nu\,\omega_0)$ we have

$$\eta_\Omega = \int_\Omega 2(-1)^{|J|}\left(\xi^\sigma\mathrm{D}_J(B^{JK}_{\sigma\nu}\cdot\mathrm{D}_K\zeta^\nu)\right)(J^{2r+1}\gamma)\,\omega_0,$$

summation over $|J|, |K| \leq r$. Calculating the expression $\mathrm{D}_J(B^{JK}_{\sigma\nu}\mathrm{D}_K\zeta^\nu)$ step by step we obtain

$$\eta_\Omega = \int_\Omega \left(2\xi^\sigma \sum_{|J|\leq r} \sum_{|K|\leq r} (-1)^{|J|} \sum_{|J_1|+|J_2|=|J|} \mathrm{D}_{K+J_1}\zeta^\nu \, \mathrm{D}_{J_2} B^{JK}_{\sigma\nu} \right) \circ (J^{2r+1}\gamma)\,\omega_0,$$

summation over $|J|, |K| \leq r$.

$$\eta_\Omega = \int_\Omega \left(2\xi^\sigma \sum_{j=1}^{r} \sum_{k=0}^{r} (-1)^j \sum_{p+l=j} \binom{j}{p} \mathrm{d}_{i_1} \cdots \mathrm{d}_{i_{k+p}}\zeta^\nu \times \mathrm{d}_{i_{k+p+1}} \cdots \mathrm{d}_{i_{k+j}} B^{i_{k+1}\dots i_{k+j}, i_1\dots i_k}_{\sigma\nu} \right) (J^s\gamma)\,\omega_0,$$

sym $(i_1, \dots, i_{k+j})$. Rearranging the summations we obtain

$$\eta_\Omega = 2\int_\Omega \left(\sum_{j=0}^{2r} \sum_{k=0}^{r} \sum_{l=j-k}^{r} (-1)^{-l} \binom{l}{j-k} \mathrm{d}_{i_{j+1}} \cdots \mathrm{d}_{i_{k+l}} \times B^{i_1\dots i_k, i_{k+1}\dots i_{k+l}}_{\sigma\nu} \xi^\sigma_{i_1\dots i_j} \zeta^\nu \right) (J^{2r+1}\gamma)\,\omega_0 \tag{27}$$

$$\eta_\Omega = \int_\Omega \left(\sum_{j=0}^{2r} \sum_{k=0}^{r} \sum_{l=j-k}^{r} (-1)^{-l} \binom{l}{j-k} \mathrm{d}_{i_{j+1}} \cdots \mathrm{d}_{i_{k+l}} \, B^{i_1\dots i_k, i_{k+1}\dots i_{k+l}}_{\sigma\nu} \right) \times \omega^\sigma_{i_1\dots i_j} \wedge \omega^\nu (J^{2r+1}\xi, J^{2r+1}\zeta)(J^{2r+1}\gamma) \wedge \omega_0$$

and finally rearrange the expression so that

$$\eta_\Omega = \int_\Omega (J^{2r+1}\gamma)^* i_{J^{2r+1}\zeta} i_{J^{2r+1}\xi} \varrho_0. \tag{28}$$

The argumentation leading to the conclusion that the representative is defined correctly (globally) is quite analogous to the one presented in the first part of the proof.

Another way to ascertain that the expressions defined locally by (16), (18) give rise to globally defined objects is to check the transformation properties of these expressions. The proof using the transformation properties can be found in the Appendix.

It remains to discuss the following problem: Find the criteria for recognizing the representatives of classes of forms in the r-th order variational sequence and the reconstruction of classes from their representatives. This problem is solved for the physically relevant part of the variational sequence in mechanics (see [9] and [16]). For the field theory the calculations are technically difficult and are not finished up to now.

Appendix. Transformation rules for representatives

Let (V, ψ) and $(\tilde{V}, \tilde{\psi})$ be any two fibered charts. We will consider the transformation properties of various objects over the intersection $V \cap \tilde{V}$.

The transformation properties of total derivatives of functions. Let $f \in \Omega_0^r(V \cap \tilde{V})$, then it holds that with the obvious notation

$$\tilde{\mathrm{d}}_j f = d_k f \cdot \frac{\partial x^k}{\partial \tilde{x}^j}.$$

We shall generalize this result.

Theorem A. *With the above used conventions it holds that*

$$\tilde{D}_J f = \sum_{|I|\leq|J|} D_I f \left(\frac{\partial^{a_1} x^{i_1}}{\partial \tilde{x}^{j_1} \cdots \partial \tilde{x}^{j_{a_1}}} \cdots \frac{\partial^{a_{|I|}} x^{i_{|I|}}}{\partial \tilde{x}^{j_{a_{|I|-1}}} \cdots \partial \tilde{x}^{j_{|J|}}} \right)_{\operatorname{ord} J}.$$

There are $\binom{|J|}{|I|}$ *summands pertaining to the given length of the multiindex* I. $\operatorname{ord} J$ *means that the summation is taken over all multiindices* J *such that* $j_1 \leq \cdots \leq j_{a_1}$, $\ldots$, $j_{a_{|I|-1}} \leq \cdots \leq j_{a_{|I|}}$, *the indices* $a_1 \leq \cdots \leq a_{|J|}$ *taking all admissible values.*

Proof is done in a straightforward manner by induction on $|J|$.

Total derivatives of products of functions. Let $f, g \in \Omega_0^r V$. Then

$$\mathrm{D}_K(f \cdot g) = \mathrm{d}_{k_l} \cdots \mathrm{d}_{k_1}(f \cdot g) = \sum_{q=0}^{l}{}' (\mathrm{d}_{k_l} \cdots \mathrm{d}_{k_{q+1}} f)(\mathrm{d}_{k_q} \cdots \mathrm{d}_{k_1} g),$$

where the primed sum runs over all indices $k_1, \ldots, k_l$ in which the ordering in the subindices of the total derivatives is decreasing. There are exactly $\binom{l}{q}$ summands for a given q.

The transformation properties of representatives of $n+1$**-forms.** The forms $p_1\varrho = P_\sigma^J \omega_J^\sigma \wedge \omega_0$ are defined in a coordinate-free way. The representative is given by (16). The transformation properties of representatives will be given by induction with respect to r. For $r = 1$ it holds

$$\varrho_0 = \left\{ \frac{\partial y^\sigma}{\partial \tilde{y}^\nu} \det\left(\frac{\partial x^j}{\partial \tilde{x}^l}\right) P_\sigma - \tilde{\mathrm{D}}_K \left[\frac{\partial y_J^\sigma}{\partial \tilde{y}_K^\nu} \det\left(\frac{\partial x^j}{\partial \tilde{x}^l}\right) P_\sigma^J \right] \right\} \tilde{\omega}^\nu \wedge \tilde{\omega}_0,$$

where the summation is taken over $|J| = 0, 1$. Using Theorem A we see directly that the coefficients

$$\frac{\partial y^\sigma}{\partial \tilde{y}^\nu} \det\left(\frac{\partial x^j}{\partial \tilde{x}^l}\right) (P_\sigma - \mathrm{d}_j P_\sigma^j)$$

have the correct transformation properties of components of $n+1$-forms of the type $Q_\nu \omega^\nu \wedge \omega_0$.

Now we can proceed by induction with respect to r.

$$\begin{aligned} \varrho_0 = \Bigg[& \sum_{k=0}^{r} (-1)^k \tilde{\mathrm{d}}_{l_k} \cdots \tilde{\mathrm{d}}_{l_1} \sum_{|K|\leq|J|} \frac{\partial y_J^\sigma}{\partial \tilde{y}_K^\nu} \det\left(\frac{\partial x}{\partial \tilde{x}}\right) P_\sigma^J \\ & + (-1)^{r+1} \tilde{\mathrm{d}}_{l_{r+1}} \cdots \tilde{\mathrm{d}}_{l_1} \sum_{|J|=r+1} \frac{\partial y_J^\sigma}{\partial \tilde{y}_K^\nu} \det\left(\frac{\partial x}{\partial \tilde{x}}\right) P_\sigma^J \Bigg] \tilde{\omega}^\nu \wedge \tilde{\omega}_0. \end{aligned}$$

The part which has been added to ϱ_0 by raising the order by 1 reads

$$\left[\sum_{|K|\le r}(-1)^{|K|}\tilde{\mathrm{D}}_K\,\frac{\partial y^\sigma_J}{\partial \tilde{y}^\nu_K}\,\det\left(\frac{\partial x}{\partial \tilde{x}}\right)P^J_\sigma\right.$$
$$\left.+\,(-1)^{r+1}\tilde{\mathrm{D}}_L\,\frac{\partial y^\sigma_I}{\partial \tilde{y}^\nu_L}\,\det\left(\frac{\partial x}{\partial \tilde{x}}\right)P^I_\sigma\right]\tilde{\omega}^\nu\wedge\tilde{\omega}_0,$$

where $|J|=|L|=r+1$. Now we shall use the result from the previous paragraph and obtain

$$\left\{\sum\sum_{\substack{|K|\le r\\|R|+|S|=|K|}}{}^{'}\;\tilde{\mathrm{D}}_R\left[\frac{\partial y^\sigma_J}{\partial \tilde{y}^\nu_{RS}}\det\left(\frac{\partial x}{\partial \tilde{x}}\right)\right]\tilde{\mathrm{D}}_S P^J_\sigma\right.$$
$$\left.+\,(-1)^{r+1}\sum_{|R|+|S|=r+1}{}^{'}\;\tilde{\mathrm{D}}_R\left[\frac{\partial y^\sigma_I}{\partial \tilde{y}^\nu_{RS}}\det\left(\frac{\partial x}{\partial \tilde{x}}\right)\right]\tilde{\mathrm{D}}_S P^I_\sigma\right\}\tilde{\omega}^\nu\wedge\tilde{\omega}_0,$$

where $|J|=r+1$ and $0\le|I|\le r$. Let us define the numbers $a_{i,j}$ for $j\le i$ recursively by $a_{i,1}=1$, $a_{i,i}=1$ and $a_{i+1,j+1}=(j+1)a_{i,j+1}+a_{i,j}$. Using the properties of the primed sum and the transformation rules for total derivatives for $\tilde{\mathrm{D}}_S P^I_\sigma$ we obtain precisely the numbers $a_{|R|,|S|}$ as coefficients in both sums. Recursively canceling the terms starting from the highest one we recover the needed additional term

$$(-1)^{r+1}\,\frac{\partial y^\sigma}{\partial \tilde{y}^\nu}\,\det\left(\frac{\partial x}{\partial \tilde{x}}\right)\mathrm{D}_J P^J_\sigma\tilde{\omega}^\nu\wedge\tilde{\omega}_0.$$

The transformation properties of representatives of $(n+2)$**-forms.** We shall procced in an analogous manner as in the case of representatives of $n+1$-forms. We again check directly that the transformation formula holds for $r=1$ and assume that it holds for orders from 1 up to r. Writing down the additional terms for $r+1$-order and using the same properties as in the case of representatives of $n+1$ forms we again recover the required transformation rules.

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