Proceedings of the Seminar on Differential Geometry
Mathematical Publications Volume 2
Silesian University in Opava, Opava, 2000, 1–13

Poisson manifolds of geodesic arcs in classical mechanics[1]

L. Klapka

Abstract. This paper is a survey of results on Poisson manifolds of geodesic arcs based on lectures given by the author at the seminar on calculus of variations held at the universities in Brno and Opava, 1986–1996. Special regards are devoted to relations between Poisson manifolds of geodesic arcs and classical Lagrangian mechanics.

Keywords and phrases. Classical mechanics, configuration in-out manifolds, convex regions, Euler–Lagrange equations, fibrations of algebras, Frobenius algebras, geodesics, geodesic arcs, Hamiltonian mechanics, Lagrangian mechanics, Levi-Civita connections, linear connections, manifolds of geodesic arcs, metric tensors, Poisson manifolds.

MS classification. 16P10, 17B63, 37J99, 49S05, 52A30, 53B05, 53C05, 53C15, 53C22, 53D17, 58D15, 70H03.

1. Introduction

This research was started from a problem of differentiable Poisson brackets of coordinates in mechanics [5]. The essence of this problem will be explained by the following simple example. Consider a Lagrange function $\lambda \in C^\infty(j^1(\mathbb{R} \times \mathbb{R}))$, defined on the first jet prolongation of the fibered manifold $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which has in global coordinates t, q, v the expression of

$$\lambda = \frac{1}{n}\, v^n,$$

[1] Research supported by Grants CEZ:J10/98:192400002, VS 96003 “Global Analysis” of the Czech Ministry of Education, Youth and Sports, and by Grant 201/00/0724 of the Czech Grant Agency.
This paper is in final form and no part of it will be published elsewhere.

where $n \neq 0$ is some real number. Using methods usual in analytical mechanics, we can calculate the integral of motion

$$Q = q - v(t - T),$$

which has the meaning of the coordinate at time $t = T$, and the Poisson bracket

$$\{Q, q\} = \frac{(t - T)^{n-1}(q - Q)^{2-n}}{n - 1}.$$

It is interesting that this Poisson bracket $\{Q, q\}$ is as a function of variables t, q, Q infinitely differentiable only in the case of $n = 2$, i.e. in the only physically important case of all considered ones. Therefore, it would be interesting to find all Lagrange functions which lead to infinitely differentiable Poisson brackets of coordinates for some configuration manifold.

Unfortunately, this example has not a good geometrical sense. That is caused by the fact that $\partial Q/\partial v = 0$ holds for $t = T$ and thus the variables t, q, Q do not form a suitable coordinate system for exploring the differentiability of Poisson brackets on $j^1(\mathbb{R} \times \mathbb{R})$. That is why the problem of differentiable Poisson brackets of coordinates does not exist from the view of jet prolongations and symplectic manifolds. Therefore, another geometrical structure must be introduced for its formulation. This paper shows that an applicable structure is the manifold of geodesic arcs.

Section 2 is devoted to smooth manifolds of geodesic arcs. Necessary and sufficient conditions for a set of arcs to be the set of all geodesic arcs of some linear connection are presented. On this set a structure of a smooth manifold is introduced.

In Section 3 we recall the well-known definition of Poisson manifolds. In Section 4 a definition of a Poisson manifold of geodesic arcs is presented and studied. Local coordinate expressions are given.

In Section 5 we recall the notion of a Frobenius algebra. In Section 6, necessary and sufficient conditions are presented under which a given Lagrange function generates a Poisson manifold of geodesic arcs. These conditions are framed in terms of tangent and cotangent Frobenius algebras. Local expressions for general Poisson manifolds of geodesic arcs are rather complicated. This section shows that they can be simplified if the Poisson manifold is generated by some Lagrange function. Further, a second simplification is found by changing contravariant velocities to covariant velocities. Expressions for the Lagrange function and the corresponding linear connection are given.

In Section 7 a geometrical formulation and a solution of the problem of differentiable Poisson brackets of coordinates in classical mechanics are presented. In this section a concept of configuration in-out manifold is introduced as a Poisson manifold which describes a relation between two configurations of a classical mechanical system. It is shown that every in-out manifold is isomorphic to some Poisson manifold of geodesic arcs. The corresponding general Hamilton function is presented.

Finally, relations between Poisson manifolds of geodesic arcs and classical Lagrangian mechanics are clarified in Section 8.

In this paper the notions of vector fields, linear connections, geodesics, Lagrangian mechanics, Poisson manifolds, and Frobenius algebras are used in the usual sense (see, e.g. [1], [2], [11], and [17]). By a manifold we mean a smooth manifold or a smooth manifold with a boundary or a smooth manifold with corners (see [13], [15]). All used mappings are smooth. In all local expressions we use the standard summation convention.

2. Geodesic arcs

Let $\mathbb{R}$ be the manifold of real numbers. We consider a closed interval $[0, 1] \subset \mathbb{R}$ and a finite-dimensional manifold X. An *arc* γ on X is a smooth mapping $\gamma\colon [0, 1] \to X$. We define a *reparametrization* as an affine mapping $[0, 1] \to [0, 1]$. Let us consider a *monoid M of all reparametrizations* with the multiplication $M \times M \ni (\mu, \nu) \to \mu \circ \nu \in M$. We say that an arc $\gamma \circ \mu$, where $\mu \in M$, is a *subarc* of the arc γ.

Let $W(X)$ be a set of arcs on X. We say that $W(X)$ is *closed on subarcs* if and only if $\gamma \in W(X)$ and $\mu \in M$ imply $\gamma \circ \mu \in W(X)$. We say that an open set $U \subset X$ is *convex with respect to* $W(X)$ if and only if

(i) for each two points $a, b \in U$ there exists a unique arc $\gamma_{ab} \in W(X)$ such that $\gamma_{ab}(0) = a$, $\gamma_{ab}(1) = b$, $\gamma_{ab}([0, 1]) \subset U$,
(ii) the mapping $U \times U \times [0, 1] \ni (a, b, w) \to \gamma_{ab}(w) \in U$ is smooth.

We say that X is *locally convex with respect to* $W(X)$ if and only if for each $c \in X$ there exists an open set U such that $c \in U \subset X$, U is convex with respect to $W(X)$.

Let us consider a smooth linear connection Γ on X. A *geodesic arc* of the connection Γ is an arc on X which is a geodesic of Γ. In 1932 J.H.C. Whitehead proved the following Theorem on convex regions [18]:

Theorem 1. (Whitehead) *Let $W(X)$ be the set of all geodesic arcs of some linear connection Γ on a manifold X. Then X is locally convex with respect to $W(X)$.*

In 1992 the author proved the following inversion of Whitehead's Theorem on convex regions [7]:

Theorem 2. *Let $W(X)$ be a set of arcs on a manifold X. The following two assertions are equivalent:*

1. *There is a linear connection Γ on X such that $W(X)$ is a set of all its geodesic arcs,*
2. *$W(X)$ is a maximal set of arcs on X satisfying the conditions:*
   (*a*) *X is locally convex with respect to $W(X)$,*
   (*b*) *$W(X)$ is closed on subarcs.*

Let us suppose that the assertion 2 of Theorem 2 is satisfied. From the condition (b) we have that the monoid M acts on $W(X)$ from right in the following way

$$\mathcal{R}\colon W(X) \times M \ni (\gamma, \mu) \to \gamma \circ \mu \in W(X).$$

The global chart $M \ni \mu \to (\mu(0), \mu(1)) \in [0, 1] \times [0, 1]$ defines a structure of a manifold on M in such a way that the multiplication is smooth. Therefore, the monoid of reparametrizations M with the above defined structure is the Lie monoid. It is known (see, e.g. [11]) that there exists a bijective mapping $\psi\colon W(X) \ni \gamma \to \dot{\gamma}(0) \in \operatorname{codom} \psi \subset TX$, where $\dot{\gamma}\colon [0, 1] \to TX$ is the prolongation of the geodesic arc γ on the tangent bundle TX. The set $W(X)$, equipped with a structure of a manifold such that ψ is a diffeomorphism, is called a *manifold of geodesic arcs*. The simplest example of a manifold of geodesic arcs is the monoid of reparametrizations $M = W([0, 1])$ in itself.

Since every mapping $\gamma \in W(X)$ is smooth, we see that the action $\mathcal{R}$ is smooth too. We shall use a partial mapping

$$\mathcal{R}_\mu : W(X) \ni \gamma \to \gamma \circ \mu \in W(X).$$

The action $\mathcal{R}$ of the Lie monoid of reparametrizations M is of fundamental importance in the theory of geodesic arcs. Theorem 2 shows that the notion of a symmetric linear connection may be defined with the help of $\mathcal{R}$.

Let us consider two manifolds of geodesic arcs $W(Y)$ and $W(X)$. Any mapping $\pi : Y \to X$, such that $\gamma \in W(Y)$ implies $\pi \circ \gamma \in W(X)$, defines a mapping

$$\mathcal{L}_\pi : W(Y) \ni \gamma \to \pi \circ \gamma \in W(X)$$

of manifolds of geodesic arcs. Moreover, if Y is a fibered manifold over the base X with the projection π then $W(Y)$ is a fibered manifold over the base $W(X)$ with the projection $\mathcal{L}_\pi$. In such a case $W(Y)$ is called a *fibered manifold of geodesic arcs* $\mathcal{L}_\pi : W(Y) \to W(X)$.

3. Poisson manifolds

Let us consider a finite-dimensional manifold P. A *Poisson algebra over* P is a Lie algebra structure on $C^\infty(P)$ that satisfies the *Leibniz condition*

$$\{FG, H\} = F\{G, H\} + G\{F, H\}.$$

A *Poisson bracket* $\{\cdot, \cdot\}$ is the Lie bracket of the corresponding Poisson algebra. A *Casimir function* in a Poisson algebra over P is a function $F \in C^\infty(P)$ such that $\{F, G\} = 0$ for all functions $G \in C^\infty(P)$. A *Poisson manifold* is a manifold P with a Poisson algebra over P (see, e.g. [4], [14], [16], [17]). A *Poisson mapping* is a homomorphism of Poisson manifolds.

A *Poisson submanifold* Q is a submanifold Q in a Poisson manifold P with a Poisson algebra $C^\infty(Q)$ for which the inclusion $Q \to P$ is a Poisson mapping.

Important particular examples of Poisson manifolds are *symplectic manifolds*. Opposite examples are *Abelian Poisson manifolds*, Poisson algebras of which are Abelian.

4. Poisson manifolds of geodesic arcs

Let $W(X)$ be a manifold of geodesic arcs. A *Poisson right M-algebra over $W(X)$* is a Poisson algebra over $W(X)$ such that for each $\mu \in M$ a mapping $C^\infty(W(X)) \ni F \to F \circ \mathcal{R}_\mu \in C^\infty(W(X))$ is an endomorphism. A *Poisson manifold of geodesic arcs* is a manifold of geodesic arcs equipped with a Poisson right M-algebra. Similarly, a *fibered Poisson manifold of geodesic arcs* is a fibered manifold of geodesic arcs equipped with a Poisson right M-algebra. In 1994 the author proved the following Theorem [8]:

Theorem 3. *Let $W(X)$ be a manifold of geodesic arcs of a symmetric linear connection Γ on X. Then there exists a bijective correspondence between the set of all Poisson right M-algebras over $W(X)$ and the set of all ordered pairs (g, h) of tensor fields on X satisfying the following three conditions:*

1. *g is a tensor field of the type* $(2, 1)$*,*
2. *h is a tensor field of the type* $(2, 0)$*,*
3. *in each local chart on X the components g^{ij}_k, h^{ij} satisfy the relations:*

(1) $$h^{ij} + h^{ji} = 0,$$

(2) $$h^{il} h^{jk}_l + h^{jl} h^{ki}_l + h^{kl} h^{ij}_l = 0,$$

(3) $$g^{il}_m g^{jk}_l - g^{jl}_m g^{ik}_l + g^{ik}_{ml} h^{lj} - g^{jk}_{ml} h^{li} = g^{lk}_m h^{ij}_l + R^k_{mlr} h^{li} h^{rj},$$

(4) $$g^{ij}_k - g^{ji}_k = h^{ij}_k,$$

(5) $$g^{ij}_{kl} + g^{ij}_{lk} + g^{ji}_{kl} + g^{ji}_{lk} = (R^i_{klm} + R^i_{lkm}) h^{mj} + (R^j_{klm} + R^j_{lkm}) h^{mi},$$

(6) $$\begin{aligned} &\tfrac{1}{2}\,(h^{ij}_{lm} + h^{ij}_{ml} + R^i_{lmk} h^{kj} + R^i_{mlk} h^{kj} - R^j_{lmk} h^{ki} - R^j_{mlk} h^{ki})_r \\ &\quad + \tfrac{1}{2}\,(h^{ij}_{mr} + h^{ij}_{rm} + R^i_{mrk} h^{kj} + R^i_{rmk} h^{kj} - R^j_{mrk} h^{ki} - R^j_{rmk} h^{ki})_l \\ &\quad + \tfrac{1}{2}(h^{ij}_{rl} + h^{ij}_{lr} + R^i_{rlk} h^{kj} + R^i_{lrk} h^{kj} - R^j_{rlk} h^{ki} - R^j_{lrk} h^{ki})_m \\ &\quad = (R^i_{mrk} + R^i_{rmk}) g^{kj}_l + (R^i_{rlk} + R^i_{lrk}) g^{kj}_m + (R^i_{lmk} + R^i_{mlk}) g^{kj}_r \\ &\quad - \,(R^j_{mrk} + R^j_{rmk}) g^{ki}_l - (R^j_{rlk} + R^j_{lrk}) g^{ki}_m - (R^j_{lmk} + R^j_{mlk}) g^{ki}_r, \end{aligned}$$

where R^i_{jkl} are components of the Riemannian tensor field of the connection Γ and the lower indices which does not belong to the indexation of tensors g, h denote the corresponding covariant derivatives.

Local coordinate expressions for Poisson manifolds of geodesic arcs are

(7) $$\{x^k, x^l\} = h^{kl},$$

(8) $$\{x^k, \dot{x}^l\} = (g^{kl}_m - h^{kn} \Gamma^l_{mn})\, \dot{x}^m,$$

(9) $$\begin{aligned} \{\dot{x}^k, \dot{x}^l\} = \big(&\tfrac{1}{2}\, h^{mk} R^l_{nrm} - \tfrac{1}{2}\, h^{ml} R^k_{nrm} - \tfrac{1}{2}\, h^{kl}_{nr} \\ &+ g^{mk}_r \Gamma^l_{nm} - g^{ml}_r \Gamma^k_{nm} + h^{ms} \Gamma^k_{mn} \Gamma^l_{sr}\big)\, \dot{x}^n\, \dot{x}^r, \end{aligned}$$

where x^k, $\dot{x}^k$ are standard local coordinates on $W(X)$ and Γ^k_{lm} are components of the connection Γ on X.

The simplest example of a Poisson manifold of geodesic arcs is the monoid of reparametrizations M. From Theorem 2 and formula (5) we get $\Gamma = 0$ and $g = \mathrm{const}$. If w is the identical coordinate on $[0, 1]$, then the Poisson structure on M is given by $\{w, \dot{w}\} = g\dot{w}$.

5. Frobenius algebras

An *algebra* $\mathbb{A}$ is a finite-dimensional $\mathbb{R}$-module $\mathbb{A}$ together with a bilinear multiplication $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$ which makes $\mathbb{A}$ into an associative ring with a unity element. A *structure tensor* of $\mathbb{A}$ is the tensor of the type $(2, 1)$ associated with this multiplication. An algebra $\mathbb{A}$ is called *commutative* if $\mathbb{A}$ is a commutative ring.

Any algebra $\mathbb{A}$ is a left $\mathbb{A}$-module. The dual $\mathbb{R}$-module $\mathbb{A}^*$, equipped with the multiplication $\mathbb{A} \times \mathbb{A}^* \ni (a, \omega) \to (\mathbb{A} \ni b \to \omega(ba) \in \mathbb{R}) \in \mathbb{A}^*$, is a left $\mathbb{A}$-module as well. An algebra $\mathbb{A}$ is a *Frobenius algebra* if and only if there exists an isomorphism $g : \mathbb{A} \to \mathbb{A}^*$ of these left $\mathbb{A}$-modules. The left $\mathbb{A}$-module $\mathbb{A}^*$, equipped with the multiplication $\mathbb{A}^* \times \mathbb{A}^* \to \mathbb{A}^*$ such that g is an isomorphism of algebras, will be called a *dual Frobenius algebra of* $\mathbb{A}$. The unity element in the dual Frobenius algebra $\mathbb{A}^*$ will be denoted by $\langle \cdot \rangle \colon \mathbb{A} \ni a \to \langle a \rangle \in \mathbb{R}$.

Let $\mathbb{A}$ be an algebra. Denote by exp the mapping that takes each point $a \in \mathbb{A}$ to $y(1) \in \mathbb{A}$, where $y\colon \mathbb{R} \to \mathbb{A}$ is the solution of the differential equation $dy/d\tau = ay$ under the condition $y(0) = 1$. The mapping exp exists and the solution y is given by $y\colon \mathbb{R} \ni \tau \to \exp(\tau a) \in \mathbb{A}$. Moreover, the mapping exp is a local diffeomorphism. This means that for any $b \in \mathbb{A}$ there is a neighborhood $V \ni b$ such that the mapping $V \ni a \to \exp a \in \exp V$ is a diffeomorphism. Therefore, we can locally define a smooth mapping $\ln\colon \exp V \to V$ by the formula $\ln \circ \exp|_V = \mathrm{id}_V$.

A vector bundle $Z \to X$ is called a *fibration of algebras* if and only if any fiber of Z is an algebra and the corresponding structure tensor field is smooth. Over the manifold X we shall consider partly a fibration of tangent algebras TX partly a fibration of cotangent algebras T^*X.

If g_k^{ij} are components of a cotangent commutative algebra structure tensor field, then the commutativity gives

$$g_k^{ij} = g_k^{ji}, \tag{10}$$

and the associativity gives

$$g_i^{ml} g_m^{jk} = g_i^{jm} g_m^{kl}. \tag{11}$$

There exists a differential invariant of a structure tensor field. This invariant is a tensor field of the type $(3, 2)$. Its components are

$$\begin{aligned} J_{jk}^{ilm} = {} & g_s^{il}\frac{\partial g_j^{sm}}{\partial x^k} + g_s^{im}\frac{\partial g_j^{sl}}{\partial x^k} + g_k^{si}\frac{\partial g_s^{lm}}{\partial x^j} + g_j^{si}\frac{\partial g_k^{lm}}{\partial x^s} + g_j^{sl}\frac{\partial g_k^{im}}{\partial x^s} + g_j^{sm}\frac{\partial g_k^{il}}{\partial x^s} \\ & - g_s^{il}\frac{\partial g_k^{sm}}{\partial x^j} - g_s^{im}\frac{\partial g_k^{sl}}{\partial x^j} - g_j^{si}\frac{\partial g_s^{lm}}{\partial x^k} - g_k^{si}\frac{\partial g_j^{lm}}{\partial x^s} - g_k^{sl}\frac{\partial g_j^{im}}{\partial x^s} - g_k^{sm}\frac{\partial g_j^{il}}{\partial x^s}, \end{aligned} \tag{12}$$

where x^i are local coordinates on X. It is easy to prove that (10), (11), (12) imply $J_{jk}^{ilm} = J_{jk}^{lim} = J_{jk}^{iml} = -J_{kj}^{ilm}$.

6. Generating Lagrange functions

Let X be a configuration manifold, TX be the corresponding tangent bundle with the projection $\pi\colon TX \to X$. Let us consider a smooth regular Lagrange function L, where $\operatorname{dom} L \subset TX$ is an open submanifold equipped with the canonical symplectic structure, $\operatorname{codom} L = \mathbb{R}$. Any mapping $[0, 1] \to X$ satisfying the corresponding Euler–Lagrange equations is called an *extremal arc* of the Lagrange function L.

Let $W_L(X)$ be the set of all extremal arcs of L. The set $W_L(X)$, equipped with a symplectic structure such that the bijective mapping $\psi_L\colon W_L(X) \ni \gamma \to \dot\gamma(0) \in \operatorname{codom} \psi_L \subset \operatorname{dom} L$ is an isomorphism of symplectic manifolds, is called a *symplectic manifold of extremal arcs*.

We say that the Lagrange function L *generates* a Poisson manifold of geodesic arcs $W_\Gamma(X)$ if and only if

(i) $W_L(X) \subset W_\Gamma(X)$ is a symplectic submanifold,
(ii) $W_L(X) \ni \gamma \to \gamma(0) \in X$ is a surjective mapping.

Let us remark that using local expressions (7)–(9) we get the following two assertions: No Poisson manifold of geodesic arcs is symplectic, so $W_L(X) \neq W_\Gamma(X)$. If L is a Lagrange function satisfying (ii), then there exists at most one Poisson manifold of geodesic arcs satisfying (i).

In 1998 the author proved the following Theorem [10]:

Theorem 4. *A given smooth Lagrange function* L, $\operatorname{dom} L \subset TX$, $\operatorname{codom} L = \mathbb{R}$, *generates a Poisson manifold of geodesic arcs if and only if the three following conditions hold:*

1. *there exists a fibration of tangent commutative Frobenius algebras* TX *such that for every* $v \in \operatorname{codom} \psi_L$

$$L(v) = \langle\, v\,(\ln\ v - 1)\,\rangle + \text{const}, \tag{13}$$

2. *there exists a fibration of dual Frobenius algebras* T^*X *such that the differential invariant* (12) *is zero,*
3. $\pi(\operatorname{codom} \psi_L) = X$.

If a given Lagrange function generates a Poisson manifold of geodesic arcs, we can, using (13), calculate local expressions for the tensors g, h, and the connection Γ satisfying (1)–(9). We get $h = 0$, g is the cotangent algebra structure tensor field, and

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2}\, g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) \\ &\quad - \frac{1}{2}\, g^{il}\, g_{jr}\, g_{ks} \left(g^m\, \frac{\partial g^{rs}_l}{\partial x^m} + \frac{\partial g^m}{\partial x^l}\, g^{rs}_m - \frac{\partial g^r}{\partial x^m}\, g^{ms}_l - \frac{\partial g^s}{\partial x^m}\, g^{rm}_l \right),\end{aligned} \tag{14}$$

where x^i are local coordinates on X, g^k are components of the tangent algebra unity element field, $g^{ij} = g^{ij}_k\, g^k$, g_{ij}'s make the solution of equations $g_{ij}\, g^{jk} = \delta^k_i$, δ^k_i is the Kronecker symbol. Formula (14) was originally proved in the paper [6], but the calculation given in [9] is easier. It is readily seen that if the Lie derivative in the second term equals zero, then Γ is the Levi-Civita connection and g^{ij} are components of a contravariant metric tensor.

The local expressions for the Poisson manifold of geodesic arcs generated by some Lagrange function are

$$\begin{aligned}\{x^k, x^l\} &= 0, \\ \{x^k, \dot{x}^l\} &= g^{kl}_r\, \dot{x}^r, \\ \{\dot{x}^k, \dot{x}^l\} &= \left(g^{km}_r\, \Gamma^l_{sm} - g^{lm}_r\, \Gamma^k_{sm}\right) \dot{x}^r\, \dot{x}^s.\end{aligned} \tag{15}$$

The paper [9] shows that they can be simplified by changing contravariant velocities $\dot{x}^k$ to covariant velocities $\dot{x}_k = g_{kl}\, \dot{x}^l$:

$$\begin{aligned}\{x^k, x^l\} &= 0, \\ \{x^k, \dot{x}_l\} &= g^{kr}_l\, \dot{x}_r, \\ \{\dot{x}_k, \dot{x}_l\} &= \frac{1}{2} \left(\frac{\partial g^{rs}_k}{\partial x^l} - \frac{\partial g^{rs}_l}{\partial x^k} \right) \dot{x}_r\, \dot{x}_s.\end{aligned}$$

7. In-out manifolds

The notion of an in-out manifold was introduced by the author in 1989 for solving the problem of finding the most general form for Poisson brackets of configuration coordinates at two different times in Hamiltonian mechanics.

Let P be a Poisson manifold. The Leibniz condition implies that the bracket operation is a derivation in each entry, and so in particular, for each function $F \in C^\infty(P)$ there is a vector field $\xi_F \in \mathfrak{X}(P)$ such that $\partial_{\xi_F} G = \{G, F\}$ for all $G \in C^\infty(P)$, where ∂_{ξ_F} denotes a Lie derivative with respect to ξ_F. The local flow of this vector field $\alpha^F \colon (-\varepsilon, +\varepsilon) \times P \to P$ is called a *local flow generated by the function* F. If the local flow α^F exists, then the partial mapping $\alpha^F_\tau \colon P \to P$ is the Poisson one for all $\tau \in (-\varepsilon, +\varepsilon)$.

A configuration of a Hamiltonian mechanical system is represented by a point of its configuration manifold X. A state of system is described by a point of a cotangent bundle T^*X. The cotangent bundle is a symplectic manifold and the configuration manifold is an Abelian Poisson manifold. Thus a natural projection $T^*X \to X$ is a Poisson mapping. A dynamical evolution manifests itself as a local flow α^H generated by a Hamiltonian function H. We need the configuration at two times. So we introduce a manifold $X \times X$ and denote natural projections on its factors as “in” and “out”.

We say that a Poisson manifold $X \times X$ is a *configuration in-out manifold* if there exists a regular Hamiltonian function $H \in C^\infty(T^*X)$ and a real number $\varepsilon > 0$ such that a mapping $T^*X \to X \times X$ defined by the commutative diagram

$$\begin{array}{ccccc} T^*X & \xleftarrow{\alpha^H_{\tau_1}} & T^*X & \xrightarrow{\alpha^H_{\tau_2}} & T^*X \\ \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow{\text{in}} & X \times X & \xrightarrow{\text{out}} & X \end{array}$$

is a Poisson mapping for any pair $\tau_1, \tau_2 \in (-\varepsilon, +\varepsilon)$. Let us remark that if $X \times X$ is a configuration in-out manifold, then all mappings in the diagram are Poisson ones.

In 1989 the author proved the following theorem [6]:

Theorem 5. *If b is a point in a diagonal of in-out manifold $X \times X$, then there is a chart $(x^k, \dot{x}^k)$ in b such that* (14) *and* (15) *hold, where x^k, g^{kl}_r, g^k are functions on X composed with the projection* out*.*

From Theorems 3 and 5, every in-out manifold $X \times X$ is isomorphic to some Poisson manifold of geodesic arcs $W(X)$. Since the Hamilton function H is regular, there exists the Lagrange function L generating $W(X)$. Using Theorem 4, we can calculate

$$H(p) = \langle \exp(p) \rangle - \text{const},$$

where exp and $\langle \cdot \rangle$ are defined on the fibration of cotangent algebras T^*X.

8. Relations to classical mechanics

For a geometric formulation of the first order Lagrangian theory the formalism of fibered manifolds and theirs lower two prolongations is used (see [12]). In the case of mechanics, we consider a fibered manifold $\pi_0\colon \mathbb{R}\times X \to \mathbb{R}$, where $\mathbb{R}$ is the manifold of real numbers and X is a finite-dimensional manifold, with its prolongations $j^1(\mathbb{R}\times X)$, $j^2(\mathbb{R}\times X)$ and projections $\pi_1\colon j^1(\mathbb{R}\times X) \to \mathbb{R}$, $\pi_{10}\colon j^1(\mathbb{R}\times X) \to \mathbb{R}\times X$, $\pi_{21}\colon j^2(\mathbb{R}\times X) \to j^1(\mathbb{R}\times X)$, $\pi_{20}\colon j^2(\mathbb{R}\times X) \to \mathbb{R}\times X$. We shall use the fiber charts with coordinates τ, x^k, $\dot{x}^k$, $\ddot{x}^k$. A time evolution of mechanical system is described by a local section σ of the fibered manifold π_0 which is a solution of Euler–Lagrange equations

$$\epsilon_k \circ j^2\sigma = 0. \tag{16}$$

Here Euler–Lagrange expressions ϵ_k are components of π_{20}-horizontal 1-contact differential 2-form on $j^2(\mathbb{R}\times X)$ such that a Lagrange function λ satisfying

$$\epsilon_k = \frac{d}{d\tau}\left(\frac{\partial\lambda}{\partial\dot{x}^k}\right) - \frac{\partial\lambda}{\partial x^k}, \tag{17}$$

where $d/d\tau$ is the total derivative, locally exists. Here, in contrast to Section 6, the Lagrange function λ is the component of the π_1-horizontal differential 1-form on $j^1(\mathbb{R}\times X)$. Note that by the usual notation practice in the classical mechanics all necessary projections are omitted (for instance, the restriction of projection π_{21} absents in the second term of the right-hand side of (17)).

Equations (16) are said to be *classical Euler–Lagrange equations* if and only if the corresponding Lagrange function has the form

$$\lambda = \tfrac{1}{2}\, g_{kl}\, \dot{x}^k\, \dot{x}^l + A_k\, \dot{x}^k + \varphi, \tag{18}$$

where g_{kl}, A_k, and φ are functions on $\mathbb{R}\times X$. This case is the most spread one in applications and all the known physically important mechanical systems can be converted into it [3].

We suppose that the Euler–Lagrange expressions are regular, that is, for each point of $j^2(\mathbb{R}\times X)$ the determinant

$$\det\left(\frac{\partial\epsilon_k}{\partial\ddot{x}^l}\right) \neq 0. \tag{19}$$

Let us consider a set $E(\mathbb{R}\times X)$ of all solutions $\sigma_{ab}\colon [a, a+b] \to \mathbb{R}\times X$ of equations (16), where $a, b \in \mathbb{R}$, $b > 0$. Then there exists a bijection $\chi\colon E(\mathbb{R}\times X) \ni \sigma_{ab} \to (b, j^1\sigma_{ab}(a)) \in \operatorname{codom}\chi \subset (0,\infty)\times j^1(\mathbb{R}\times X)$, where $\operatorname{codom}\chi$ is an open subset. The Poisson structure of this manifold can be found by the methods of classical mechanics. From classical canonical relations

$$\{x^k, x^l\} = 0, \qquad \left\{x^k, \frac{\partial\lambda}{\partial\dot{x}^l}\right\} = \delta^k_l, \qquad \left\{\frac{\partial\lambda}{\partial\dot{x}^k}, \frac{\partial\lambda}{\partial\dot{x}^l}\right\} = 0$$

we obtain

$$\{x^k, x^l\} = 0,$$

$$\{x^k, \dot{x}^m\}\, \frac{\partial\epsilon_l}{\partial\ddot{x}^m} = \delta^k_l,$$

$$\{\dot{x}^k, \dot{x}^l\}\, \frac{\partial\epsilon_r}{\partial\ddot{x}^k}\frac{\partial\epsilon_s}{\partial\ddot{x}^l} = \frac{1}{2}\left(\frac{\partial\epsilon_s}{\partial\dot{x}^r} - \frac{\partial\epsilon_r}{\partial\dot{x}^s}\right).$$

Note that in the classical mechanics the coordinates on $\mathbb{R}$ and $(0, \infty)$ are considered as Casimir functions. The set $E(\mathbb{R} \times X)$, equipped with a Poisson algebra such that χ is an isomorphism of Poisson manifolds, will be called a *Poisson manifold of solutions of Euler–Lagrange equations*.

In 1995 the author proved the following Theorem on the relation between Poisson manifolds of solutions of Euler–Lagrange equations and Poisson manifolds of geodesic arcs in classical mechanics [8]:

Theorem 6. *Let $W(\mathbb{R} \times X) \to W(\mathbb{R})$ be a fibered Poisson manifold of geodesic arcs such that all fibers are Poisson submanifolds, fibers over constant arcs are Abelian, and fibers over non-constant arcs are symplectic. Then $W(\mathbb{R} \times X)$ is the union of three disjoint Poisson submanifolds such that the two of them are isomorphic to a Poisson manifold of solutions of classical Euler–Lagrange equations and the third of them is an Abelian fibration of manifolds of geodesic arcs of Levi-Civita connections on X.*

Proof. Let us consider a local chart x on $\mathbb{R} \times X$ such that x^0 is a coordinate on $\mathbb{R}$, $x^1, x^2, \dots, x^{\dim X}$ are coordinates on X, and $\operatorname{codom} x$ is an open ball in $\mathbb{R}^{\dim X+1}$. Since $W(\mathbb{R} \times X)$ is a fibered manifold of geodesic arcs over the base $W(\mathbb{R})$, we get $R^i_{jkl} = 0$ for $i = 0$. Hence, the coordinate x^0 may be chosen in such a way that for $i = 0$

$$\Gamma^i_{jk} = 0. \tag{20}$$

Since the fibers over constant arcs are Abelian, by (7) and (8) we obtain $h^{ij} = 0$, $g^{ij}_k = 0$, where $k \neq 0$. Since the fibers over non-constant arcs are Poisson submanifolds, by (8) we obtain $g^{i0}_0 = g^{0i}_0 = 0$.

Throughout all the following text we will not consider the zero value of indices i, j, k, l, m, r, s. We denote $g^{ij}_0 = g^{ij}$ and $x^0 = \tau$. Taking into account that the fibers over non-constant arcs are symplectic we obtain

$$\det(g^{ij}) \neq 0 \tag{21}$$

at each point of X. Relations (1)–(3) are fulfilled identically. Relation (4) gives $g^{ij} = g^{ji}$. Relation (5) can be expressed in the form

$$\frac{\partial g^{ij}}{\partial x^k} + \Gamma^i_{km} g^{mj} + \Gamma^j_{km} g^{im} = 0, \qquad \frac{\partial g^{ij}}{\partial \tau} + \Gamma^i_{0m} g^{mj} + \Gamma^j_{0m} g^{im} = 0.$$

From this, by a straightforward computation we get

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right), \tag{22}$$

$$\Gamma^i_{j0} = \Gamma^i_{0j} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial \tau} + \Phi_{lj} \right), \tag{23}$$

where g_{ij}'s make the solution of equations $g_{ij}\, g^{jk} = \delta^k_i$ and Φ_{ij} are functions on $\operatorname{dom} x$ such that $\Phi_{ij} + \Phi_{ji} = 0$. Relation (6) can be written according to (22) and (23) in the form

$$\frac{\partial \Phi_{ij}}{\partial x^k} + \frac{\partial \Phi_{jk}}{\partial x^i} + \frac{\partial \Phi_{ki}}{\partial x^j} = 0, \qquad \frac{\partial \Phi_{ij}}{\partial \tau} + \frac{\partial F_j}{\partial x^i} - \frac{\partial F_i}{\partial x^j} = 0,$$

where $F_i = g_{ij}\Gamma^j_{00}$. Thence the Poincaré lemma enables us to show that there exist functions A_i, φ on $\operatorname{dom} x$ satisfying

$$\Phi_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}, \qquad F_i = \frac{\partial A_i}{\partial \tau} - \frac{\partial \varphi}{\partial x^i}. \tag{24}$$

Since the Lagrange function (18) exists on $j^1(\operatorname{dom} x)$, the components

$$\epsilon_k = g_{kl}(\ddot{x}^l + \Gamma^l_{rs}\dot{x}^r\dot{x}^s) + \left(\frac{\partial g_{kl}}{\partial \tau} + \Phi_{kl}\right)\dot{x}^l + F_k$$

of a π_{20}-horizontal 1-contact differential 2-form on $j^2(\mathbb{R}\times X)$ are Euler–Lagrange expressions. According to (19) and (21) they are regular. Let us denote the corresponding manifold of solutions of Euler–Lagrange equations by $E(\mathbb{R}\times X)$.

In view of (20) $W(\mathbb{R}) = W_1(\mathbb{R})\cup W_2(\mathbb{R})\cup W_3(\mathbb{R})$, where $W_1(\mathbb{R})$ is the open submanifold of all increasing geodesic arcs, $W_2(\mathbb{R})$ is the open submanifold of all decreasing geodesic arcs, and $W_3(\mathbb{R})$ is the closed submanifold of all constant geodesic arcs. Let us decompose the manifold $W(\mathbb{R}\times X)$ into the three disjoint submanifolds $W_1(\mathbb{R}\times X) = \mathcal{L}^{-1}_{\pi_0}(W_1(\mathbb{R}))$, $W_2(\mathbb{R}\times X) = \mathcal{L}^{-1}_{\pi_0}(W_2(\mathbb{R}))$, and $W_3(\mathbb{R}\times X) = \mathcal{L}^{-1}_{\pi_0}(W_3(\mathbb{R}))$, where $\mathcal{L}_{\pi_0}$ is the projection $W(\mathbb{R}\times X)\ni\gamma\to\pi_0\circ\gamma\in W(\mathbb{R})$. These submanifolds are Poisson ones because of $W_1(\mathbb{R}\times X)$, $W_2(\mathbb{R}\times X)$ are open ones and $W_3(\mathbb{R}\times X)$ is by assumption the union of Abelian Poisson manifolds over constant geodesic arcs. Clearly, $W_3(\mathbb{R}\times X)$ is an Abelian submanifold too.

Let us suppose $\gamma\in W_1(\mathbb{R}\times X)$. Then there exists a unique mapping σ such that $\operatorname{dom}\sigma = \operatorname{codom}(\pi_0\circ\gamma)$, $\gamma = \sigma\circ\pi_0\circ\gamma$. It holds $\sigma\in E(\mathbb{R}\times X)$. The induced mapping $W_1(\mathbb{R}\times X)\to E(\mathbb{R}\times X)$ is an isomorphism of Poisson manifolds.

Let us suppose $\gamma\in W_2(\mathbb{R}\times X)$ and define the mapping $\mu\colon[0,1]\ni w\to(1-w)\in[0,1]$. In such a case $\gamma\circ\mu = \mathcal{R}_\mu(\gamma)\in W(\mathbb{R}\times X)$. Since $\pi_0\circ\gamma$ and μ are decreasing mappings, $\pi_0\circ\gamma\circ\mu$ is an increasing mapping and so $\gamma\circ\mu\in W_1(\mathbb{R}\times X)$. Since $\mathcal{R}_\mu$ is a Poisson mapping and since $\mathcal{R}^{-1}_\mu = \mathcal{R}_\mu$ holds, the mapping $W_2(\mathbb{R}\times X)\ni\gamma\to\gamma\circ\mu\in W_1(\mathbb{R}\times X)$ is an isomorphism of Poisson manifolds. Consequently, the fact that there is an isomorphism between $W_1(\mathbb{R}\times X)$ and $E(\mathbb{R}\times X)$ implies the existence of isomorphism between $W_2(\mathbb{R}\times X)$ and $E(\mathbb{R}\times X)$.

For each $a\in\mathbb{R}$ the set $\pi_0^{-1}(a)\subset\mathbb{R}\times X$ is a geodesic submanifold diffeomorphic to X. According to (22) the connection on $\pi_0^{-1}(a)$ is the Levi-Civita one. Let $W(\pi_0^{-1}(a))$ be the manifold of geodesic arcs of this connection. Taking into account

$$W_3(\mathbb{R}\times X) = \bigcup_{a\in\mathbb{R}} W\big(\pi_0^{-1}(a)\big),$$

we see that $W_3(\mathbb{R}\times X)$ is the Abelian fibration of manifolds of geodesic arcs of Levi-Civita connections on X over the base $\mathbb{R}$. This completes the proof.

Theorem 7. *Let X be a simply connected manifold, $W(\mathbb{R}\times X)\to W(\mathbb{R})$ be a fibered Poisson manifold of geodesic arcs satisfying the conditions of Theorem* 6. *Then there exists a Lagrange function generating a Poisson manifold of geodesic arcs $W(\mathbb{R}\times X\times[0,1])$ such that*

1. *$W(\mathbb{R}\times X\times[0,1])\to W(\mathbb{R}\times X)$ is a fibered Poisson manifold of geodesic arcs,*
2. *the fiber over any constant arc is an Abelian Poisson submanifold isomorphic to the monoid of reparametrizations.*

Proof. Let $W_1(\mathbb{R}) \subset W(\mathbb{R})$ be the open submanifold of all increasing geodesic arcs, $W_1(\mathbb{R} \times X) \subset W(\mathbb{R} \times X)$ be the open submanifold over $W_1(\mathbb{R})$, $W_L(\mathbb{R} \times X \times [0,1]) \subset W(\mathbb{R} \times X \times [0,1])$ be the submanifold of all geodesic arcs

$$[0,1] \ni w \to (\gamma(w), \mu(w)) \in (\mathbb{R} \times X) \times [0,1],$$

where $\gamma \in W_1(\mathbb{R} \times X)$, $\mu \in M$. Let $\operatorname{dom} L \subset T(\mathbb{R} \times X \times [0,1])$ be the submanifold such that $W_L(\mathbb{R} \times X \times [0,1]) \ni \gamma \to \dot{\gamma}(0) \in \operatorname{dom} L$ is a bijection. Let us consider a Lagrange function $L\colon \operatorname{dom} L \to \mathbb{R}$ given by local coordinate expressions

$$L = \dot{w} \ln \dot{\tau} + \frac{\frac{1}{2} g_{kl}\, \dot{x}^k \dot{x}^l + A_k\, \dot{x}^k\, \dot{\tau} + \varphi\, \dot{\tau}^2}{\dot{\tau}}, \tag{25}$$

where $\dot{\tau}, \dot{x}^k, \dot{w}$ are the standard tangent coordinates associated with local coordinates τ, x^k, w on $\mathbb{R} \times X \times [0,1]$, such that τ is a geodesic coordinate on $\mathbb{R}$, x^k are coordinates on X, and w is the identical coordinate on $[0,1]$, functions g_{kl}, A_k, φ are solutions of equations $g_{ij}\, g^{jk} = \delta_i^k$ and (24). Since X is a simply connected manifold, the mentioned Lagrange function on $\operatorname{dom} L$ globally exists.

By a straightforward computation it follows that the Lagrange function (25) generates a Poisson manifold of geodesic arcs $W(\mathbb{R} \times X \times [0,1])$ satisfying conditions 1 and 2. This completes the proof.

Finally note that the bilinear multiplication $\star$ in fibration of cotangent Frobenius algebras is given by relations

$$\begin{array}{lll} d\tau \star d\tau = 0, & d\tau \star dx^j = 0, & d\tau \star dw = d\tau, \\ dx^i \star d\tau = 0, & dx^i \star dx^j = g^{ij} d\tau, & dx^i \star dw = dx^i, \\ dw \star d\tau = d\tau, & dw \star dx^j = dx^j, & dw \star dw = dw, \end{array} \tag{26}$$

and the tangent algebra unity element field has the form

$$\frac{\partial}{\partial \tau} + A^i \frac{\partial}{\partial x^i} + \left(\frac{1}{2} A_i\, A^i - \varphi \right) \frac{\partial}{\partial w}, \tag{27}$$

where $A^i = g^{ij} A_j$. By (13), formula (25) is equivalent to (26) and (27).

Acknowledgments

The author is indebted to Professor Demeter Krupka for his continual interest. Discussions with him have contributed significantly to the progress of this work and are gratefully acknowledged.

References

[1] R. Abraham, J.E. Marsden, *Foundations of Mechanics, 2nd Ed.*, The Benjamin/Cummings Publ. Comp., Reading, 1978.

[2] C.W. Curtis, I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, New York, 1962.

[3] R.P. Feynman, Space-time approach to non-relativistic quantum mechanics, Rev. Mod. Phys 20 (1948) 327–346.

[4] A.A. Kirillov, Local Lie Algebras, Russian Math. Surveys 31 (1976) 56–75.

[5] L. Klapka, The problem of differentiable Poisson bracket of coordinates in mechanics, in: *Differential Geometry and Its Applications, Communications*, Proc. Conf. Brno, Czechoslovakia, 1986 (J.E. Purkyně Univ., Brno, 1987) 167–174.

[6] L. Klapka, Configuration in-out manifolds in mechanics, in: *Differential Geometry and Its Applications*, Proc. Conf. Brno, Czechoslovakia, 1989 (World Scientific, Singapore, 1990) 336–340.

[7] L. Klapka, The inversion of Whitehead's theorem on convex regions, in: *Proc. Conf. Diff. Geom. and Its Applications* (Silesian University, Opava, Czech Republic, 1993) 71–74.

[8] L. Klapka, Poisson manifolds of geodesic arcs, in: *Differential Geometry and Applications*, Proc. Conf. Brno, Czech Republic, 1995 (Masaryk University, Brno, 1996) 603–610.

[9] L. Klapka, Local expressions for Poisson manifolds of geodesic arcs in Lagrangian mechanics, in: *Differential Geometry and Applications*, Proc. Conf. Brno, Czech Republic, 1998 (Masaryk University, Brno, 1999) 503–509.

[10] L. Klapka, Lagrange functions generating Poisson manifolds of geodesic arcs, Rend. Circ. Mat. Palermo, Serie II, Suppl. 63 (2000) 113–119.

[11] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry, Volume I*, Interscience Publishers, New York, 1963.

[12] D. Krupka, A geometric theory of ordinary first order variational problems in fibered manifolds, I. Critical sections, J. Math. Anal. Appl. 49 (1975) 180–206.

[13] S. Lang, *Introduction to Differentiable Manifolds*, Interscience Publishers, New York, 1962.

[14] A. Lichnerowicz, Les variètes de Poisson et leur algébres de Lie associées, J. Diff. Geom. 12 (1977) 253–300.

[15] P.W. Michor, *Manifolds of Differentiable Mappings*, Shiva Publishing Limited, Kent, 1980.

[16] A.M. Vinogradov, I.S. Krasil'shchik, What is the Hamiltonian formalism? Russian Math. Surveys 30 (1975) 177–202.

[17] A. Weinstein, The local structure of Poisson manifolds, J. Diff. Geom. 18 (1983) 523–557.

[18] J.H.C. Whitehead, Convex regions in the geometry of paths, Quart. J. Math. 3 (1932) 33–42.

Lubomír Klapka

Mathematical Institute
Silesian University in Opava
Bezručovo nám. 13, 746 01 Opava
Czech Republic

E-mail: Lubomir.Klapka@math.slu.cz

Received 20 September 2000