Proceedings of the Seminar on Differential Geometry
Mathematical Publications Volume 2
Silesian University in Opava, Opava, 2000, 39–85

Jets and contact elements[1]

D. Krupka and M. Krupka

Abstract. The purpose of this research-expository work is to introduce basic concepts of the theory of jets, and to study their general properties. An r-jet of a real function of several real variables at a point is simply the collection of the coefficients of the r-th Taylor polynomial of f at this point. The concept of an r-jet is easily generalized to differentiable mappings of smooth manifolds in terms of charts. The structure of the following manifolds of jets is discussed:
(a) higher order differential groups,
(b) jets of mappings of a Euclidean space into a manifold, with source at the origin (velocities, regular velocities, higher order frames),
(c) manifolds of contact elements (higher order Grassmann prolongations of a manifold, i.e., the quotients of manifolds of regular velocities by the differential groups acting on them).
(d) jet prolongations of fibered manifolds and fibrations,
(e) jet prolongations of Lie groups, Lie group actions, principal and associated bundles.

Keywords and phrases. Jet, regular jet, velocity, differential groups, frame, contact element, Grassmann prolongation, prolongation of a fibered manifold, prolongation of a Lie group, prolongation of a principal bundle, prolongation of an associated bundle.

MS classification. 58A20.

Introduction

In this work, we present a self-contained introduction to the theory of jets, suitable for a deeper, systematic study of the subject. We explain basic ideas, and give proofs of all assertions. The choice of topic we discuss corresponds with the use of the theory

[1] Research supported by Grants CEZ:J10/98:192400002, VS 96003 “Global Analysis” of the Czech Ministry of Education, Youth and Sports, and by Grant 201/00/0724 of the Czech Grant Agency.
This paper is in final form and no part of it will be published elsewhere.

of jets in differential geometry (natural bundles, differential invariants), the calculus of variations on smooth manifolds (Lagrange theory, natural variational principles), and in mathematical physics (higher order mechanics and field theory).

Our basic references are [3], [4], [5], [6], and [8]. It is not our aim to simplify, or to shorten the exposition to a minimum. Instead, we insist on a deeper, active understanding of basic motions, as well as techniques of working with jets. We do not discuss possible generalizations of the theory to more abstract categories than the basic ones of the smooth differential geometry (the categories of smooth manifolds and fiber bundles). Recent developments in this direction can be found in [5]; neither it is our goal to discuss applications (see e.g. [6], [7], [11]). Numerous references to all these subjects can be found in [5], [8], [10], and [11].

1. Jets of smooth mappings

1.1. The higher order chain rule. Let n and k be positive integers. As usual, we denote by $D_i f = \partial f/\partial x^i$ the i-the partial derivative of a function $f : \mathbf{R}^n \to \mathbf{R}$. If $I = \{i_1, i_2, \dots, i_k\}$ is a set of positive integers such that $1 \le i_1, i_2, \dots, i_k \le n$, we denote

$$D_I = D_{i_1} D_{i_2} \cdots D_{i_k}. \tag{1}$$

Since the partial derivative operators commute, the symbol on the left hand side is correctly defined. The following explicit formula has numerous applications.

Lemma 1. *Let $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$ be open sets, let $f : V \to \mathbf{R}$ be a smooth function, and let $g = (g^\sigma)$, $1 \le \sigma \le m$, be a smooth mapping of U into V. Then*

$$\begin{aligned} &D_{i_s} \cdots D_{i_2} D_{i_1}(f \circ g)(t) \\ &\quad = \sum_{k=1}^{s} \sum_{(I_1, I_2, \dots, I_k)} D_{\sigma_k} \cdots D_{\sigma_2} D_{\sigma_1} f(g(t))\, D_{I_k} g^{\sigma_k}(t) \\ &\quad \cdots D_{I_2} g^{\sigma_2}(t) D_{I_1} g^{\sigma_1}(t), \end{aligned} \tag{2}$$

where the second sum is understood to be extended to all partitions $(I_1, I_2, \dots, I_k)$ of the set $\{i_1, i_2, \dots, i_s\}$.

Proof. To prove (2), we proceed by induction. We have

$$\begin{aligned} &D_{i_1}(f \circ g)(t) = D_\sigma f(g(t)) D_{i_1} g^\sigma(t), \\ &D_{i_2} D_{i_1}(f \circ g)(t) = D_{\sigma_2} D_{\sigma_1} f(g(t)) D_{i_2} g^{\sigma_2}(t) D_{i_1} g^{\sigma_1}(t) \\ &\quad + D_\sigma f(g(t)) D_{i_1 i_2} g^\sigma(t). \end{aligned} \tag{3}$$

Now assuming that

$$\begin{aligned} &D_{i_{s-1}} \cdots D_{i_2} D_{i_1}(f \circ g)(t) \\ &\quad = \sum_{k=1}^{s} \sum_{(J_1, J_2, \dots, J_k)} D_{\sigma_k} \cdots D_{\sigma_2} D_{\sigma_1} f(g(t))\, D_{J_k} g^{\sigma_k}(t) \\ &\quad \cdots D_{J_2} g^{\sigma_2}(t) D_{J_1} g^{\sigma_1}(t) \end{aligned} \tag{4}$$

we obtain

$$
\begin{aligned}
&D_{i_s} D_{i_{s-1}} \cdots D_{i_2} D_{i_1} (f \circ g)(t) \\
&= \sum_{k=1}^{s} \sum_{(J_1, J_2, \dots, J_k)} D_{\sigma_s} D_{\sigma_k} \cdots D_{\sigma_2} D_{\sigma_1} f(g(t))\, D_{i_s} g^{\sigma_s}(t) D_{J_k} g^{\sigma_k}(t) \\
&\cdots D_{J_2} g^{\sigma_2}(t) D_{J_1} g^{\sigma_1}(t) \\
&+ \sum_{k=1}^{s} \sum_{(J_1, J_2, \dots, J_k)} \big(D_{\sigma_k} \cdots D_{\sigma_2} D_{\sigma_1} f(g(t))\, (D_{J_k} g^{\sigma_k}(t) \\
&\cdots D_{J_2} g^{\sigma_2}(t) D_{i_s} D_{J_1} \alpha^{\sigma_1}(t) + D_{J_k} g^{\sigma_k}(t) \cdots D_{i_s} D_{J_2} g^{\sigma_2}(t) D_{J_1} g^{\sigma_1}(t) \\
&+ \cdots + D_{i_s} D_{J_k} g^{\sigma_k}(t) \cdots D_{J_2} g^{\sigma_2}(t) D_{J_1} g^{\sigma_1}(t) \big) \\
&= \sum_{k=1}^{s} \sum_{(I_1, I_2, \dots, I_k)} D_{\sigma_k} \cdots D_{\sigma_2} D_{\sigma_1} f(g(t))\, D_{I_k} g^{\sigma_k}(t) \cdots D_{I_2} g^{\sigma_2}(t) D_{I_1} g^{\sigma_1}(t)
\end{aligned}
\tag{5}
$$

which gives (2).

Formula (2) is called the *higher order chain rule*, or simply the *chain rule*.

1.2. Jets of smooth mappings. Let X and Y be two manifolds, $x \in X$ a point, W_1, W_2 two neighborhoods of x. We say that two mappings of class C^0 $f_1 : W_1 \to Y$, and $f_2 : W_2 \to Y$ are *tangent of order* 0 at x, if $f_1(x) = f_2(x)$. If $r \geq 1$ is an integer, we say that two mappings of class C^r $f_1 : W_1 \to Y$ and $f_2 : W_2 \to Y$ are *tangent to the r-th order at x*, if they are tangent of order 0 (as mappings of class C^0), and there exist a chart (U, φ), $\varphi = (x^i)$, at x and a chart (V, ψ), $\psi = (y^\sigma)$, at $f_1(x) = f_2(x)$ such that $U \subset W_1 \cap W_2$, $f_1(U)$, $f_2(U) \subset V$, and

$$D^k(\psi f_1 \varphi^{-1})(\varphi(x)) = D^k(\psi f_2 \varphi^{-1})(\varphi(x)) \tag{1}$$

for all $k \leq r$. We say that two mappings of class C^∞ $f_1 : W_1 \to Y$ and $f_2 : W_2 \to Y$ are *tangent to order* ∞ at x, if they are tangent to order r for every r.

Let $r \geq 1$. If in components, $\psi f_1 \varphi^{-1} = (y^\sigma f_1 \varphi^{-1})$, $\psi f_2 \varphi^{-1} = (y^\sigma f_2 \varphi^{-1})$, then f_1 and f_2 are tangent to order r at x if and only if $f_1(x) = f_2(x)$ and

$$D_{i_1} D_{i_2} \cdots D_{i_k} (y^\sigma f_1 \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \cdots D_{i_k} (y^\sigma f_2 \varphi^{-1})(\varphi(x)) \tag{2}$$

for all $k = 1, 2, \dots, r$, where $1 \leq i_1, i_2, \dots, i_k \leq n$, $1 \leq \sigma \leq m$.

If f_1, f_2 are tangent to order r at x, then for any chart $(\bar{U}, \bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$, at x and any chart $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, at $f_1(x) = f_2(x)$,

$$D^k(\bar{\psi} f_1 \bar{\varphi}^{-1})(\bar{\varphi}(x)) = D^k(\bar{\psi} f_2 \bar{\varphi}^{-1})(\bar{\varphi}(x)) \tag{3}$$

for all $k = 1, 2, \dots, r$. To see it we express the derivative

$$
\begin{aligned}
&D_{i_1} D_{i_2} \cdots D_{i_k} (\bar{y}^\sigma f_1 \bar{\varphi}^{-1})(\bar{\varphi}(x)) \\
&\quad = D_{i_1} D_{i_2} \cdots D_{i_k} (\bar{y}^\sigma \psi^{-1} \circ \psi f_1 \varphi^{-1} \circ \varphi \bar{\varphi}^{-1})(\bar{\varphi}(x))
\end{aligned}
\tag{4}
$$

as a polynomial in the variables $D_{j_1}(y^\nu f_1 \varphi^{-1})(\varphi(x))$, $D_{j_1} D_{j_2}(y^\nu f_1 \varphi^{-1})(\varphi(x))$, ..., $D_{j_1} D_{j_2} \cdots D_{j_k}(y^\nu f_1 \varphi^{-1})(\varphi(x))$ (Section 1.1, Lemma 1). Then the derivative $D_{i_1} D_{i_2} \cdots D_{i_k}$

$(\bar{y}^\sigma f_2\bar{\varphi}^{-1})(\bar{\varphi}(x))$ is expressed by the same polynomial in the variables $D_{j_1}(y^\nu f_2\varphi^{-1})(\varphi(x))$, $D_{j_1}D_{j_2}(y^\nu f_2\varphi^{-1})(\varphi(x)), \dots, D_{j_1}D_{j_2}\cdots D_{j_k}(y^\nu f_2\varphi^{-1})(\varphi(x))$. Now (3) follows from (2).

Let $r \geq 0$ be an integer, or $r = \infty$. Fix two points $x \in X$, $y \in Y$, and denote by $C^r_{(x,y)}(X, Y)$ the set of mappings of class C^r $f : W \to Y$, where W is a neighborhood of x, such that $f(x) = y$ (W is not fixed). The relation "*f, g are tangent to order r at x*" on $C^r_{(x,y)}(X, Y)$ is obviously *reflexive*, *transitive*, and *symmetric*, so it is an equivalence. Equivalence classes of this equivalence are called *r-jets* with *source x* and *target y*. The r-jet whose *representative* is a mapping $f \in C^r_{(x,y)}(X, Y)$ is called the *r-jet of f at x*, and is denoted by $J^r_x f$. If there is no danger of confusion we call an r-jet with source x and target y simply an *r-jet*, or a *jet*.

The set of r-jets with source $x \in X$ and target $y \in Y$ is denoted by $J^r_{(x,y)}(X, Y)$. Clearly, $J^0_x f = (x, y)$, and $J^0_{(x,y)}(X, Y) = \{(x, y)\}$.

Let $r \geq 0$ be an integer, or $r = \infty$. Let $f \in C^r_{(x,y)}(X, Y)$, $f : W \to Y$. If U is a neighborhood of the point $x \in X$ and V is a neighborhood of $y \in Y$, we may, using continuity arguments, restrict the range and the domain of f to V and to U. Let $\iota_{V,Y} : V \to Y$ and $\iota_{U\cap f^{-1}(V),W} : U \cap f^{-1}(V) \to W$ be the canonical inclusions. We define $f' \in C^r_{(x,y)}(U, V)$ by the formula $f' = \iota^{-1}_{V,Y} \circ f \circ \iota_{U\cap f^{-1}(V)}$, which induces a mapping ν of $J^r_{(x,y)}(X, Y)$ into $J^r_{(x,y)}(U, V)$. Conversely, if $f' \in C^r_{(x,y)}(U, V)$, we define f by $f = \iota_{V,Y} \circ f' \circ \iota^{-1}_{U\cap f^{-1}(V),W}$, which induces a mapping ι of $J^r_{(x,y)}(U, V)$ into $J^r_{(x,y)}(X, Y)$. Explicitly,

$$\begin{aligned} \nu(J^r_x f) &= J^r_x\big(\iota^{-1}_{V,Y} \circ f \circ \iota_{U\cap f^{-1}(V)}\big), \\ \iota(J^r_x f') &= J^r_x\big(\iota_{V,Y} \circ f' \circ \iota^{-1}_{U\cap f^{-1}(V),W}\big). \end{aligned} \tag{5}$$

Both ν and ι are bijections, and $\nu = \iota^{-1}$ is its inverse. The mappings ι, ν are called the *canonical identifications* of $J^r_{(x,y)}(U, V)$ and $J^r_{(x,y)}(X, Y)$.

Now we introduce a C^r structure on the set $J^r_{(x,y)}(X, Y)$. Let (U, φ), $\varphi = (x^i)$ be a chart at x, and let (V, ψ), $\psi = (y^\sigma)$, be a chart at y. We set for every $J^r_x \in J^r_{(x,y)}(X, Y)$

$$y^\sigma_{i_1 i_2\cdots i_k}(J^r_x f) = D_{i_1}D_{i_2}\cdots D_{i_k}(y^\sigma f\varphi^{-1})(\varphi(x)), \tag{6}$$

where $1 \leq k \leq r$, $1 \leq \sigma \leq m$, and $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$. $y^\sigma_{i_1 i_2\cdots i_k}$ are real-valued functions on $J^r_{(x,y)}(X, Y)$. Then we set

$$\chi^r_{\varphi,\psi}(J^r_x f) = \big(y^\sigma_{i_1}(J^r_x f), y^\sigma_{i_1 i_2}(J^r_x f), \dots, y^\sigma_{i_1 i_2\cdots i_r}(J^r_x f)\big). \tag{7}$$

This defines (in components) a mapping $\chi^r_{\varphi,\psi} : J^r_{(x,y)}(X, Y) \to \mathbf{R}^N$, where

$$N = m\left(\binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+r-1}{r}\right) = m\left(\binom{n+r}{n} - 1\right). \tag{8}$$

In connection with the use of Section 1.1, Lemma 1, we also apply a different notation. If $I = \{i_1, i_2, \dots, i_k\}$ is a set of positive integers such that $1 \leq i_1, i_2, \dots, i_k \leq n$, we denote

$$y^\sigma_I(J^r_x f) = D_I(y^\sigma f\varphi^{-1})(\varphi(x)), \tag{9}$$

where D_I is given by Section 1.1, (1).

Lemma 2. *Let X and Y be two smooth manifolds. There exists one and only one smooth structure on $J^r_{(x,y)}(X,Y)$ such that for every chart (U,φ), $\varphi = (x^i)$, at x and every chart (V,ψ), $\psi = (y^\sigma)$, at y, $(J^r_{(x,y)}(X,Y), \chi^r_{\varphi,\psi})$, $\chi^r_{\varphi,\psi}(J^r_x f) = (y^\sigma_{i_1 i_2 \cdots i_k})$ is a chart on $J^r_{(x,y)}(X,Y)$.*

Proof. First we show that the mapping $\chi^r_{\varphi,\psi} : J^r_{(x,y)})(X,Y) \to \mathbf{R}^N$ is a bijection. It follows immediately from the definition of an r-jet that $\chi^r_{\varphi,\psi}$ is injective. To show that it is surjective, choose a point $A = (A^\sigma_{i_1}, A^\sigma_{i_1 i_2}, \dots, A^\sigma_{i_1 i_2 \cdots i_r}) \in \mathbf{R}^N$; here we assume that $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$ for every $k = 1, 2, \dots, r$. We extend the system A to *all* sequences $(j_1, j_2, \dots, j_k)$ putting $A^\sigma_{j_1 j_2 \cdots j_k} = A^\sigma_{i_1 i_2 \cdots i_k}$ whenever $(j_1, j_2, \dots, j_k)$ is a permutation of $(i_1, i_2, \dots, i_k)$, and define a mapping $g : \mathbf{R}^n \to \mathbf{R}^m$, $g = (g^\sigma)$, by the formula

$$\begin{aligned} g^\sigma(x^1, x^2, \dots, x^n) &= y^\sigma_0 + A^\sigma_{j_1}(x^{j_1} - x^{j_1}_0) + \frac{1}{2!} A^\sigma_{j_1 j_2}(x^{j_1} - x^{j_1}_0)(x^{j_2} - x^{j_2}_0) \\ &+ \cdots + \frac{1}{r!} A^\sigma_{i_1 i_2 \cdots i_k}(x^{j_1} - x^{j_1}_0)(x^{j_2} - x^{j_2}_0) \cdots (x^{j_r} - x^{j_r}_0), \end{aligned} \tag{10}$$

where $x_0 = (x^j_O) = \varphi(x)$, $y_0 = (y^\sigma_0) = \psi(y)$. Then $\psi^{-1} g \varphi(x) = y$. Putting

$$f = \psi^{-1} g \varphi \tag{11}$$

we obtain a smooth mapping defined on a neighborhood of x, such that $f(x) = y$. Therefore, $J^r_x f \in J^r_{(x,y)}(X,Y)$, and by (7), $\chi^r_{\varphi,\psi}(J^r_x f) = (D_{i_1} g^\sigma(x_0), D_{i_1} D_{i_2} g^\sigma(x_0), \dots, D_{i_1} D_{i_2} \cdots D_{i_k} g^\sigma(x_0)) = A$. This proves that $\chi^r_{\varphi,\psi}$ is surjective and completes the proof that it is bijective.

Let (U,φ), $\varphi = (x^i)$, and $(\bar{U}, \bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$, be two charts at x and let (V,ψ), $\psi = (y^\sigma)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, be two charts at y. We have for every $J^r_x f \in J^r_{(x,y)}(X,Y)$

$$\bar{y}^\sigma_{i_1 i_2 \cdots i_k}(J^r_x f) = D_{i_1} D_{i_2} \cdots D_{i_k}(\bar{y}^\sigma f \bar{\varphi}^{-1})(\bar{\varphi}(x)). \tag{12}$$

Expressing the right-hand side as in (4), and using Section 1.1, Lemma 1, we obtain a polynomial in $y^\nu_{j_1}(J^r_x f), y^\nu_{j_1 j_2}(J^r_x f), \dots, y^\nu_{j_1 j_2 \cdots j_k}(J^r_x f)$. Since these polynomials are components of the mapping $\chi^r_{\bar{\varphi},\bar{\psi}} \circ (\chi^r_{\varphi,\psi})^{-1}$, this mapping is smooth. This proves compatibility of the charts $(J^r_{(x,y)}(X,Y), \chi^r_{\varphi,\psi})$, $(J^r_{(x,y)}(X,Y), \chi^r_{\bar{\varphi},\bar{\psi}})$.

The chart $(J^r_{(x,y)}(X,Y), \chi^r_{\varphi,\psi})$ is said to be *associated* with the pair of charts (U,φ), (V,ψ).

Remark 1. The manifold topology on $J^r_{(x,y)}(X,Y)$ is the topology of the Euclidean space $\mathbf{R}^N$.

Remark 2 (geometric interpretation of an r-jet). We denote

$$L^r_{n,m} = J^r_{(0,0)}(\mathbf{R}^n, \mathbf{R}^m), \tag{13}$$

and define for every $J^r_0 f \in L^r_{n,m}$,

$$a^\sigma_{i_1 i_2 \cdots i_k}(J^r_x f) = D_{i_1} D_{i_2} \cdots D_{i_k} f^\sigma(0), \tag{14}$$

where $f = (f^\sigma)$, $1 \le \sigma \le m$, $1 \le k \le r$, $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$. The real-valued functions $a^\sigma_{i_1 i_2 \cdots i_k}$ define a chart on $J^r_{(0,0)}(\mathbf{R}^n, \mathbf{R}^m)$ (in this case $\varphi = \mathrm{id}_{\mathbf{R}^n}$, $\psi = \mathrm{id}_{\mathbf{R}^m}$). This chart, as well as its coordinate functions (14), are called *canonical*.

Consider the product

(15) $$L(\mathbf{R}^n, \mathbf{R}^m) \times L^2_{(s)}(\mathbf{R}^n, \mathbf{R}^m) \times \cdots \times L^r_{(s)}(\mathbf{R}^n, \mathbf{R}^m),$$

where $L(\mathbf{R}^n, \mathbf{R}^m)$ is the vector space of linear mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$, and $L^k_{(s)}(\mathbf{R}^n, \mathbf{R}^m)$ is the vector space of k-linear, *symmetric* mappings from $\mathbf{R}^n \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n$ (k factors) to $\mathbf{R}^m$. Using the canonical bases of $\mathbf{R}^n$ and $\mathbf{R}^m$, we can identify vectors in $L(\mathbf{R}^n, \mathbf{R}^m)$ (resp. $L^k_{(s)}(\mathbf{R}^n, \mathbf{R}^m)$) with their matrices (A^σ_i) (resp. $(A^\sigma_{i_1 i_2 \cdots i_k})$), where $1 \le \sigma \le m$, $1 \le i_1, i_2, \ldots, i_k \le n$. The matrix $(A^\sigma_{i_1 i_2 \cdots i_k})$ is *symmetric* in the subscripts, so that the dimension of the vector space (15) is N (8).

Clearly, (15) carries *canonical* topological and smooth structures of a finite-dimensional vector space.

Since $J^r_{(x,y)}(X, Y)$, $L^r_{n,m}$, and the vector space (15) are diffeomorphic with $\mathbf{R}^N$, they are all diffeomorphic. A diffeomorphism of $L^r_{n,m}$ and the vector space (15) is obtained by extending the set of canonical coordinates $a^\sigma_{i_1 i_2 \cdots i_k}$ to *all* (not necessarily non-decreasing) sequences $(j_1, j_2, \ldots, j_k)$ by putting $a^\sigma_{j_1 j_2 \cdots j_k} = a^\sigma_{i_1 i_2 \cdots i_k}$ whenever $(j_1, j_2, \ldots, j_k)$ is a permutation of $(i_1, i_2, \ldots, i_k)$. The diffeomorphism obtained in this way is called the *canonical identification*, and gives us a geometric interpretation of the r-jets belonging to the set $L^r_{n,m}$.

Let X and Y be smooth manifolds $n = \dim X$, $m = \dim Y$. We denote

(16) $$\begin{aligned} &J^0_x(X, Y) = \{x\} \times Y, \quad J^0(X, Y) = X \times Y, \\ &J^r_x(X, Y) = \bigcup_{y \in Y} J^r_{(x,y)}(X, Y), \quad J^r(X, Y) = \bigcup_{x \in X} J^r_x(X, Y), \quad r \le 1. \end{aligned}$$

For every $J^r_x f \in J^r(X, Y)$, $P = J^r_x f$, we set

(17) $$\rho^{r,s}(J^r_x f) = J^s_x f, \quad 0 \le s \le r, \quad \mu^r(J^r_x f) = x, \quad \nu^r(J^r_s f) = f(x).$$

These formulas define the *canonical r-jet projections* $\rho^{r,s} : J^r(X, Y) \to J^s(X, Y)$, $\mu^r : J^r(X, Y) \to X$, and $\nu^r : J^r(X, Y) \to Y$. μ^r (resp. ν^r) is sometimes called the *source* (resp. *target*) *projection*. The r-jet projections restrict naturally to the subsets $J^r_{(x,y)}(X, Y)$ and $J^r_x(X, Y)$ of $J^r(X, Y)$.

We introduce a C^r structure on the sets $J^r_{(x,y)}(X, Y)$, $J^r_x(X, Y)$, and $J^r(X, Y)$. Let (U, φ), $\varphi = (x^i)$, be a chart on X, and let (V, ψ), $\psi = (y^K)$, be a chart on Y. We set

(18) $$W^r = (\rho^{r,0})^{-1}(U \times V), \qquad \chi^r_{\varphi,\psi} = \left(x^i, y^K, \chi^K_{i_1}, \chi^K_{i_1 i_2}, \ldots, \chi^K_{i_1 i_2 \cdots i_r}\right),$$

where $1 \le k \le r$, $1 \le K \le m$, $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$, and $\chi^K_{i_1 i_2 \cdots i_k}$ are real-valued functions on W^r defined by

(19) $$\chi^K_{i_1 i_2 \cdots i_k}(J^r_x f) = D_{i_1} D_{i_2} \cdots D_{i_k}(y^K f \varphi^{-1})(\varphi(x)).$$

Clearly, $\chi^r_{\varphi,\psi}$ is a mapping of W^r into $\varphi(U) \times \psi(V) \times \mathbf{R}^N$, where

(20) $$N = m\left(\binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+r-1}{r}\right) = m\left(\binom{n+r}{n} - 1\right).$$

Sometimes it is convenient to use an alternative notation. If $I = \{i_1, i_2, \ldots, i_k\}$ is a set of positive integers such that $1 \le i_1, i_2, \ldots, i_k \le n$, we denote

(21) $$\chi^K_I(J^r_x f) = D_I(y^K f \varphi^{-1})(\varphi(x)).$$

where $D_I = D_{i_1} D_{i_2} \cdots D_{i_k}$ (see Section 1.1). Then in components, $\chi^r_{\varphi,\psi} = (x^i, y^K, \chi^K_I)$.

Lemma 3. *Let X and Y be smooth manifolds.*

(a) *There exists one and only one smooth structure on $J^r(X, Y)$ such that for every chart (U, φ) on X and every chart (V, ψ) on Y, $(W^r, \chi^r_{\varphi,\psi})$ is a chart on $J^r(X, Y)$. In this smooth structure, the r-jet projections are smooth surjective submersions.*

(b) *For every $x \in X$, the set $J^r_x(X, Y)$ is a submanifold of $J^r(X, Y)$. If (U, φ) is a chart at x, and (V, ψ) is a chart on Y, then the chart $(W^r, \chi^r_{\varphi,\psi})$ is adapted to $J^r_x(X, Y)$.*

(c) *For every $(x, y) \in X \times Y$, the set $J^r_{(x,y)}(X, Y)$ is a submanifold of $J^r(X, Y)$. If (U, φ) is a chart at x, and (V, ψ) is a chart at y, then the chart $(W^r, \chi^r_{\varphi,\psi})$ is adapted to $J^r_{(x,y)}(X, Y)$.*

Proof. (a) First we show that $\chi^r_{\varphi,\psi}$ is a bijection. It follows immediately from the definition of an r-jet that $\chi^r_{\varphi,\psi}$ is injective. To show that it is surjective, choose $x_0 \in \varphi(U)$, $y_0 \in \psi(V)$, and a point $P = (P^K_{i_1}, P^K_{i_1 i_2}, \ldots, P^K_{i_1 i_2 \cdots i_r}) \in \mathbf{R}^N$; here $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$ for every $k = 1, 2, \ldots, r$. We extend P to *all* sequences $(j_1, j_2, \ldots, j_k)$ putting $P^K_{j_1 j_2 \cdots j_k} = P^K_{i_1 i_2 \cdots i_k}$ whenever $(j_1, j_2, \ldots, j_k)$ is a permutation of $(i_1, i_2, \ldots, i_k)$, and define a mapping $g : \mathbf{R}^n \to \mathbf{R}^m$, $g = (g^K)$, by the formula

$$
\begin{aligned}
g^K(x^1, x^2, \ldots, x^n) &= y^K_0 + P^\sigma_{j_1}(x^{j_1} - x^{j_1}_0) + \frac{1}{2!} P^K_{j_1 j_2}(x^{j_1} - x^{j_1}_0)(x^{j_2} - x^{j_2}_0) \\
&\quad + \cdots + \frac{1}{r!} P^K_{i_1 i_2 \cdots i_k}(x^{j_1} - x^{j_1}_0)(x^{j_2} - x^{j_2}_0) \cdots (x^{j_r} - x^{j_r}_0)
\end{aligned} \tag{22}
$$

where $x_0 = (x^j_0)$, $y_0 = (y^K_0)$. Then $x = \varphi^{-1}(x_0) \in U$, $y = \psi^{-1}(y_0) \in V$, and $g(x_0) = y_0$. Putting $f = \psi^{-1} g \varphi$, we obtain a smooth mapping defined on a neighborhood of x, such that $f(x) = y$. Since the chart expression of f satisfies $\psi f \varphi^{-1} = g$, we have

$$
\begin{aligned}
\chi^r_{\varphi,\psi}(J^r_x f) &= \big(x^i(x), y^K(y), D_{i_1} g^K(x_0), D_{i_1} D_{i_2} g^K(x_0), \\
&\quad \ldots, D_{i_1} D_{i_2} \cdots D_{i_k} g^K(x_0)\big) \\
&= \big(x^i_0, y^K_0, P^K_{i_1}, P^K_{i_1 i_2}, \ldots, P^K_{i_1 i_2 \cdots i_r}\big).
\end{aligned} \tag{23}
$$

This proves that $\chi^r_{\varphi,\psi}$ is surjective and completes the proof that it is bijective.

Let (U, φ), $\varphi = (x^i)$, and $(\bar{U}, \bar{\varphi})$, $\bar{\varphi} = (\bar{x}^i)$, be two charts on X such that $U \cap \bar{U} \neq \emptyset$, and let (V, ψ), $\psi = (y^K)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^K)$ be two charts at on Y such that $V \cap \bar{V} \neq \emptyset$. Define $(\bar{W}^r, \chi^r_{\bar{\varphi},\bar{\psi}})$, $\chi^r_{\bar{\varphi},\bar{\psi}} = (\bar{x}^i, \bar{y}^K, \bar{\chi}^K_{i_1 i_2 \cdots i_k})$ by (18) and (19). We have for every $J^r_x f \in W^r \cap \bar{W}^r$

$$
\begin{aligned}
\bar{\chi}^K_{i_1 i_2 \cdots i_k}(J^r_x f) &= D_{i_1} D_{i_2} \cdots D_{i_k}(\bar{y}^K f \varphi^{-1})(\bar{\varphi}(x)) \\
&= D_{i_1} D_{i_2} \cdots D_{i_k}(\bar{y}^K \psi^{-1} \circ \psi f \varphi^{-1} \circ \varphi \bar{\varphi}^{-1})(\bar{\varphi}(x)).
\end{aligned} \tag{24}
$$

Using the higher order chain rule (Section 1.1, Lemma 1), we obtain $\bar{\chi}^K_{i_1 i_2 \cdots i_k}(J^r_x f)$ as a polynomial in $\chi^L_{j_1}(J^r_x f)$, $\chi^L_{j_1 j_2}(J^r_x f), \ldots, \chi^L_{j_1 j_2 \cdots j_k}(J^r_x f)$. Since these polynomials are components of the mapping $\chi^r_{\bar{\varphi},\bar{\psi}} \circ (\chi^r_{\varphi,\psi})^{-1}$, this mapping is smooth. This proves compatibility of the charts $(W^r, \chi^r_{\varphi,\psi})$, $(\bar{W}^r, \chi^r_{\bar{\varphi},\bar{\psi}})$.

It is immediately seen that the jet projections (17) are expressed in the charts $(W^r, \chi^r_{\varphi,\psi})$ as the Cartesian projections. This shows that the jet projections are smooth.

(b) The set $W^r \cap J^r_x(X, Y)$ is expressed by equations of the form $x^i = a^i$, where $a^i \in \mathbf{R}$ are some constants. This proves (b).

(c) The set $W^r \cap J^r_{(xy)}(X, Y)$ is expressed by equations of the form $x^i = a^i$, $y^K = b^K$, where $a^i, b^K \in \mathbf{R}$ are some constants. This proves (c).

The chart $(W^r, \chi^r_{\varphi,\psi})$ is said to be *associated* with the charts (U, φ), (V, ψ).

Remark 3. Note that we have some canonical identifications. The r-jet $J^r_{\varphi(x)}\psi f \varphi^{-1}$ is by definition the equivalence class expressed in the *canonical coordinates* on $\mathbf{R}^n$ and $\mathbf{R}^{n+m}$ by the collection of real numbers $x^i(x), y^K(f(x))\ D_{j_1}(y^K f\varphi^{-1})(\varphi(x))$, $D_{j_1}D_{j_2}(y^K f\varphi^{-1})(\varphi(x)), \dots, D_{j_1}D_{j_2}\cdots D_{j_k}(y^\nu f\varphi^{-1})(\varphi(x))$, i.e., by the same collection as $J^r_x f$ in the associated chart $(W^r, \chi^r_{\varphi,\psi})$. Thus, we have

$$\chi^r_{\varphi,\psi}(J^r_x f) = J^r_{\varphi(x)}\psi f \varphi^{-1}. \tag{25}$$

Let X and Y be smooth manifolds, $W \subset X$ an open set, and $f : W \to Y$ a smooth mapping. Setting

$$J^r f(x) = J^r_x f \tag{26}$$

we define a mapping $J^r f : W \to J^r(X, Y)$. This mapping is called the *r-jet prolongation*, or simply the *jet prolongation* of f.

Let (U, φ), $\varphi = (x^i)$ (resp. (V, ψ), $\psi = (y^K)$) be a chart at x (resp. at $y = f(x)$), and let $(W^r, \chi^r_{\varphi,\psi})$ be the associated chart on $J^r(X, Y)$. Then $J^r f$ is expressed by

$$\begin{aligned}(27)\qquad &(\chi^r_{\varphi,\psi} \circ J^r f \circ \varphi^{-1})(x') \\ &\quad = \big(x, (y^K f\varphi^{-1})(x'), D_{i_1}D_{i_2}\cdots D_{i_k}(y^K f\varphi^{-1})(x')\big),\end{aligned}$$

and is therefore smooth.

1.3. The composition of jets. Let X, Y, and Z be three real, finite-dimensional smooth manifolds. We say that r-jets $P \in J^r_{(x,u)}(X, Y)$, $Q \in J^r_{(y,z)}(Y, Z)$ are *composable*, if any representatives of P and Q are composable (as mappings). Clearly, P and Q are composable if and only if the target of P coincides with the source of Q, i.e., if $u = y$.

Let P (resp. Q) be represented by f (resp. g), i.e., $P = J^r_x f$, $Q = J^r_y g$. Assume that P, Q are composable. Shrinking the domain of definition of f if necessary, we may assume that the composed mapping $g \circ f$ is defined. Then also the r-jet $J^r_x(g \circ f)$ is defined. It is easy to determine the coordinates of $J^r_x(g \circ f)$ in terms of the coordinates of P and Q.

Let (U, φ), $\varphi = (x^i)$ (resp. (V, ψ), $\psi = (y^\sigma)$, resp. (W, η), $\eta = (z^A)$) be a chart at x (resp. $y = f(x)$, resp. $z = g(y)$). We have in the chart $(J^r_{(x,z)}(X, Z), \chi^r_{\varphi,\eta})$, $\chi^r_{\varphi,\eta}(J^r_x(g \circ f)) = (w^A_{i_1 i_2\cdots i_k})$ (Section 1.2, Lemma 2),

$$\begin{aligned}(1)\qquad \chi^r_{\varphi,\eta}(J^r_x(g \circ f)) = \big(&D_{i_1}(z^A g f\varphi^{-1})(\varphi(x)), D_{i_1}D_{i_2}(z^A g f\varphi^{-1})(\varphi(x)), \\ &\dots, D_{i_1}D_{i_2}\cdots D_{i_r}(z^A g f\varphi^{-1})(\varphi(x))\big),\end{aligned}$$

i.e., for every $k = 1, 2, \dots, r$,

$$\begin{aligned}(2)\qquad w^A_{i_1 i_2\cdots i_k}(J^r_x(g \circ f)) &= D_{i_1}D_{i_2}\cdots D_{i_k}(z^A g f\varphi^{-1})(\varphi(x)) \\ &= D_{i_1}D_{i_2}\cdots D_{i_k}(z^A g\psi^{-1} \circ \psi f\varphi^{-1})(\varphi(x)).\end{aligned}$$

We apply the higher order chain rule to this expression (Section 1.1, Lemma 1). Denote the corresponding associated charts by $(J^r_{(x,y)}(X,Y), \chi^r_{\varphi,\psi})$, $\chi^r_{\varphi,\psi}(J^r_x f) = (y^\sigma_{i_1 i_2 \cdots i_k})$ $(J^r_{(y,z)}(Y,Z), \chi^r_{\psi,\eta})$, $\chi^r_{\psi,\eta}(J^r_x g) = (z^A_{\sigma_1\sigma_2\cdots\sigma_k})$. Then

$$w^A_{i_1 i_2 \cdots i_s}\big(J^r_x(g\circ f)\big) = \sum_{k=1}^{s} \sum_{(I_1,I_2,\dots,I_k)} z^A_{\sigma_1\sigma_2\cdots\sigma_k}(J^r_x g) y^{\sigma_k}_{I_k}(J^r_x f)\cdots y^{\sigma_2}_{I_2}(J^r_x f) y^{\sigma_1}_{I_1}(J^r_x f), \tag{3}$$

i.e., with obvious simplification,

$$w^A_{i_1 i_2 \cdots i_s} = \sum_{k=1}^{s} \sum_{(I_1,I_2,\dots,I_k)} z^A_{\sigma_1\sigma_2\cdots\sigma_k} y^{\sigma_k}_{I_k}\cdots y^{\sigma_2}_{I_2} y^{\sigma_1}_{I_1}. \tag{4}$$

Now by Section 1.2, (2), if $J^r_y g = J^r_y g'$ and $J^r_x f = J^r_x f'$, then $J^r_x(g\circ f) = J^r_x(g'\circ f')$ which means that the r-jet $J^r_x(g\circ f)$ depends on P and Q only.

If P and Q are *composable* r-jets, $P = J^r_x f$, $Q = J^r_y g$, we define

$$Q\circ P = J^r_x(g\circ f) \tag{5}$$

and call the r-jet $Q\circ P$ the *composite* of P and Q. The mapping $(P,Q)\to Q\circ P$ of $J^r_{(x,y)}(X,Y)\times J^r_{(y,z)}(Y,Z)$ into $J^r_{(x,z)}(X,Z)$ where $y = f(x)$, $z = g(y)$, is called the *composition* of r-jets. The composition of r-jets is *associative*.

Equation (3), or (4), is the r-jet *composition formula*.

In particular, we have the following result.

Lemma 4. *The composition of r-jets is smooth.*

Proof. By (3), the coordinates of the r-jet $Q\circ P$ depend polynomially on the coordinates of the r-jets P, Q.

1.4. Regular jets, invertible jets. Let id_X (resp. id_Y) be the identity mapping of a manifold X (resp. Y), $x\in X$ (resp. $y\in Y$) a point. Then $J^r_x \mathrm{id}_X \in J^r_{(x,x)}(X,X)$ and $J^r_y \mathrm{id}_Y \in J^r_{(y,y)}(Y,Y)$. For any r-jet $P\in J^r_{(x,y)}(X,Y)$, $P = J^r_x f$, the composites $J^r_y \mathrm{id}_Y\circ P = J^r_y \mathrm{id}_Y\circ J^r_x f$, $P\circ J^r_x \mathrm{id}_X = J^r_x f\circ J^r_x \mathrm{id}_X$, are defined, and

$$J^r_y \mathrm{id}_Y\circ P = P, \qquad P\circ J^r_x \mathrm{id}_X = P. \tag{1}$$

An r-jet $P\in J^r_{(x,y)}(X,Y)$ is called *regular*, if there exists an r-jet $Q\in J^r_{(y,x)}(Y,X)$, such that

$$Q\circ P = J^r_x \mathrm{id}_X\,. \tag{2}$$

P is called *invertible*, if there exists $Q\in J^r_{(y,x)}(Y,X)$ such that

$$Q\circ P = J^r_x \mathrm{id}_X, \qquad P\circ Q = J^r_y \mathrm{id}_Y\,. \tag{3}$$

Lemma 5. (a) *An r-jet $P\in J^r_{(x,y)}(X,Y)$ is regular if and only if every of its representatives is an immersion at the point x.*

(b) *An r-jet $P\in J^r_{(x,y)}(X,Y)$ is invertible if and only if every of its representatives is a diffeomorphism on a neighborhood of x.*

Proof. (a) Let f be a representative of an r-jet $P = J^r_x f$. Assume that we have an r-jet $Q = J^r_y g$ satisfying (2), and its representative g. Then the mappings $g \circ f$ and id_X represent the same r-jet with source and target x, and we have for any chart (U, φ) at x and any chart (V, ψ) at y

(4) $$D^1(\varphi g f \varphi^{-1})(\varphi(x)) = D^1(\varphi g \psi^{-1})(\psi f(x)) \circ D^1(\psi f \varphi^{-1})(\varphi(x)) = \mathrm{id}_{\mathbf{R}^n}.$$

In particular, $\operatorname{rank} D^1(\varphi g f \varphi^{-1})(\varphi(x)) = n$ which is the dimension of the image of the linear mapping $D^1(\varphi g f \varphi^{-1})(\varphi(x)) : \mathbf{R}^n \to \mathbf{R}^n$. This implies that $\operatorname{rank} D^1(\psi f \varphi^{-1})$ $(\varphi(x))$ must be equal to n. Therefore, f is an immersion at x, by the rank theorem. Conversely, if a representative f of P is an immersion at x, then we apply the rank theorem again.

(b) If P is invertible we easily find, using similar arguments, that $\dim Y = m$ must be equal to $\dim X = n$, and then we apply the rank theorem. The converse is obvious.

The set of regular r-jets in $J^r_{(x,y)}(X, Y)$, is denoted by $\operatorname{imm} J^r_{(x,y)}(X, Y)$; it is an open subset of $J^r_{(x,y)}(X, Y)$. Obviously, using continuity of the determinant function we easily show that the set W of points $(w^\sigma_{i_1}, w^\sigma_{i_1 i_2}, \dots, w^\sigma_{i_1 i_2 \cdots i_r}) \in \mathbf{R}^N$ such that the matrix (w^ν_j) is of maximal rank n, is open in $\mathbf{R}^N$. Then using a chart (U, φ), $\varphi = (x^i)$, at x, a chart (V, ψ), $\psi = (y^\sigma)$, at y, and the associated chart $(J^r_{(x,y)}(X, Y), \chi^r_{\varphi,\psi})$, $\chi^r_{\varphi,\psi} = (y^\sigma_{i_1}, y^\sigma_{i_1 i_2}, \dots, y^\sigma_{i_1 i_2 \cdots i_r})$ on $J^r_{(x,y)}(X, Y)$, we obtain the set $\operatorname{imm} J^r_{(x,y)}(X, Y)$ as the inverse image of W by the continuous mapping $\chi^r_{\varphi,\psi}$.

$\operatorname{imm} J^r_{(x,y)}(X, Y) \neq \emptyset$ if and only if $\dim X = n \leq \dim Y = m$.

If $n = m$, then the set $\operatorname{imm} J^r_{(x,y)}(X, Y)$ consists of *invertible* r-jets. Conversely, if the set $\operatorname{imm} J^r_{(x,y)}(X, Y)$ contains an invertible r-jet, then the points x and y have neighborhoods of the same dimension.

2. Jet manifolds

2.1. Differential groups. Let r, n be positive integers. We denote

(1) $$L^r_n = \operatorname{imm} J^r_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n).$$

Thus, L^r_n is the set of *invertible* r-jets in the jet manifold $J^r_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n)$. Restricting the canonical coordinates $a^i_{j_1 j_2 \cdots j_k}$ on $J^r_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n)$ (Section 1.2, (14)) to L^r_n we obtain the *canonical coordinates* on L^r_n

(2) $$a^i_{j_1 j_2 \cdots j_k}(J^r_x \alpha) = D_{j_1} D_{j_2} \cdots D_{j_k} \alpha^i(0),$$

where $\alpha = (\alpha^i)$, $1 \leq i \leq n$, $1 \leq k \leq r$, $1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n$. In these coordinates $L^r_n = \{J^r_0 \alpha \in J^r_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n) \mid \det a^i_j(J^r_0 \alpha) \neq 0\}$.

The canonical coordinates (2) will be also written by means of the convention introduced in Section 1.2, (9). Namely, if $I = \{i_1, i_2, \dots, i_k\}$ is a set of positive integers such that $1 \leq i_1, i_2, \dots, i_k \leq n$, we also write

(3) $$a^i_I = a^i_{j_1 j_2 \cdots j_k}.$$

The composition of jets (see Section 1.3, (5)) defines an *operation*

(4) $$L^r_n \times L^r_n \ni (A, B) \to A \circ B \in L^r_n$$

on the set L^r_n. This operation is *associative*, the r-jet $J^r_0 \,\mathrm{id}_{\mathbf{R}^n} \in L^r_n$ is the *unity*, and every r-jet $A \in L^r_n$, $A = J^r_0\alpha$ has a unique inverse $A^{-1} = J^r_0\alpha^{-1}$. Thus, (4) defines a *group structure* on L^r_n. Since the composition of r-jets is smooth (Section 1.3, Lemma 3), L^r_n is a *Lie group*. We call this Lie group the r-th *differential group of* $\mathbf{R}^n$, or simply a *differential group*. From Section 1.2, (8) we derive that

$$\dim L^r_n = n\left(\binom{n+r}{n} - 1\right). \tag{5}$$

Note that L^1_n can be cannonically identified with the *general linear group* $GL_n(\mathbf{R})$.

Using the r-jet composition formula (Section 1.3, (4)) and the canonical coordinates (2), (3), we can describe the group operation (4) explicitly. If $A, B \in L^r_n$, $A = J^r_0\alpha$, $B = J^r_0\beta$, and $C = A \circ B = J^r_0(\alpha \circ \beta)$, and $a^k_{i_1 i_2 \cdots i_s} = a^k_{i_1 i_2 \cdots i_s}(J^r_0\alpha)$, $b^k_{i_1 i_2 \cdots i_s} = a^k_{i_1 i_2 \cdots i_s}(J^r_0\beta)$, $c^k_{i_1 i_2 \cdots i_s} = a^k_{i_1 i_2 \cdots i_s}(J^r_0(\alpha \circ \beta))$, then

$$c^k_{i_1 i_2 \dots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} a^k_{j_1 j_2 \dots j_p} b^{j_1}_{I_1} b^{j_2}_{I_2} \cdots b^{j_p}_{I_p}, \tag{6}$$

where the second sum is extended to all partitions $(I_1, I_2, \dots, I_p)$ of the set $(i_1, i_2, \dots, i_s)$.

Example 1 (group operation in L^3_n). In applications explicit chart expressions for group operation in differential groups are needed. Using the definition, we derive the corresponding formulas for the group L^3_n in the canonical coordinates. Let $A, B \in L^3_n$ be two 3-jets. Let $U, V, W \subset \mathbf{R}^n$ be three neighborhoods of the origin $0 \in \mathbf{R}^n$, $\alpha : U \to V$, $\beta : V \to W$ two diffeomorphisms such that $A = J^3_0\alpha$, $B = J^3_0\beta$. Denote by (x^i) the canonical coordinates on $\mathbf{R}^n$ (as well as on U, V, and W). Write in components $\alpha = (x^i\alpha)$, $\beta = (x^i\beta)$, and consider the diffeomorphism $\gamma = \beta \circ \alpha$ of U into W, $\gamma = (x^i\gamma)$. Then the product of A and B in L^3_n is the 3-jet $C = J^3_0\gamma$. To obtain the canonical coordinates of C we should compute all partial derivatives of the components of γ up to the 3-rd order at the point $0 \in \mathbf{R}^n$. Differentiating components of this diffeomorphism at a point $x \in U$, we obtain

$$\begin{aligned}
D_{j_1}(x^i\gamma)(x) &= D_{j_1}(x^i\beta \circ \alpha)(x) \\
&= D_k(x^i\beta)(\alpha(x)) D_{j_1}(x^k\alpha)(x), \\
D_{j_2}D_{j_1}(x^i\gamma)(x) &= D_{j_2}D_{j_1}(x^i\beta \circ \alpha)(x) \\
&= D_{k_2}D_{k_1}(x^i\beta)(\alpha(x)) D_{j_2}(x^{k_2}\alpha)(x) D_{j_1}(x^{k_1}\alpha)(x) \\
&\quad + D_k(x^i\beta)(\alpha(x)) D_{j_2}D_{j_1}(x^k\alpha)(x), \\
D_{j_3}D_{j_2}D_{j_1}(x^i\gamma)(x) &= D_{j_3}D_{j_2}D_{j_1}(x^i\beta \circ \alpha)(x) \\
&= D_{k_3}D_{k_2}D_{k_1}(x^i\beta)(\alpha(x)) D_{j_3}(x^{k_3}\alpha)(x) D_{j_2}(x^{k_2}\alpha)(x) D_{j_1}(x^{k_1}\alpha)(x) \\
&\quad + D_{k_2}D_{k_1}(x^i\beta)(\alpha(x)) D_{j_3}D_{j_2}(x^{k_2}\alpha)(x) D_{j_1}(x^{k_1}\alpha)(x) \\
&\quad + D_{k_2}D_{k_1}(x^i\beta)(\alpha(x)) D_{j_2}(x^{k_2}\alpha)(x) D_{j_3}D_{j_1}(x^{k_1}\alpha)(x) \\
&\quad + D_{k_2}D_{k_1}(x^i\beta)(\alpha(x)) D_{j_3}(x^{k_2}\alpha)(x) D_{j_2}D_{j_1}(x^{k_1}\alpha)(x) \\
&\quad + D_k(x^i\beta)(\alpha(x)) D_{j_3}D_{j_2}D_{j_1}(x^k\alpha)(x).
\end{aligned} \tag{7}$$

Substituting $x = \alpha(x) = 0$, we get

$$(8)\quad \begin{aligned} &D_{j_1}(x^i\gamma)(0) = D_k(x^i\beta)(0)D_{j_1}(x^k\alpha)(0),\\ &D_{j_2}D_{j_1}(x^i\gamma)(0) = D_{k_2}D_{k_1}(x^i\beta)(0)D_{j_2}(x^{k_2}\alpha)(0)D_{j_1}(x^{k_1}\alpha)(0)\\ &\quad + D_k(x^i\beta)(0)D_{j_2}D_{j_1}(x^k\alpha)(0),\\ &D_{j_3}D_{j_2}D_{j_1}(x^i\gamma)(0)\\ &\quad = D_{k_3}D_{k_2}D_{k_1}(x^i\beta)(0)\cdot D_{j_3}(x^{k_3}\alpha)(0)\cdot D_{j_2}(x^{k_2}\alpha)(0)\cdot D_{j_1}(x^{k_1}\alpha)(0)\\ &\quad + D_{k_2}D_{k_1}(x^i\beta)(0)\cdot D_{j_3}D_{j_2}(x^{k_2}\alpha)(0)\cdot D_{j_1}(x^{k_1}\alpha)(0)\\ &\quad + D_{k_2}D_{k_1}(x^i\beta)(0)\cdot D_{j_2}(x^{k_2}\alpha)(0)\cdot D_{j_3}D_{j_1}(x^{k_1}\alpha)(0)\\ &\quad + D_{k_2}D_{k_1}(x^i\beta)(0)\cdot D_{j_3}(x^{k_2}\alpha)(0)\cdot D_{j_2}D_{j_1}(x^{k_1}\alpha)(0)\\ &\quad + D_k(x^i\beta)(0)\cdot D_{j_3}D_{j_2}D_{j_1}(x^k\alpha)(0), \end{aligned}$$

or, which is the same,

$$(9)\quad \begin{aligned} &a^i_{j_1}(J^3_0\gamma) = a^i_k(J^3_0\beta)\cdot a^k_{j_1}(J^3_0\alpha),\\ &a^i_{j_2j_1}(J^3_0\gamma) = a^i_{k_2k_1}(J^3_0\beta)\cdot a^{k_2}_{j_2}(J^3_0\alpha)\cdot a^{k_1}_{j_1}(J^3_0\alpha)\\ &\quad + a^i_k(J^3_0\beta)\cdot a^k_{j_2j_1}(J^3_0\alpha),\\ &a^i_{j_3j_2j_1}(J^3_0\gamma) = a^i_{k_3k_2k_1}(J^3_0\beta)\cdot a^{k_3}_{j_3}(J^3_0\alpha)\cdot a^{k_2}_{j_2}(J^3_0\alpha)\cdot a^{k_1}_{j_1}(J^3_0\alpha)\\ &\quad + a^i_{k_2k_1}(J^3_0\beta)\cdot a^{k_2}_{j_3j_2}(J^3_0\alpha)\cdot a^{k_1}_{j_1}(J^3_0\alpha)\\ &\quad + a^i_{k_2k_1}(J^3_0\beta)\cdot a^{k_2}_{j_2}(J^3_0\alpha)\cdot a^{k_1}_{j_3j_1}(J^3_0\alpha)\\ &\quad + a^i_{k_2k_1}(J^3_0\beta)\cdot a^{k_2}_{j_3}(J^3_0\alpha)\cdot a^{k_1}_{j_2j_1}(J^3_0\alpha) + a^i_k(J^3_0\beta)\cdot a^k_{j_3j_2j_1}(J^3_0\alpha). \end{aligned}$$

We usually abbreviate these formulas by writing

$$(10)\quad \begin{aligned} &c^i_{j_1} = b^i_k a^k_{j_1},\\ &c^i_{j_2j_1} = b^i_{k_2k_1}a^{k_2}_{j_2}a^{k_1}_{j_1} + b^i_k a^k_{j_2j_1},\\ &c^i_{j_3j_2j_1} = b^i_{k_3k_2k_1}a^{k_3}_{j_3}a^{k_2}_{j_2}a^{k_1}_{j_1} + b^i_{k_2k_1}a^{k_2}_{j_3j_2}a^{k_1}_{j_1} + b^i_{k_2k_1}a^{k_2}_{j_2}a^{k_1}_{j_3j_1}\\ &\quad + b^i_{k_2k_1}a^{k_2}_{j_3}a^{k_1}_{j_2j_1} + b^i_k a^k_{j_3j_2j_1} \end{aligned}$$

with obvious meaning of the symbols. These formulas represent *equations of the group operation* in the differential group L^3_n in canonical coordinates.

Now we compute the chart expression of the mapping $L^3_n \ni A \to A^{-1} \in L^3_n$. We take in (10) $B = A^{-1}$, $C = J^3_0\,\mathrm{id}_{\mathbf{R}^n}$. Then

$$(11)\quad c^i_{j_1} = \delta^i_{j_1},\quad c^i_{j_1j_2} = 0,\quad c^i_{j_1j_2j_3} = 0,$$

and equations (10) reduce to

$$(12)\quad \begin{aligned} &b^i_k a^k_{j_1} = \delta^i_{j_1},\\ &b^i_{k_2k_1}a^{k_2}_{j_2}a^{k_1}_{j_1} + b^i_k a^k_{j_2j_1} = 0,\\ &b^i_{k_3k_2k_1}a^{k_3}_{j_3}a^{k_2}_{j_2}a^{k_1}_{j_1} + b^i_{k_2k_1}a^{k_2}_{j_3j_2}a^{k_1}_{j_1} + b^i_{k_2k_1}a^{k_2}_{j_2}a^{k_1}_{j_3j_1}\\ &\quad + b^i_{k_2k_1}a^{k_2}_{j_3}a^{k_1}_{j_2j_1} + b^i_k a^k_{j_3j_2j_1} = 0. \end{aligned}$$

The first equation determines b^i_k as elements of the inverse matrix to the matrix (a^k_j). Using this fact we get

$$(13)\qquad \begin{aligned} &b^i_k a^k_{j_1} = \delta^i_{j_1},\\ &b^i_{p_2p_1} = -a^k_{j_2j_1} b^i_k b^{j_2}_{p_2} b^{j_1}_{p_1},\\ &b^i_{p_3p_2p_1} = -\big(b^i_{k_2k_1}(a^{k_1}_{j_1}a^{k_2}_{j_3j_2} + a^{k_2}_{j_2}a^{k_1}_{j_3j_1} + a^{k_2}_{j_3}a^{k_1}_{j_2j_1}) + b^i_k a^k_{j_3j_2j_1}\big) b^{j_3}_{p_3} b^{j_2}_{p_2} b^{j_1}_{p_1}, \end{aligned}$$

where it is assumed that we substitute for b^i_k from the first equation into the second and the third ones, and then for $b^i_{k_2k_1}$ from the second equation into the third one. We conclude that the mapping $A \to A^{-1}$, expressed in canonical coordinates by (13), is represented by *rational* functions.

Remark 1. Sometimes it is useful to use the *second canonical coordinates* on L^r_n, defined by

$$(14)\qquad b^i_{j_1j_2\cdots j_k}(A) = a^i_{j_1j_2\cdots j_k}(A^{-1}),$$

where $1 \le i \le n$, $1 \le k \le r$, $1 \le j_1 \le j_2 \le \cdots \le j_k \le n$.

2.2. Velocities. Throughout this section, $m, n \ge 1$ and $r \ge 0$ are integers, and Y is a smooth manifold of dimension $n + m$.

By an n*-velocity of order* r at a point $y \in Y$ we mean an r-jet $P \in J^r_{(0,y)}(\mathbf{R}^n, Y)$, $P = J^r_0\zeta$. When there is no danger of confusion, we omit n and r, and speak simply of a velocity. We denote

$$(1)\qquad T^r_n Y = \bigcup_{y\in Y} J^r_{(0,y)}(\mathbf{R}^n, Y),$$

and define surjective mappings $\tau^{r,s}_n : T^r_n Y \to T^s_n Y$, where $0 \le s \le r$, by

$$(2)\qquad \tau^{r,s}_n(J^r_0\zeta) = J^s_0\zeta.$$

The set $T^r_n Y$ is endowed with a right action of the differential group L^r_n, defined by the jet composition

$$(3)\qquad T^r_n Y \times L^r_n \ni (P, A) \to P \circ A \in T^r_n Y.$$

This action is said to be *canonical*.

Let (V, ψ), $\psi = (y^K)$, be a chart on Y. We set

$$(4)\qquad V^r_n = (\tau^{r,0}_n)^{-1}(V), \qquad \psi^r_n = \big(y^K, y^K_{i_1}, y^K_{i_1i_2}, \dots, y^K_{i_1i_2\cdots i_r}\big),$$

where $1 \le K \le n + m$, $1 \le i_1 \le i_2 \le \cdots \le i_r \le n$, and for every $P \in V^r_n$, $P = J^r_0\zeta$,

$$(5)\qquad y^K_{i_1i_2\cdots i_k}(P) = D_{i_1}D_{i_2}\cdots D_{i_k}(y^K\zeta)(0),$$

where $0 \le k \le r$.

Note that formula (5) can be written in a slightly different way. To this purpose we denote by $\mathrm{tr}_\xi : \mathbf{R}^{n+m} \to \mathbf{R}^{n+m}$ the *translation* sending a vector $\xi \in \mathbf{R}^{n+m}$ to the origin $0 \in \mathbf{R}^{n+m}$. By definition,

$$(6)\qquad \mathrm{tr}_\xi(x) = x - \xi.$$

Now writing in components $\mathrm{tr}_\xi = (\mathrm{tr}_\xi^K)$, we have

(7) $$y^K_{j_1 j_2 \cdots j_s}(P) = D_{j_1} D_{j_2} \cdots D_{j_s} (\mathrm{tr}^K_{\psi\zeta(0)}\, \psi\zeta)(0).$$

In the following theorem we use the set of r-jets $L^r_{n,m} = J^r_{(0,0)}(\mathbf{R}^n, \mathbf{R}^m)$ with source at $0 \in \mathbf{R}^n$ and target at $0 \in \mathbf{R}^m$ (Section 1.2, Remark 2). Elements of this set are called *standard n-velocities of order r* in $\mathbf{R}^m$.

Theorem 1. *Let $m, n \geq 1$ and $r \geq 0$ be integers, and let Y be a smooth manifold of dimension $n + m$.*

There exists one and only one smooth structure on $T^r_n Y$ such that for any chart (V, ψ), $\psi = (y^K)$, on Y, the pair (V^r_n, ψ^r_n), $\psi^r_n = (y^K, y^K_{i_1}, y^K_{i_1 i_2}, \ldots, y^K_{i_1 i_2 \cdots i_r})$, is a chart on $T^r_n Y$. The dimension of $T^r_n Y$ is given by

(8) $$N = (n + m)\binom{n + r}{n}.$$

In this smooth structure, the canonical right action of L^r_n on $T^r_n Y$ is smooth, and $T^r_n Y$ is a fibration with base Y, projection $\tau^{r,0}_n$, and fiber $L^r_{n,n+m}$.

Proof. Using (7) we can see at once that ψ^r_n is a bijection of V^r_n onto the open set $\psi(V) \times L^r_{n,n+m} \subset \mathbf{R}^{n+m} \times \mathbf{R}^N$, where N is determined by Section 1.2, (8). Thus, (V^r_n, ψ^r_n) is a chart on $T^r_n Y$. Let (V, ψ), $\psi = (y^K)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^K)$, be two charts on Y such that $V \cap \bar{V} \neq \emptyset$. Using the higher order chain rule (Section 1.1, Lemma 1), it is easy to see that the corresponding coordinate transformation from (V^r_n, ψ^r_n) to $(\bar{V}^r_n, \bar{\psi}^r_n)$ is polynomial in the coordinates $y^K_{i_1}, y^K_{i_1 i_2}, \ldots, y^K_{i_1 i_2 \cdots i_r}$ hence smooth.

Therefore, the charts (V^r_n, ψ^r_n), $(\bar{V}^r_n, \bar{\psi}^r_n)$ are compatible.

Since the equations of the mapping $\tau^{r,s}_n : T^r_n Y \to T^s_n Y$ in terms of the charts (V^r_n, ψ^r_n), (V^s_n, ψ^s_n) are given by

(9) $$y^K_{i_1 i_2 \cdots i_k} \circ \tau^{r,s}_n = y^K_{i_1 i_2 \cdots i_k},$$

where $0 \leq k \leq s$, $\tau^{r,s}_n$ is a submersion.

The smoothness of the right action follows from the polynomiality of the composition of jets (Section 1.3, (3)).

The set $T^r_n Y$ endowed with the smooth structure defined in Theorem 1, and with the canonical right action (3) of L^r_n is called the *manifold of n-velocities of order r* over Y. The chart (V^r_n, ψ^r_n) on $T^r_n Y$ is said to be *associated* with the chart (V, ψ).

The canonical group action (3) can be easily determined in the canonical coordinates $a^i_{j_1 j_2 \cdots j_k}$ on L^r_n (Section 2.1, (2), (3)), and in a chart (V, ψ), $\psi = (y^K)$, on Y. Using the associated chart (V^r_n, ψ^r_n), (3) is expressed by the equations

(10) $$\bar{y}^K = y^K, \qquad \bar{y}^K_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \ldots, I_p)} y^K_{j_1 j_2 \cdots j_p} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p},$$

where the second sum is extended to all partitions $(I_1, I_2, \ldots, I_p)$ of the set $(i_1, i_2, \ldots, i_s)$ (see Section 1.3, (4)).

Example 2 (the action of L^2_n on $T^2_n Y$). In our standard notation, let $P = J^2_0 \zeta$, $A = J^2_0 \alpha$. By (3), we consider the mapping $t \to (y^K \zeta \circ \alpha)(t)$.

Since

$$D_i(y^K\zeta\circ\alpha)(t) = D_k(y^K\zeta)(\alpha(t))D_i\alpha^k(t),$$
$$D_iD_j(y^K\zeta\circ\alpha)(t) = D_lD_k(y^K\zeta)(\alpha(t))D_j\alpha^l(t)D_i\alpha^k(t)$$
$$+ D_k(y^K\zeta)(\alpha(t))D_iD_j\alpha^k(t),$$

we have the following equations of the action of L^2_n on T^2_nY

$$\bar{y}^K = y^K, \qquad \bar{y}^K_i = y^K_k a^k_i, \qquad \bar{y}^K_{ij} = y^K_{kl}a^k_i a^l_j + y^K_k a^k_{ij}.$$

It is clear from these formulas how to obtain equations of the action of L^r_n on T^r_nY by a process of a formal differentiation.

Let γ be a smooth mapping of an open set $U \subset \mathbf{R}^n$ into Y. Then for any $t \in U$, the mapping $x \to \gamma\circ \mathrm{tr}_{-t}(x)$ is defined on a neighborhood of the origin $0 \in \mathbf{R}^n$ so that the r-jet $J^r_0(\gamma\circ\mathrm{tr}_{-t})$ is defined. The mapping

(11) $$U \ni t \to (T^r_n\gamma)(t) = J^r_0(\gamma\circ\mathrm{tr}_{-t}) \in T^r_nY$$

is called the r-*prolongation*, or simply the *prolongation* of γ (for terminology, compare with Section 1.2). Since $y^K_{i_1i_2\cdots i_k}\circ T^r_n\gamma(t) = D_{i_1}D_{i_2}\cdots D_{i_k}(y^K(\gamma\circ\mathrm{tr}_{-t}))(0)$ and $D_i(y^K(\gamma\circ\mathrm{tr}_{-t}))(x) = D_i(y^K\gamma)(x+t)$, we get for the chart expression of (11)

(12) $$(y^K_{i_1i_2\cdots i_k}\circ T^r_n\gamma)(t) = D_{i_1}D_{i_2}\cdots D_{i_k}(y^K\gamma)(t).$$

In particular, $T^r_n\gamma$ is a smooth mapping.

Assume that we have an element $P \in T^r_nY$, $P = J^r_0\zeta$. A representative ζ of P defines the tangent mapping $T_0T^{r-1}_n\zeta$, which sends a tangent vector $\xi \in T_0\mathbf{R}^n$ to the tangent vector $T_0T^{r-1}_n\zeta\cdot\xi$ of T^{r-1}_nY at $\tau^{r,r-1}_n(P) = J^{r-1}_0\zeta$. If $\xi = \xi^i(\partial/\partial t^i)_0$, then by (12),

(13) $$\begin{aligned} T_0T^{r-1}_n\zeta\cdot\xi &= \sum_{k=0}^{r-1}\sum_{i_1\le i_2\le\cdots\le i_k}\left(\frac{\partial(y^K_{i_1i_2\cdots i_k}\circ T^{r-1}_n\zeta)}{\partial t^i}\right)_0 \xi^i\left(\frac{\partial}{\partial y^K_{i_1i_2\cdots i_k}}\right)_{J^{r-1}_0\zeta} \\ &= \sum_{k=0}^{r-1}\sum_{i_1\le i_2\le\cdots\le i_k} y^K_{i_1i_2\cdots i_k i}(J^r_0\zeta)\,\xi^i\left(\frac{\partial}{\partial y^K_{i_1i_2\cdots i_k}}\right)_{J^{r-1}_0\zeta} \\ &= \xi^i d_i(P), \end{aligned}$$

where

(14) $$d_i = \sum_{k=0}^{r-1}\sum_{i_1\le i_2\le\cdots\le i_k} y^K_{i_1i_2\cdots i_k i}\frac{\partial}{\partial y^K_{i_1i_2\cdots i_k}}$$

is a *morphism* $T^r_n \ni P \to d_i(P) \in TT^{r-1}_nY$ over T^{r-1}_nY. Indeed, the tangent vectors $d_i(P)$ are defined independently of the chosen chart: If $(\overline{V},\overline{\psi})$, $\overline{\psi} = (\bar{y}^K)$, is some other chart at $y = \zeta(0)$, then

(15) $$\bar{d}_i = \sum_{k=0}^{r-1}\sum_{j_1\le j_2\le\cdots\le j_k} \bar{y}^K_{j_1j_2\cdots j_k i}\frac{\partial}{\partial \bar{y}^K_{j_1j_2\cdots j_k}},$$

and by (13),

(16) $$\bar{d}_i = d_i.$$

d_i is called the *i-th formal derivative morphism*.

Remark 2. In (14), $\partial/\partial y^K_{i_1 i_2 \cdots i_k}$ are understood as tangent vectors to $T^{r-1}_n Y$. Formula (14) does *not* define a vector field on $T^r_n Y$ since it is not invariant when the tangent vectors $\partial/\partial y^K_{i_1 i_2 \cdots i_k}$ are subject to coordinate transformations on $T^r_n Y$.

Let $f : V^{r-1}_n \to \mathbf{R}$ be a smooth function. We define the *i-th formal derivative* $d_i f : V^r_n \to \mathbf{R}$ by

(17) $$d_i f = \sum_{k=0}^{r-1} \sum_{j_1 \le j_2 \le \cdots \le j_k} y^K_{j_1 j_2 \cdots j_k i} \frac{\partial f}{\partial y^K_{j_1 j_2 \cdots j_k}}.$$

Then by (12)

(18) $$\begin{aligned} &D_p(f \circ T^{r-1}_n \gamma)(t) \\ &= \sum_{k=0}^{r-1} \sum_{j_1 \le j_2 \le \cdots \le j_k} \left(\frac{\partial (f \circ T^{r-1}_n \gamma)}{\partial y^K_{j_1 j_2 \cdots j_k}} \right)_{(T^{r-1}_n \gamma)(t)} D_p\big(y^K_{j_1 j_2 \cdots j_k} \circ T^{r-1}_n \gamma\big)(t) \\ &= \sum_{k=0}^{r-1} \sum_{j_1 \le j_2 \le \cdots \le j_k} \left(\frac{\partial (f \circ (\psi^r_n)^{-1})}{\partial y^K_{j_1 j_2 \cdots j_k}} \right)_{(T^{r-1}_n \gamma)(t)} D_p D_{j_1} D_{j_2} \cdots D_{j_k} (y^K \gamma)(t) \\ &= \sum_{k=0}^{r-1} \sum_{j_1 \le j_2 \le \cdots \le j_k} \big(y^K_{j_1 j_2 \cdots j_k p} (T^r_n \gamma)(t)\big) \left(\frac{\partial (f \circ (\psi^{r-1}_n)^{-1})}{\partial y^K_{j_1 j_2 \cdots j_k}} \right)_{(T^{r-1}_n \gamma)(t)} \\ &= (d_p f \circ T^r_n \gamma)(t), \end{aligned}$$

i.e.,

(19) $$d_p f \circ T^r_n \gamma = D_p(f \circ T^{r-1}_n \gamma).$$

In particular, $D_q D_p (f \circ T^{r-1}_n \gamma) = D_q (d_p f \circ T^r_n \gamma) = d_q d_p f \circ T^{r+1}_n \gamma$, i.e.,

(20) $$d_q d_p f = d_p d_q f.$$

Note that if we take $f = y^A_{j_1 j_2 \cdots j_k}$ in (17), we get

(21) $$d_i y^A_{j_1 j_2 \cdots j_k} = y^A_{j_1 j_2 \cdots j_k i}.$$

Our aim now will be to derive explicit transformation formulas between the induced charts on $T^r_n Y$. Let us write the transformation equations from (V, ψ) to $(\overline{V}, \overline{\psi})$ in the form

(22) $$\bar{y}^K = F^K(y^L).$$

We wish to determine the functions $F^K_{i_1}, F^K_{i_1 i_2}, \ldots, F^K_{i_1 i_2 \cdots i_r}$ defining the corresponding transformation

(23) $$\bar{y}^K_{i_1 i_2 \cdots i_k} = F^K_{i_1 i_2 \cdots i_k}\big(y^L, y^L_{j_1}, y^L_{j_1 j_2}, \ldots, y^L_{j_1 j_2 \cdots j_k}\big), \quad 1 \le k \le r,$$

from (V_n^r, ψ_n^r) to $(\overline{V}_n^r, \overline{\psi}_n^r)$.

Note that by (21) and (16),

$$\bar{y}^K_{j_1 j_2 \cdots j_k j_{k+1}} = \bar{d}_{j_{k+1}} \bar{y}^K_{j_1 j_2 \cdots j_k} = d_{j_{k+1}} \bar{y}^K_{j_1 j_2 \cdots j_k} = \cdots = d_{j_{k+1}} \cdots d_{j_2} d_{j_1} \bar{y}^K. \tag{24}$$

This formula may be applied whenever the transformation rules (22) for the coordinate transformations on Y are known.

Lemma 1. *The following formula holds*

$$F^K_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \cdots, I_p)} y^{L_1}_{I_1} y^{L_2}_{I_2} \cdots y^{L_p}_{I_p} \frac{\partial^p F^K}{\partial y^{L_1} \partial y^{L_2} \cdots \partial y^{L_p}}, \tag{25}$$

where the second summation is extended over all partitions $(I_1, I_2, \dots, I_p)$ *of the set* $(i_1, i_2, \dots, i_s)$.

Proof. We proceed by induction.

1. First consider the case $r = 1$. We have $V_n^1 = (\tau_n^{1,0})^{-1}(V)$ and $\psi_n^1 = (y^K, y_i^K)$ where $1 \le K \le n+m$, $1 \le i \le n$, and by definition,

$$y^K(J_0^1 \zeta) = y^K(\zeta(0)), \qquad y_i^K(J_0^1 \zeta) = D_i(y^K \zeta)(0). \tag{26}$$

Obviously,

$$\bar{y}^K = F^K(y^L), \qquad \bar{y}_i^K = \frac{\partial F^K}{\partial y^L} y_i^L \tag{27}$$

on $V_n^1 \cap \overline{V}_n^1$ or, which is the same, $F^K_{i_1} = d_{i_1} F^K$.

2. Now assume that $s > 1$, and

$$F^K_{i_1 i_2 \cdots i_{s-1}} = \sum_{p=1}^{s-1} \sum_{(I_1, I_2, \dots, I_p)} y^{L_1}_{I_1} y^{L_2}_{I_2} \cdots y^{L_p}_{I_p} \frac{\partial^p F^K}{\partial y^{L_1} \partial y^{L_2} \cdots \partial y^{L_p}}. \tag{28}$$

Then by (21)

$$\begin{aligned}
F^K_{i_1 i_2 \cdots i_{s-1} i_s} &= d_{i_s} F^K_{i_1 i_2 \cdots i_{s-1}} \\
&= \sum_{p=1}^{s-1} \sum_{(I_1, I_2, \dots, I_p)} \left(d_{i_s} y^{L_1}_{I_1} y^{L_2}_{I_2} \cdots y^{L_p}_{I_p} + y^{L_1}_{I_1} d_{i_s} y^{L_2}_{I_2} \cdots y^{L_p}_{I_p} + \cdots + y^{L_1}_{I_1} y^{L_2}_{I_2} \cdots d_{i_s} y^{L_p}_{I_p} \right) \\
&\quad \cdot \frac{\partial^p F^K}{\partial y^{L_1} \partial y^{L_2} \cdots \partial y^{L_p}} + \sum_{p=1}^{s-1} \sum_{(I_1, I_2, \dots, I_p)} y^{L_1}_{I_1} y^{L_2}_{I_2} \cdots y^{L_p}_{I_p} d_{i_s} \frac{\partial^p F^K}{\partial y^{L_1} \partial y^{L_2} \cdots \partial y^{L_p}} \\
&= \sum_{p=1}^{s-1} \sum_{(I_1, I_2, \dots, I_p)} \left(y^{L_1}_{I_1 i_s} y^{L_2}_{I_2} \cdots y^{L_p}_{I_p} + y^{L_1}_{I_1} y^{L_2}_{I_2 i_s} \cdots y^{L_p}_{I_p} + \cdots + y^{L_1}_{I_1} y^{L_2}_{I_2} \cdots y^{L_p}_{I_p i_s} \right) \\
&\quad \cdot \frac{\partial^p F^K}{\partial y^{L_1} \partial y^{L_2} \cdots \partial y^{L_p}} + \sum_{p=1}^{s-1} \sum_{(I_1, I_2, \dots, I_p)} y^{L_1}_{I_1} y^{L_2}_{I_2} \cdots y^{L_p}_{I_p} y^{L_{p+1}}_{i_s} \frac{\partial^{p+1} F^K}{\partial y^{L_1} \partial y^{L_2} \cdots \partial y^{L_p} \partial y^{L_{p+1}}} \\
&= \sum_{p=1}^{s} \sum_{(J_1, J_2, \dots, J_p)} y^{L_1}_{J_1} y^{L_2}_{J_2} \cdots y^{L_p}_{J_p} \frac{\partial^p F^K}{\partial y^{L_1} \partial y^{L_2} \cdots \partial y^{L_p}},
\end{aligned} \tag{29}$$

and the formula (25) is verified.

2.3. Regular velocities. Let $m, n \geq 1$ be fixed integers. We need a convention regarding partitions of the sequence $(1, 2, \dots, n, n+1, \dots, n+m)$ in two complementary subsequences. A *subsequence* $(i_1, i_2, \dots, i_n)$ of the sequence $(1, 2, \dots, n, n+1, \dots, n+m)$, consisting of n elements, is called an *n-subsequence*. Indeed, one has exactly

$$(1) \qquad \binom{n+m}{n}$$

different n-subsequences. Every n-subsequence $(i_1, i_2, \dots, i_n)$ has a unique *complementary* subsequence $(\sigma_1, \sigma_2, \dots, \sigma_m)$. Note that since we consider *subsequences,* we always assume that $i_1 < i_2 < \dots < i_n$, $\sigma_1 < \sigma_2 < \dots < \sigma_m$.

We write $(K) = (1, 2, \dots, n, n+1, \dots, n+m)$, $(i) = (i_1, i_2, \dots, i_n)$, and $(\sigma) = (\sigma_1, \sigma_2, \dots, \sigma_m)$, to express that $K = 1, 2, \dots, n, n+1, \dots, n+m$, $i = i_1, i_2, \dots, i_n$, and $\sigma = \sigma_1, \sigma_2, \dots, \sigma_m$, respectively. We also write, with obvious meaning, $\nu \in (\sigma)$, $j \in (i)$, etc.

Let $r \geq 0, m, n \geq 1$ be integers, let Y be a smooth manifold of dimension $n+m$, and let $T^r_n Y$ be the manifold of n-velocities of order r over Y. We shall consider the set of regular n-velocities of order r in $T^r_n Y$, denoted by $\operatorname{imm} T^r_n Y$. Recall that a velocity $P \in T^r_n Y$, $P = J^r_0 \zeta$ is called *regular,* if there exists an r-jet $Q \in J^r_{(y,0)}(Y, \mathbf{R}^n)$, such that

$$(2) \qquad Q \circ P = J^r_0 \operatorname{id}_{\mathbf{R}^n}.$$

P is regular if and only if every representative ζ of P is an immersion at $0 \in \mathbf{R}^n$ (Section 2.1, Lemma 5, (a)) or, equivalently, if and only if there exist a chart (V, ψ), $\psi = (y^K)$, at $y = \zeta(0)$, and an n-subsequence $(i) = (i_1, i_2, \dots, i_n)$ of the sequence $(K) = (1, 2, \dots, n, n+1, \dots, n+m)$ such that

$$(3) \qquad \det\left(y^i_j(P)\right) = \det\left(D_j(y^i \circ \zeta)(0)\right) \neq 0.$$

Recall that $T^r_n Y$ is endowed with the *canonical right action* of the differential group L^r_n, defined by

$$(4) \qquad Q = P \circ A,$$

(see Section 2.2, (3), Theorem 1, (b)). Let (V, ψ), $\psi = (y^K)$, be a chart on Y, and let (V^r_n, ψ^r_n), $\psi^r_n = (y^K, y^K_{i_1}, y^K_{i_1 i_2}, \dots, y^K_{i_1 i_2 \cdots i_r})$, be the associated chart on $\operatorname{imm} T^r_n Y$. (4) is expressed by the equations

$$(5) \qquad \bar{y}^K = y^K, \qquad \bar{y}^K_{i_1 i_2 \cdots i_s} = \sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} y^K_{j_1 j_2 \cdots j_p} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p},$$

where the second sum is extended to all partitions $(I_1, I_2, \dots, I_p)$ of the set $(i_1, i_2, \dots, i_s)$ (see Section 2.3, (10)). Clearly, here $y^L, y^L_{p_1}, y^L_{p_1 p_2}, \dots, y^L_{p_1 p_2 \cdots p_r}$ (resp. $\bar{y}^K, \bar{y}^K_{i_1}, \bar{y}^K_{i_1 i_2}, \dots, \bar{y}^K_{i_1 i_2 \cdots i_r}$, resp. a^j_I) are the coordinates of a point $P \in \operatorname{imm} T^r_n Y$ (resp. its image $Q \in T^r_n Y$, resp. $A \in L^r_n$).

The following lemma says that formula (4), or equivalently, (5), induces a right action on $\operatorname{imm} T^r_n Y$.

Lemma 2. *The set* $\operatorname{imm} T_n^r Y$ *is an open, dense,* L_n^r*-invariant subset of* $T_n^r Y$.

Proof. Let $P \in \operatorname{imm} T_n^r Y$, $P = J_0^r \zeta$, let (V, ψ), $\psi = (y^K)$, be a chart at $y = \zeta(0)$, and let (V_n^r, ψ_n^r), $\psi_n^r = (y^K, y_{i_1}^K, y_{i_1 i_2}^K, \dots, y_{i_1 i_2 \cdots i_r}^K)$, be the associated chart at P. Since ζ is an immersion at $0 \in \mathbf{R}^n$, the matrix formed by $y_i^K(P) = D_i(y^K \zeta)(0)$ is of maximal rank equal to n. Assume that $\det(y_i^j(P)) \neq 0$ for an n-subsequence $(i) = (i_1, i_2, \dots, i_n)$ of the sequence $(K) = (1, 2, \dots, n, n+1, \dots, n+m)$. Then since the mapping $V_n^r \ni P \to \det(y_i^j(P)) \in \mathbf{R}$ is continuous, P has a neighborhood on which this function is nonzero. This verifies that the set $\operatorname{imm} T_n^r Y \subset T_n^r Y$ is open.

If $P \in \operatorname{imm} T_n^r Y$, $P = J_0^r \zeta$, and $A \in L_n^r$, $A = J_0^r \alpha$, then $P \circ A = J_0^r(\zeta \circ \alpha)$, where $\zeta \circ \alpha$ is obviously an immersion at $0 \in \mathbf{R}^n$. This proves invariance.

The set $\operatorname{imm} T_n^r Y$ is called the *manifold of regular n-velocities of order r* over Y.

Now we wish to analyze the equivalence $\mathcal{R} \subset \operatorname{imm} T_n^r Y \times \operatorname{imm} T_n^r Y$, *associated* with the canonical group action (4).

We set for every n-subsequence (i) of the sequence $(1, 2, \dots, n, n+1, \dots, n+m)$

$$W^{(i)} = \left\{ P \in V_n^r \mid \det(y_j^i(P)) \neq 0 \right\}, \tag{6}$$

where (V_n^r, ψ_n^r) is the chart on $\operatorname{imm} T_n^r Y$ associated with (V, ψ). In (6), $i \in (i)$, and $1 \leq j \leq n$. $W^{(i)}$ is an open subset of V_n^r. It is easily seen that $W^{(i)}$ is L_n^r-invariant. Indeed, if $P \in W^{(i)}$ is a point, then by (5), for every $A \in L_n^r$, $y_j^K(P \circ A) = \bar{y}_j^K = y_p^K(P) a_j^p(A)$ and $\det(y_j^i(P \circ A)) = \det(y_p^i(P) a_j^p(A)) = \det(y_j^i(P)) \det A \neq 0$, i.e., $P \circ A \in W^{(i)}$. Shrinking the canonical coordinates $(y^K, y_{i_1}^K, y_{i_1 i_2}^K, \dots, y_{i_1 i_2 \cdots i_r}^K)$ to $W^{(i)}$ we obtain a *chart* denoted by $(W^{(i)}, \chi^{(i)})$. We have

$$\bigcup_{(i)} W^{(i)} = V_n^r, \tag{7}$$

which implies that the charts $(W^{(i)}, \chi^{(i)})$ form an *atlas* on the manifold $\operatorname{imm} T_n^r Y$. The coordinate transformation from $(W^{(i)}, \chi^{(i)})$ to $(W^{(j)}, \chi^{(j)})$ coincides with the restriction of the identity mapping of V_n^r to $W^{(i)} \cap W^{(j)}$.

We introduce a collection of functions $z_i^k : W^{(i)} \to \mathbf{R}$, by

$$z_i^k y_j^i = \delta_j^k, \tag{8}$$

where $i \in (i)$, and $1 \leq j, k \leq n$. Existence of these functions is guaranteed by the condition (6). z_i^k is a rational function of y_j^i, and is therefore smooth.

Now consider equations (4), and the equivalence $\mathcal{R}$ on $\operatorname{imm} T_n^r Y$ "*there exists* $A \in L_n^r$ *such that* $Q = P \circ A$".

Lemma 3. *Let* $(P, Q) \in \operatorname{imm} T_n^r Y \times \operatorname{imm} T_n^r Y$ *be a point. The following conditions are equivalent:*

(a) $(P, Q) \in \mathcal{R}$.

(b) *There exist a chart* (V, ψ), $\psi = (y^K)$, *on* Y *and an n-subsequence* (i) *of the sequence* $(1, 2, \dots, n+m)$ *such that* $P, Q \in W^{(i)}$, *and the coordinates* $y_{i_1 i_2 \cdots i_s}^K$ *(resp.* $\bar{y}_{i_1 i_2 \cdots i_s}^K$, *resp.* a_I^j*) of* P *(resp.* Q, *resp.* A*) satisfy*

$$\bar{y}^K = y^K, \qquad \bar{y}_{i_1 i_2 \cdots i_s}^\sigma = \sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} y_{j_1 j_2 \cdots j_p}^\sigma a_{I_1}^{j_1} a_{I_2}^{j_2} \cdots a_{I_p}^{j_p}, \tag{9}$$

$$1 \leq i_1, i_2, \dots, i_s, j_1, j_2, \dots, j_p \leq n, 1 \leq s \leq r,$$

and the recurrent formula

$$a^q_{k_1k_2\cdots k_s} = z^q_i\left(\bar{y}^i_{k_1k_2\cdots k_s} - \sum_{p=2}^{s}\sum_{(I_1,I_2,\dots,I_p)} a^{j_1}_{I_1}a^{j_2}_{I_2}\cdots a^{j_p}_{I_p}y^i_{j_1j_2\cdots j_p}\right), \quad i \in (i). \tag{10}$$

where (σ) *is the complementary subsequence of* (i).

Proof. 1. Assume that (a) is satisfied. Then there exist (V,ψ), $\psi = (y^K)$, and (i), such that P, Q, and A satisfy (4) hence (5) and $P, Q \in W^{(i)}$. If (σ) is the complementary subsequence, we can split (5) in two subsystems, taking $K = i$, and $K = \sigma$. Then the first subsystem reduces to the condition $\bar{y}^i = y^i$, and to the recurrent formula (10), which determines the canonical coordinates of the group element A as certain rational functions of $y^i_{j_1j_2\cdots j_s}$, $\bar{y}^i_{j_1j_2\cdots j_s}$. The second subsystem of (5), together with the condition $\bar{y}^i = y^i$, gives (9).

2. If (b) is satisfied, conditions (9) and (10) imply (5), therefore, P and Q belong to the same L^r_n-orbit.

Let (V,ψ), $\psi = (y^K)$, be a chart at a point $y \in Y$, and let $(i) = (i_1, i_2, \dots, i_n)$ be an n-subsequence of the sequence $(1, 2, \dots, n, n+1, \dots, n+m)$. We need a formula expressing the formal derivative morphism (Section 2.2, (14)) in terms of the chart $(W^{(i)}, \chi^{(i)})$. Note that (i) defines a splitting of the sequence of the coordinate functions $(y^1, y^2, \dots, y^{n+m})$ into two subsequences (y^i) and (y^σ). Setting $\psi^{(i)} = (y^i)$ defines, in components, a mapping of V onto a set U in $\mathbf{R}^n$. Since $\psi^{(i)}$ is the composite of ψ and the Cartesian projection of $\mathbf{R}^{n+m}$ onto $\mathbf{R}^n$, which is an open mapping, U is open.

An r-jet $P \in V^r_n$, $P = J^r_0\zeta$, belongs to $W^{(i)}$ if and only if the mapping $\zeta^{(i)} = \psi^{(i)}\circ\zeta$ of a neighborhood W of $0 \in \mathbf{R}^n$ into $\mathbf{R}^n$, sending $0 \in \mathbf{R}^n$ into the point $\psi^{(i)}(\zeta(0)) \in U$, is a diffeomorphism at $0 \in \mathbf{R}^m$.

Let $P \in W^{(i)}$, $P = J^r_0\zeta$. Recall that a representative ζ of P defines the $(r-1)$-*prolongation* of ζ,

$$W \ni t \to (T^{r-1}_n\zeta)(t) = J^{r-1}_0(\zeta\circ \mathrm{tr}_{-t}) \in \mathrm{imm}\, T^{r-1}_n Y \tag{11}$$

(Section 2.2, (11)). Then the composite $T^{r-1}_n\zeta \circ (\zeta^{(i)})^{-1}\circ\psi^{(i)}\circ\tau^{r,0}$ is defined on a neighborhood of P in $W^{(i)}$, and takes values in $\tau^{r,r-1}(W^{(i)}) \subset \mathrm{imm}\, T^{r-1}_n Y$. Let $\xi \in T_P\, \mathrm{imm}\, T^r_n Y$ be a tangent vector at P,

$$\xi = \sum_{s=0}^{r}\xi^K_{i_1i_2\cdots i_s}\left(\frac{\partial}{\partial y^K_{i_1i_2\cdots i_s}}\right)_P, \tag{12}$$

and consider the tangent vector

$$h^{(i)}(\xi) = T_0\big(T^{r-1}_n\zeta\circ(\zeta^{(i)})^{-1}\circ\psi^{(i)}\circ\tau^{r,0}\big)\cdot\xi \tag{13}$$

of $\mathrm{imm}\, T^{r-1}_n Y$ at $\tau^{r,r-1}(P) \in \tau^{r,r-1}(W^{(i)}) \subset \mathrm{imm}\, T^{r-1}_n Y$. We get

$$\begin{aligned} h^{(i)}(\xi) &= \big(T_{((\zeta^{(i)})^{-1}\circ\psi^{(i)}\circ\tau^{r,0})(P)}T^{r-1}_n\zeta\circ T_P((\zeta^{(i)})^{-1}\circ\psi^{(i)}\circ\tau^{r,0})\big)\cdot\xi \\ &= T_{((\zeta^{(i)})^{-1}\circ\psi^{(i)}\circ\zeta)(0)}T^{r-1}_n\zeta\circ T_P\big((\zeta^{(i)})^{-1}\circ\psi^{(i)}\circ\tau^{r,0}\big)\cdot\xi \\ &= T_0T^{r-1}_n\zeta\circ T_P\big((\zeta^{(i)})^{-1}\circ\psi^{(i)}\circ\tau^{r,0}\big)\cdot\xi. \end{aligned} \tag{14}$$

But

$$(15)\qquad \begin{aligned}&\big(t^q \circ (\zeta^{(i)})^{-1} \circ \psi^{(i)} \circ \tau^{r,0} \circ (\psi^r_n)^{-1}\big)\big(y^K, y^K_{p_1}, y^K_{p_1p_2}, \dots, y^K_{p_1p_2\cdots p_r}\big)\\ &\quad = \big(t^q \circ (\zeta^{(i)})^{-1}\big)(y^{i_1}, y^{i_2}, \dots, y^{i_n}),\end{aligned}$$

where $(t^1, t^2, \dots, t^n)$ are the canonical coordinates on $\mathbf{R}^n$. Since

$$(16)\qquad \begin{aligned}&T_P\big((\zeta^{(i)})^{-1} \circ \psi^{(i)} \circ \tau^{r,0}\big) \cdot \xi\\ &\quad = \left(\frac{\partial(t^q \circ (\zeta^{(i)})^{-1} \circ \psi^{(i)} \circ \tau^{r,0} \circ (\psi^r_n)^{-1})}{\partial y^K_{q_1q_2\cdots q_s}}\right)_{\psi^r_n(P)} \cdot \xi^K_{i_1i_2\cdots i_s}\left(\frac{\partial}{\partial t^q}\right)_0\\ &\quad = \left(\frac{\partial(t^q \circ (\zeta^{(i)})^{-1}}{\partial y^i}\right)_{\psi^{(i)}(y)} \cdot \xi^i\left(\frac{\partial}{\partial t^q}\right)_0\\ &\quad = z^q_i(P)\xi^i\left(\frac{\partial}{\partial t^q}\right)_0,\end{aligned}$$

we have, comparing this expression with Section 2.2, (13) (14), $h^{(i)}(\xi) = z^q_i(P)\,\xi^i\, d_q(P)$, where d_q is the formal derivative morphism. Denoting

$$(17)\qquad \Delta_i = z^q_i d_q,$$

we get the formula

$$(18)\qquad h^{(i)}(\xi) = \xi^i \Delta_i.$$

Notice that in (17) and (18), summation through $i \in (i)$ takes place.

Lemma 4. (a) *For every* $i, j \in (i)$,

$$(19)\qquad \Delta_i\Delta_j = \Delta_j\Delta_i.$$

(b) *If* (V, ψ), $\psi = (y^K)$, *and* $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^K)$, *are two charts, and* (i), (j) *are two* n*-subsequences of* $(1, 2, \dots, n+m)$, *then*

$$(20)\qquad \bar{\Delta}_j = \bar{z}^s_j y^i_s \Delta_i.$$

Proof. (a) First note that the relation $y^i_s z^s_j = \delta^i_j$ implies $d_p y^i_s z^s_j + y^i_s d_p z^s_j = 0$ hence $z^q_i d_p y^i_s z^s_j + z^q_i y^i_s d_p z^s_j = 0$, and $z^q_i d_p y^i_s z^s_j + d_p z^q_j = 0$. Thus, for any smooth function $f : \tau^{r,r-1}(W^{(i)}) \to \mathbf{R}$,

$$(21)\qquad \begin{aligned}\Delta_j\Delta_i f &= z^p_j d_p(z^q_i d_q f) = z^p_j d_p z^q_i d_q f + z^p_j z^q_i d_p d_q f\\ &= -z^p_j z^q_k y^k_{sp} z^s_i d_q f + z^p_j z^q_i d_p d_q f.\end{aligned}$$

This formula together with Section 2.2, (20), proves (19).

(b) We have, with our standard notation, $\bar{\Delta}_j = \bar{z}^s_j \bar{d}_s = \bar{z}^s_j d_s = \bar{z}^s_j \delta^p_s d_p = \bar{z}^s_j y^i_s z^p_i d_p = \bar{z}^s_j y^i_s \Delta_i$, where $i \in (i)$, $j \in (j)$.

Remark 3. Lemma 4(b) shows that the morphisms Δ_i span a subbundle of the tangent space $T\mathrm{imm}\, T^{r-}_n Y$, determined independently of charts.

Using the charts $(W^{(i)}, \chi^{(i)})$, we can construct new charts on $\operatorname{imm} T^r_n Y$ adapted to the canonical right action of L^r_n. In the following theorem, these charts are described by means of the morphism $\Delta_i = z^q_i d_q$, (17).

Theorem 1. (a) *Let (i) be an n-subsequence of the sequence $(1, 2, \dots, n+m)$, and let (σ) be the complementary subsequence. There exist unique functions w^σ, $w^\sigma_{i_1}$, $w^\sigma_{i_1 i_2}$, $\dots$, $w^\sigma_{i_1 i_2 \cdots i_r}$, where $i_1, i_2, \dots, i_r \in (i)$ and $\sigma \in (\sigma)$, defined on $W^{(i)}$, symmetric in the subscripts, such that*

$$(22) \qquad y^\sigma = w^\sigma, \qquad y^\sigma_{p_1 p_2 \cdots p_k} = \sum_{q=1}^{k} \sum_{(I_1, I_2, \dots, I_q)} y^{i_1}_{I_1} y^{i_2}_{I_2} \cdots y^{i_q}_{I_q} w^\sigma_{i_1 i_2 \cdots i_q}.$$

The pair $(W^{(i)}, \Psi^{(i)})$, where

$$(23) \qquad \Psi^{(i)} = \left(y^i, y^i_{p_1}, y^i_{p_1 p_2}, \dots, y^i_{p_1 p_2 \cdots p_r}, w^\sigma, w^\sigma_{i_1}, w^\sigma_{i_1 i_2}, \dots, w^\sigma_{i_1 i_2 \cdots i_r}\right),$$

is a chart on $\operatorname{imm} T^r_n Y$. *The functions w^σ, $w^\sigma_{i_1}$, $w^\sigma_{i_1 i_2}$, $\dots$, $w^\sigma_{i_1 i_2 \cdots i_r}$ satisfy the recurrent formula*

$$(24) \qquad w^\sigma_{i_1 i_2 \cdots i_k i_{k+1}} = \Delta_{i_{k+1}} w^\sigma_{i_1 i_2 \cdots i_k},$$

and are L^r_n-invariant.

(b) *The canonical group action on* $\operatorname{imm} T^r_n Y$ *is described on $W^{(i)}$ by the equations*

$$(25) \qquad \begin{aligned} \bar{y}^i &= y^i, \\ \bar{y}^i_{k_1 k_2 \cdots k_s} &= \sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^i_{j_1 j_2 \cdots j_p}, \\ \overline{w}^\sigma_{i_1 i_2 \cdots i_s} &= w^\sigma_{i_1 i_2 \cdots i_s}, \end{aligned}$$

where $i, i_1, i_2, \dots, i_s \in (i)$, $\sigma \in (\sigma)$, $0 \le s \le r$. Equations

$$(26) \qquad w^\sigma_{i_1 i_2 \cdots i_s} = c^\sigma_{i_1 i_2 \cdots i_s},$$

where $c^\sigma_{i_1 i_2 \cdots i_s} \in \mathbf{R}$, are equations of the orbits of this action.

Proof. (a) We proceed in three steps.

1. To prove existence of t w^σ, $w^\sigma_{i_1}$, $w^\sigma_{i_1 i_2}$, $\dots$, $w^\sigma_{i_1 i_2 \cdots i_r}$, we proceed by induction.

First we prove that the assertion (a) is true for $r = 1$. Consider the pair $(W^{(i)}, \Psi^{(i)})$, $\Psi^{(i)} = (y^i, y^i_p, w^\sigma, w^\sigma_i)$, where by (22), $w^\sigma = y^\sigma$, $y^\sigma_p = y^i_p w^\sigma_i$. Obviously $w^\sigma_j = z^p_j y^\sigma_p$, where $j \in (i)$, which shows that $(W^{(i)}, \Psi^{(i)})$ is a new chart. Moreover, $w^\sigma_i = z^p_i d_p y^\sigma = z^p_i d_p w^\sigma$. It remains to show that the functions w^σ, w^σ_i are L^1_n-invariant. Since the group action (4) is represented by the equations $\bar{y}^i = y^i$, $\bar{y}^\sigma = y^\sigma$, $\bar{y}^i_p = a^j_p y^i_j$, $\bar{y}^\sigma_p = a^j_p y^\sigma_j$, the inverse of the matrix $\bar{y}^i_p = a^j_p y^i_j$ is $\bar{z}^p_q = z^s_q b^p_s$, where $q \in (i)$, and b^p_s stands for the inverse of a^p_s. Hence $\overline{w}^\sigma = w^\sigma$ and $\overline{w}^\sigma_i = \bar{z}^k_i \bar{y}^\sigma_k = z^s_i b^k_s a^p_k y^\sigma_p = z^p_i y^\sigma_p = w^\sigma_i$ proving invariance.

Now we apply induction. Consider (22) with symmetric $w^\sigma_{i_1 i_2 \cdots i_q}$, $1 \le q \le k$. Using the formal derivative morphism we get

$$(27) \qquad \begin{aligned} y^\sigma_{p_1 p_2 \cdots p_k p_{k+1}} &= d_{p_{k+1}} y^\sigma_{p_1 p_2 \cdots p_k} \\ &= \sum_{q=1}^{k} \sum_{(I_1, I_2, \dots, I_q)} \left(d_{p_{k+1}} (y^{i_1}_{I_1} y^{i_2}_{I_2} \cdots y^{i_q}_{I_q})\, w^\sigma_{i_1 i_2 \cdots i_q} + y^{i_1}_{I_1} y^{i_2}_{I_2} \cdots y^{i_q}_{I_q} d_{p_{k+1}} w^\sigma_{i_1 i_2 \cdots i_q}\right) \end{aligned}$$

$$= \sum_{q=1}^{k} \sum_{(I_1,I_2,\dots,I_q)} d_{p_{k+1}}(y^{i_1}_{I_1} y^{i_2}_{I_2} \cdots y^{i_q}_{I_q}) w^{\sigma}_{i_1 i_2 \cdots i_q}$$
$$+ \sum_{q=1}^{k-1} \sum_{(I_1,I_2,\dots,I_q)} y^{i_1}_{I_1} y^{i_2}_{I_2} \cdots y^{i_q}_{I_q} y^{i_{q+1}}_{p_{k+1}} \Delta_{i_{q+1}} w^{\sigma}_{i_1 i_2 \cdots i_q}$$
$$+ \sum_{(I_1,I_2,\dots,I_q)} y^{i_1}_{p_1} y^{i_2}_{p_2} \cdots y^{i_k}_{p_k} y^{i_{k+1}}_{p_{k+1}} \Delta_{i_{k+1}} w^{\sigma}_{i_1 i_2 \cdots i_k}.$$

Now we apply the induction hypothesis (24) to the second summand. We get

$$\begin{aligned} y^{\sigma}_{p_1 p_2 \cdots p_k p_{k+1}} &= \sum_{q=1}^{k} \sum_{(I_1,I_2,\dots,I_q)} d_{p_{k+1}}(y^{i_1}_{I_1} y^{i_2}_{I_2} \cdots y^{i_q}_{I_q}) w^{\sigma}_{i_1 i_2 \cdots i_q} \\ &+ \sum_{q=1}^{k-1} \sum_{(I_1,I_2,\dots,I_q)} y^{i_1}_{I_1} y^{i_2}_{I_2} \cdots y^{i_q}_{I_q} y^{i_{q+1}}_{p_{k+1}} w^{\sigma}_{i_1 i_2 \cdots i_q i_{q+1}} \\ &+ y^{i_1}_{p_1} y^{i_2}_{p_2} \cdots y^{i_k}_{p_k} y^{i_{k+1}}_{p_{k+1}} \Delta_{i_{k+1}} w^{\sigma}_{i_1 i_2 \cdots i_k}. \end{aligned} \tag{28}$$

In this formula, we sum through partitions $(I_1, I_2, \dots, I_q)$ of the set $\{p_1, p_2, \dots, p_k\}$, and $i_1, i_2, \dots, i_q, i_{q+1} \in (i)$. We want to sum through partitions $(J_1, J_2, \dots, J_q)$ of the set $\{p_1, p_2, \dots, p_k, p_{k+1}\}$. Note that such partitions arise in two possible ways, either by adding p_{k+1} to an element of some partition $(I_1, I_2, \dots, I_q)$, or as a partition of the form $(I_1, I_2, \dots, I_q, \{p_{k+1}\})$. Then, however, if we denote

$$w^{\sigma}_{i_1 i_2 \cdots i_k i_{k+1}} = \Delta_{i_{k+1}} w^{\sigma}_{i_1 i_2 \cdots i_k}, \tag{29}$$

the expression (28) can be written in the form

$$y^{\sigma}_{p_1 p_2 \cdots p_k p_{k+1}} = \sum_{q=1}^{k} \sum_{(J_1,J_2,\dots,J_q)} y^{i_1}_{J_1} y^{i_2}_{J_2} \cdots y^{i_q}_{J_q} w^{\sigma}_{i_1 i_2 \cdots i_q} \tag{30}$$

where the summation is taking place through partitions of the set $\{p_1, p_2, \dots, p_k, p_{k+1}\}$. This proves existence of the functions $w^{\sigma}, w^{\sigma}_{i_1}, w^{\sigma}_{i_1 i_2}, \dots, w^{\sigma}_{i_1 i_2 \cdots i_r}$. By Lemma 4, the functions $w^{\sigma}_{i_1 i_2}, w^{\sigma}_{i_1 i_2 i_3}, \dots, w^{\sigma}_{i_1 i_2 \cdots i_r}$ are *symmetric* in the subscripts.

2. To prove uniqueness of the functions $w^{\sigma}, w^{\sigma}_{i_1}, w^{\sigma}_{i_1 i_2}, \dots, w^{\sigma}_{i_1 i_2 \cdots i_r}$, one rewrites (22) similarly as in (28), and determines $w^{\sigma}_{i_1 i_2 \cdots i_k}$ using regularity of the matrix y^i_p.

3. It remains to prove invariance condition $\overline{w}^{\sigma}_{j_1 j_2 \cdots j_s} = w^{\sigma}_{j_1 j_2 \cdots j_s}$ stating that the functions (24) are constant along the L^r_n-orbits in $(W^{(i)}, \Psi^{(i)})$.

Consider equations (5). If P is a point of $\operatorname{imm} T^r_n Y$, and $Q = P \circ A$, then by Lemma 3, there exist a chart (V, ψ), $\psi = (y^K)$, and an n-subsequence (i) of $(1, 2, \dots, n+m)$ such that $P, Q \in W^{(i)}$. If (σ) is the complementary subsequence, the coordinates of P, Q, and A satisfy (9) and (10).

Using (22) we can write

$$\begin{aligned} \bar{y}^{\sigma}_{i_1 i_2 \cdots i_s} &= \sum_{p=1}^{s} \sum_{(I_1,I_2,\dots,I_p)} \bar{y}^{j_1}_{I_1} \bar{y}^{j_2}_{I_2} \cdots \bar{y}^{j_p}_{I_p} \overline{w}^{\sigma}_{j_1 j_2 \cdots j_p}, \\ y^{\sigma}_{j_1 j_2 \cdots j_p} &= \sum_{l=1}^{p} \sum_{(J_1,J_2,\dots,J_l)} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_l}_{J_l} w^{\sigma}_{t_1 t_2 \cdots t_l}, \end{aligned} \tag{31}$$

where $(I_1, I_2, \dots, I_p)$ is a partition of the set $\{i_1, i_2, \dots, i_s\}$, and $(J_1, J_2, \dots, J_l)$ is a partition of the set $\{j_1, j_2, \dots, j_p\}$. Then by (9)

$$\begin{aligned}
&\sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} \bar{y}^{j_1}_{I_1} \bar{y}^{j_2}_{I_2} \cdots \bar{y}^{j_p}_{I_p} \overline{w}^{\sigma}_{j_1 j_2 \cdots j_p} \\
&\quad = \sum_{p=1}^{s} \sum_{(I_1, I_2, \dots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} \left(\sum_{l=1}^{p} \sum_{(J_1, J_2, \dots, J_l)} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_l}_{J_l} w^{\sigma}_{t_1 t_2 \cdots t_l} \right).
\end{aligned} \tag{32}$$

Now we wish to determine the terms $w^{\sigma}_{t_1 t_2 \cdots t_p}$ on the right side with fixed p. Changing the notation of the indices, we get the expression

$$\sum_{q=1}^{s} \sum_{(I_1, I_2, \dots, I_q)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_q}_{I_q} \left(\sum_{p=1}^{q} \sum_{(J_1, J_2, \dots, J_p)} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p} w^{\sigma}_{t_1 t_2 \cdots t_p} \right) \tag{33}$$

from which we see that $w^{\sigma}_{t_1 t_2 \cdots t_p}$ are contained in every summand with $q \geq p$. Thus, the required terms are given by

$$\left(\sum_{q=p}^{s} \sum_{(I_1, I_2, \dots, I_q)} \sum_{(J_1, J_2, \dots, J_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_q}_{I_q} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p} \right) w^{\sigma}_{t_1 t_2 \cdots t_p}. \tag{34}$$

In this formula, $(I_1, I_2, \dots, I_q)$ is a partition of $(i_1, i_2, \dots, i_s)$, and $(J_1, J_2, \dots, J_p)$ is a partition of $(j_1, j_2, \dots, j_q)$.

Now we adopt the following notation. If $I = (i_1, i_2, \dots, i_s)$ is a multi-index, then the symbol $(I_1, I_2, \dots, I_p) \sim I$ means that $(I_1, I_2, \dots, I_p)$ is a partition of the set $\{i_1, i_2, \dots, i_s\}$.

As before, let $I = (i_1, i_2, \dots, i_s)$, and p be fixed. Consider the expression

$$\left(\sum_{(I_1, I_2, \dots, I_p)} \bar{y}^{t_1}_{I_1} \bar{y}^{t_2}_{I_2} \cdots \bar{y}^{t_p}_{I_p} \right) w^{\sigma}_{t_1 t_2 \cdots t_p}. \tag{35}$$

We wish to show that this expression is equal to (34), i.e.

$$\begin{aligned}
&\left(\sum_{(I_1, I_2, \dots, I_p)} \bar{y}^{t_1}_{I_1} \bar{y}^{t_2}_{I_2} \cdots \bar{y}^{t_p}_{I_p} \right) w^{\sigma}_{t_1 t_2 \cdots t_p} \\
&\quad = \left(\sum_{q=p}^{s} \sum_{(I_1, I_2, \dots, I_q)} \sum_{(J_1, J_2, \dots, J_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_q}_{I_q} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p} \right) w^{\sigma}_{t_1 t_2 \cdots t_p}.
\end{aligned} \tag{36}$$

Write formula (10) of Section 2.2 in the form

$$\begin{aligned}
&\bar{y}^{K}_{I} = \sum_{p=1}^{|I|} \sum_{(I_1, I_2, \dots, I_p)} a^{j_1}_{I_1} a^{j_2}_{I_2} \cdots a^{j_p}_{I_p} y^{K}_{j_1 j_2 \cdots j_p}, \\
&\quad (I_1, I_2, \dots, I_p) \sim I.
\end{aligned} \tag{37}$$

Using the same notation, we have

$$
(38)\qquad
\begin{aligned}
&\bar{y}^{t_1}_{I_1} = \sum_{q_1=1}^{|I_1|} \sum_{(I_{1,1},I_{1,2},\ldots,I_{1,q_1})} a^{j_{1,1}}_{I_{1,1}} a^{j_{1,2}}_{I_{1,2}} \cdots a^{j_{1,q_1}}_{I_{1,q_1}} y^{t_1}_{j_{1,1}j_{1,2}\cdots j_{1,q_1}},\\
&\quad (I_{1,1}, I_{1,2}, \ldots, I_{1,q_1}) \sim I_1,\\
&\bar{y}^{t_2}_{I_2} = \sum_{q_2=1}^{|I_2|} \sum_{(I_{2,1},I_{2,2},\ldots,I_{2,q_2})} a^{j_{2,1}}_{I_{2,1}} a^{j_{2,2}}_{I_{2,2}} \cdots a^{j_{2,q_2}}_{I_{2,q_2}} y^{t_2}_{j_{2,1}j_{2,2}\cdots j_{2,q_2}},\\
&\quad (I_{2,1}, I_{2,2}, \ldots, I_{2,q_2}) \sim I_2,\\
&\ldots\\
&\bar{y}^{t_p}_{I_p} = \sum_{q_p=1}^{|I_p|} \sum_{(I_{p,1},I_{p,2},\ldots,I_{p,q_p})} a^{j_{p,1}}_{I_{p,1}} a^{j_{p,2}}_{I_{p,2}} \cdots a^{j_{p,q_p}}_{I_{p,q_p}} y^{t_p}_{j_{p,1}j_{p,2}\cdots j_{p,q_p}},\\
&\quad (I_{p,1}, I_{p,2}, \ldots, I_{p,q_p}) \sim I_p,
\end{aligned}
$$

where $(I_1, I_2, \ldots, I_p) \sim I$. Thus,

$$
(39)\qquad
\begin{aligned}
&\left(\sum_{(I_1,I_2,\ldots,I_p)} \bar{y}^{t_1}_{I_1} \bar{y}^{t_2}_{I_2} \cdots \bar{y}^{t_p}_{I_p}\right) w^{\sigma}_{t_1t_2\cdots t_p}\\
&\quad = \left(\sum_{q_1=1}^{|I_1|} \sum_{(I_{1,1},I_{1,2},\ldots,I_{1,q_1})} a^{j_{1,1}}_{I_{1,1}} a^{j_{1,2}}_{I_{1,2}} \cdots a^{j_{1,q_1}}_{I_{1,q_1}} y^{t_1}_{J_1}\right)\\
&\quad \cdot \left(\sum_{q_2=1}^{|I_2|} \sum_{(I_{2,1},I_{2,2},\ldots,I_{2,q_2})} a^{j_{2,1}}_{I_{2,1}} a^{j_{2,2}}_{I_{2,2}} \cdots a^{j_{2,q_2}}_{I_{2,q_2}} y^{t_2}_{J_2}\right)\\
&\quad \ldots\\
&\quad \cdot \left(\sum_{q_p=1}^{|I_p|} \sum_{(I_{p,1},I_{p,2},\ldots,I_{p,q_p})} a^{j_{p,1}}_{I_{p,1}} a^{j_{p,2}}_{I_{p,2}} \cdots a^{j_{p,q_p}}_{I_{p,q_p}} y^{t_p}_{J_p}\right) w^{\sigma}_{t_1t_2\cdots t_p},
\end{aligned}
$$

where $J_1 = (j_{1,1}, j_{1,2}, \ldots, j_{1,q_1})$, $J_2 = (j_{2,1}, j_{2,2}, \ldots, j_{2,q_2})$, $\ldots$, $J_p = (j_{p,1}, j_{p,2}, \ldots, j_{p,q_p})$. This expression can be written in a different way. Notice that since

$$
(40)\qquad
\begin{aligned}
&(I_{1,1}, I_{1,2}, \ldots, I_{1,q_1}) \sim I_1, (I_{2,1}, I_{2,2}, \ldots, I_{2,q_2}) \sim I_2,\\
&\quad \ldots, (I_{p,1}, I_{p,2}, \ldots, I_{p,q_p}) \sim I_p,
\end{aligned}
$$

then

$$
(41)\qquad (I_{1,1}, I_{1,2}, \ldots, I_{1,q_1}, I_{2,1}, I_{2,2}, \ldots, I_{2,q_2}, \ldots, I_{p,1}, I_{p,2}, \ldots, I_{p,q_p}) \sim I,
$$

where $|I_1| + |I_2| + \cdots + |I_p| = |I| = s$ and if we define

$$
(42)\qquad q = q_1 + q_2 + \cdots + q_p,
$$

we get

$$
(43)\qquad p \leq q \leq |I_1| + |I_2| + \cdots + |I_p| = |I| = s.
$$

Now, having in mind the corresponding summation ranges,

$$\begin{aligned}
&\left(\sum_{(I_1,I_2,\ldots,I_p)} \bar{y}^{t_1}_{I_1}\bar{y}^{t_2}_{I_2}\cdots\bar{y}^{t_p}_{I_p}\right) w^{\sigma}_{t_1t_2\cdots t_p}\\
&\quad = \sum a^{j_{1,1}}_{I_{1,1}} a^{j_{1,2}}_{I_{1,2}} \cdots a^{j_{1,q_1}}_{I_{1,q_1}} a^{j_{2,1}}_{I_{2,1}} a^{j_{2,2}}_{I_{2,2}} \cdots a^{j_{2,q_2}}_{I_{2,q_2}} \cdots a^{j_{p,1}}_{I_{p,1}} a^{j_{p,2}}_{I_{p,2}} \cdots a^{j_{p,q_p}}_{I_{p,q_p}}\\
&\quad \cdot y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p} w^{\sigma}_{t_1t_2\cdots t_p}.
\end{aligned} \tag{44}$$

Denoting

$$\begin{aligned}
&(s_1, s_2, \ldots s_q)\\
&\quad = (j_{1,1}, j_{1,2}, \ldots, j_{1,q_1}, j_{2,1}, j_{2,2}, \ldots, j_{2,q_2}, \ldots, j_{p,1}, j_{p,2}, \ldots, j_{p,q_p}),\\
&(P_1, P_2, \ldots, P_q)\\
&\quad = (I_{1,1}, I_{1,2}, \ldots, I_{1,q_1}, I_{2,1}, I_{2,2}, \ldots, I_{2,q_2}, \ldots, I_{p,1}, I_{p,2}, \ldots, I_{p,q_p}),
\end{aligned} \tag{45}$$

we get $(P_1, P_2, \ldots, P_q) \sim I$, and

$$\begin{aligned}
&\left(\sum_{(I_1,I_2,\ldots,I_p)} \bar{y}^{t_1}_{I_1}\bar{y}^{t_2}_{I_2}\cdots\bar{y}^{t_p}_{I_p}\right) w^{\sigma}_{t_1t_2\cdots t_p}\\
&\quad = \left(\sum_{q=p}^{s} \sum_{(P_1,P_2,\ldots,P_q)} \sum_{(J_1,J_2,\ldots,J_p)} a^{s_1}_{P_1} a^{s_2}_{P_2} \cdots a^{s_q}_{P_q} y^{t_1}_{J_1} y^{t_2}_{J_2} \cdots y^{t_p}_{J_p}\right) w^{\sigma}_{t_1t_2\cdots t_p}.
\end{aligned} \tag{46}$$

This proves (36).

Returning to (32), and substituting from (36) we get a basic formula

$$\sum_{p=1}^{s} \sum_{(I_1,I_2,\ldots,I_p)} \bar{y}^{j_1}_{I_1}\bar{y}^{j_2}_{I_2}\cdots\bar{y}^{j_p}_{I_p}\left(\overline{w}^{\sigma}_{j_1j_2\cdots j_p} - w^{\sigma}_{j_1j_2\cdots j_p}\right) = 0. \tag{47}$$

Now it is easy to show that $\overline{w}^{\sigma}_{j_1j_2\cdots j_s} = w^{\sigma}_{j_1j_2\cdots j_s}$ provided $\overline{w}^{\sigma}_{j_1j_2\cdots j_k} = w^{\sigma}_{j_1j_2\cdots j_k}$ for all $k \le s-1$.

If $s = 1$, we get $\bar{y}^{j_1}_{i_1}(\overline{w}^{\sigma}_{j_1} - w^{\sigma}_{j_1}) = 0$, and since the matrix $\bar{y}^{j}_{i}$ is regular, $\overline{w}^{\sigma}_{j} = w^{\sigma}_{j}$.

If $s = 2$, we have

$$\bar{y}^{j_1}_{i_1i_2}(\overline{w}^{\sigma}_{j_1} - w^{\sigma}_{j_1}) + \bar{y}^{j_1}_{i_1}\bar{y}^{j_2}_{i_2}(\overline{w}^{\sigma}_{j_1j_2} - w^{\sigma}_{j_1j_2}) = \bar{y}^{j_1}_{i_1}\bar{y}^{j_2}_{i_2}(\overline{w}^{\sigma}_{j_1j_2} - w^{\sigma}_{j_1j_2}) = 0,$$

which implies, again using regularity of the matrix $\bar{y}^{j}_{i}$, that $\overline{w}^{\sigma}_{j_1j_2} = w^{\sigma}_{j_1j_2}$.

Now assume that $\overline{w}^{\sigma}_{j_1j_2\cdots j_k} = w^{\sigma}_{j_1j_2\cdots j_k}$ for all $k \le s-1$. Then (39) reduces to

$$\bar{y}^{j_1}_{i_1}\bar{y}^{j_2}_{i_2}\cdots\bar{y}^{j_s}_{i_s}\left(\overline{w}^{\sigma}_{j_1j_2\cdots j_s} - w^{\sigma}_{j_1j_2\cdots j_s}\right) = 0, \tag{48}$$

which gives us $\overline{w}^{\sigma}_{j_1j_2\cdots j_s} - w^{\sigma}_{j_1j_2\cdots j_s} = 0$ as required.

(b) This assertion is immediate.

The charts of the form $(W^{(i)}, \Psi^{(i)})$ are referred to as the *adapted charts* to the canonical group action of L^r_n on imm $T^r_n Y$.

We can now easily prove the following result.

Theorem 2. *If Y is Hausdorff, then the canonical right action of L^r_n defines on* $\operatorname{imm} T^r_n Y$ *the structure of a right principal L^r_n-bundle.*

Proof. We have to show that the equivalence $\mathcal{R}$ "*there exists $A \in L^r_n$ such that* $P = Q \circ A$" is a closed submanifold of the product manifold $\operatorname{imm} T^r_n Y \times \operatorname{imm} T^r_n Y$, and that the group action (4) is free.

But $\mathcal{R}$ is obviously a submanifold, by Theorem 1, (b). To prove that $\mathcal{R}$ is closed, consider a point $(P, Q) \in \operatorname{imm} T^r_n Y \times \operatorname{imm} T^r_n Y$ such that $(P, Q) \neq \mathcal{R}$. Then $P \neq Q$, and we distinguish two possibilities: (1) $\tau^{r,0}_n(P) = \tau^{r,0}_n(Q)$, (2) $\tau^{r,0}_n(P) \neq \tau^{r,0}_n(Q)$.

In the case (1), $P, Q \in V^r_n$ for any chart (V, ψ) on Y, Clearly, because Y is Hausdorff, in both cases the points P, Q can be separated by open sets. The product of these open sets does not intersect $\mathcal{R}$, proving that $\mathcal{R}$ is closed.

To show that the action (4) is free, we assume that $P = P \circ A$ for some r-velocity $P \in \operatorname{imm} T^r_n Y$ and some $A \in L^r_n$. Since P is *regular*, there exists $Q \in J^r_{(y,0)}(Y, \mathbf{R}^n)$, where y is the target of P, such that $Q \circ P = J^r_0 \operatorname{id}_{\mathbf{R}^n}$ (see (2)), which implies $A = J^r_0 \operatorname{id}_{\mathbf{R}^n}$.

2.4. Frames. Let X be a smooth n-dimensional manifold. An *invertible* n-velocity of order r at a point $x \in X$ is called an *r-frame* at x. Obviously, the set of r-frames at the points of X coincides with the subset $\operatorname{imm} T^r_n X$ of $T^r_n X$ formed by the r-jets $P = J^r_0 \zeta$ with source $0 \in \mathbf{R}^n$ and target in X, such that for any representative ζ of P, and any chart (U, φ), $\varphi = (x^i)$,

(1) $$\det\big(D_i(x^j \zeta)(0)\big) \neq 0.$$

We denote

(2) $$F^r X = \operatorname{imm} T^r_n X,$$

We have the *canonical jet projections* $\tau^{r,s} : F^r X \to F^s X$ and $\tau^r : F^r X \to X$, defined as the restrictions of the canonical jet projections $\tau^{r,s}_n : T^r_n X \to T^s_n X$ and $\tau^{r,0}_n : T^r_n X \to X$ to the set $F^s X$ (Section 2.2, (2)). Note that Theorem 2 can be applied to $F^r X$.

Theorem 3. (a) *The set is an open, dense, L^r_n-invariant subset of $T^r_n X$.*

(b) *The canonical right action $(P, A) \to P \circ A$ of L^r_n on $F^r X$ defines the structure of a right principal L^r_n-bundle over X.*

Proof. (a) This follows from the condition (1).

(b) We have to show that the right action $(P, A) \to P \circ A$ is free, and the orbit space $F^r X / L^r_n$ has a smooth structure such that the quotient projection of $F^r X$ onto $F^r X / L^r_n$ is a submersion.

Let $P \in F^r X$ and $A, B \in L^r_n$ be such that $P \circ A = P \circ B$. Since P is invertible, we have $J^r_0 \operatorname{id}_{\mathbf{R}^n} = P^{-1} \circ P \circ A = P^{-1} \circ P \circ B$ hence $A = B$.

It is clear that the L^r_n-orbits in $F^r X$ coincide with the sets $(\tau^r)^{-1}(x) \subset F^r X$. In particular, the equivalence on $F^r X$ defined by the group action $(P, A) \to P \circ A$ coincides with the equivalence associated with the jet projection τ^r. Therefore, we may take $F^r X / L^r_n = X$.

$F^r X$, considered as a right principal L^r_n-bundle over X, is referred to as the *bundle of r-frames* over X.

Let X (resp. Y) be an n-dimensional (resp. m-dimensional) smooth manifold. We wish to describe the manifold of r-jets $J^r(X, Y)$ as an associated fiber bundle.

Consider the manifold $L^r_{n,m}$ of r-jets with source at $0 \in \mathbf{R}^n$ and target at $0 \in \mathbf{R}^m$ (Section 1.2, (13)). $L^r_{n,m}$ is endowed with natural actions of the differential groups L^r_n and L^r_m defined by the composition of jets $\circ$, and of the product of differential groups $L^r_n \times L^r_m$. The group operation in $L^r_n \times L^r_m$ is defined by

(3) $$(A, H) \cdot (A', H') = (A \circ A', H \circ H'),$$

$L^r_n \times L^r_m$ acts to the left on $L^r_{n,m}$ by

(4) $$(A, G) \cdot P = G \circ P \circ A^{-1}.$$

$L^r_n \times L^r_m$ also acts to the right on the product of the r-frame bundles $F^r X \times F^r Y$ by

(5) $$(S, T) \cdot (A, H) = (S \circ A, T \circ H).$$

We have the following assertion.

Theorem 4. (a) *$F^r X \times F^r Y$ with the action* (5) *is a principal* $(L^r_n \times L^r_m)$*-bundle with base $X \times Y$.*

(b) *The mappings*

(6) $$F^r X \times F^r Y \times L^r_{n,m} \ni ((S, T), P) \to T \circ P \circ S^{-1} \in J^r(X, Y),$$

(7) $$F^r X \times T^r_n Y \ni (S, R) \to R \circ S^{-1} \in J^r(X, Y),$$

are frame mappings.

Proof. (a) The canonical projection of $F^r X \times F^r Y$ onto $X \times Y$ is obviously a surjective submersion. To show that the action (5) is free, assume that $(S, T) \cdot (A, H) = (S, T)$. Then $S \circ A = S$, $T \circ H = T$, and we use invertibility of S and T.

(b) Consider e.g. (7). If $Q \in J^r(X, Y)$, then for any $S \in F^r X$, equation $R \circ S^{-1} = Q$ has a solution $R = Q \circ S$. Thus, (7) is surjective. To verify invariance, choose an element $A \in L^r_n$. Then for any $(S, R) \in F^r X \times T^r_n Y$, $(R \circ A) \circ (S \circ A)^{-1} = R \circ A \circ A^{-1} \circ S^{-1} = R \circ S^{-1}$, proving L^r_n-invariance.

Corollary 1. (6) *defines on $J^r(X, Y)$ the structure of a fiber bundle with fiber $L^r_{n,m}$, associated with the principal $L^r_n \times L^r_m$-bundle $F^r X \times F^r Y$.*

Corollary 2. (7) *defines on $J^r(X, Y)$ the structure of a fiber bundle with fiber $T^r_n Y$, associated with the principal L^r_n-bundle $F^r X$.*

Remark 4 (linear frames). Let X be an n-dimensional manifold. The principal L^1_n-bundle $F^1 X$, denoted by FX, is usually called the *bundle of linear frames*, or simply the *bundle of frames* over X.

FX can alternatively be defined as follows. The elements of the set FX are *bases*, or *frames*, of the tangent spaces $T_x X$, where x runs through X. We have the mapping $\pi : FX \to X$, assigning to a basis $\Xi = (\xi_1, \xi_2, \dots, \xi_n)$ at $x \in X$ the point x. If (U, φ), $\varphi = (x^i)$, is a chart at x, then the *associated chart* (V, ψ), $\psi = (x^i, x^i_j)$, is defined as follows. We take $V = \pi^{-1}(U)$, and $\Xi \in V$. Then the coordinates $x^i(\Xi)$ are taken to be $x^i(\pi(\Xi))$, and $x^i_j(\Xi)$ are defined by the decomposition

$$(8) \qquad \xi_j = x^k_j(\Xi)\left(\frac{\partial}{\partial x^k}\right)_x$$

in the tangent space $T_x X$. The associated charts are taken to define a smooth structure on FX.

FX is endowed with a right action of the general linear group $GL_n(\mathbf{R}) = L^1_n$. If $A \in GL_n(\mathbf{R})$, $A = (a^i_j)$, then $\Xi \cdot A = (\xi_i a^i_1, \xi_i a^i_2, \dots, \xi_i a^i_n)$. In coordinates,

$$(9) \qquad x^i(\Xi \cdot A) = x^i(\Xi), \qquad x^k_j(\Xi \cdot A) = x^k_p(\Xi)a^p_j.$$

This action defines on FX the structure of a principal $GL_n(\mathbf{R})$-bundle.

An invertible 1-jet with source $0 \in \mathbf{R}^n$ and target $x \in X$ is canonically identified with a linear isomorphism from $\mathbf{R}^n$ to $T_x X$, and also with the basis of $T_x X$, consisting of the images of the vectors of the canonical basis of $\mathbf{R}^n$ under this linear isomorphism.

3. Jet prolongations of smooth manifolds

3.1. Contact elements. Let $r \geq 0$, $m, n \geq 1$ be integers, let X be a smooth manifold of dimension n, and let Y be a smooth manifold of dimension $n + m$.

Denote by $C^r_{(x,y)}(X, Y)$ the set of mappings of class C^r $f : W \to Y$, where W is a neighborhood of x, such that $f(x) = y$ (Section 1.2), and consider the set

$$(1) \qquad C^r(X, Y) = \bigcup_{(x,y)\in X\times Y} C^r_{(x,y)}(X, Y).$$

We say that two mappings $f, g \in C^r(X, Y)$ have *contact to order* r at (x_1, x_2), if f is defined at x_1, g is defined at x_2, and there exist charts (U_1, φ_1) at x_1 and (U_2, φ_2) at x_2 such that

$$(2) \qquad J^r_0\big(f\varphi_1^{-1}\,\mathrm{tr}_{-\varphi_1(x_1)}\big) = J^r_0\big(f\varphi_2^{-1}\,\mathrm{tr}_{-\varphi_2(x_2)}\big).$$

The relation "*f and g have contact to order r at* (x_1, x_2)" is an *equivalence* on $C^r(X, Y)$. Equivalence classes of this equivalence are called *contact elements of order* r with *target* y, or simply *contact elements*. The contact element whose *representative* is a mapping $f \in C^r_{(x,y)}(X, Y)$ is called the contact element *of* f at x, and is denoted by $G^r_x f$. The set of contact elements with target y is denoted by $G^r_y(X, Y)$.

The equivalence on $C^r(X, Y)$ "*f and g have contact to order r at* (x_1, x_2)" induces an equivalence on the set of r-jets $J^r(X, Y)$ "*there exist charts* (U_1, φ_1) *and* (U_2, φ_2) *such that* $J^r_0(f\varphi_1^{-1}\,\mathrm{tr}_{-\varphi_1(x_1)}) = J^r_0(f\varphi_2^{-1}\,\mathrm{tr}_{-\varphi_2(x_2)})$". When we express a contact element as the *class* of jets, we denote

$$(3) \qquad G^r_x f = [J^r_x f].$$

Clearly, $J^r_{x_1} f, J^r_{x_2} g \in J^r(X, Y)$ are equivalent if and only if there exists a diffeomorphism $\alpha : U \to V$, where U is a neighborhood of x_1 and V is a neighborhood of x_2 sending x_1 to x_2, such that $J^r_{x_1} f = J^r_{x_2} g \circ J^r_{x_1}\alpha$. Indeed, we can take α in the form $\alpha = \varphi_2^{-1}\,\mathrm{tr}_{-\varphi_2(x_2)} \circ \mathrm{tr}_{\varphi_1(x_1)}\,\varphi_1$.

3.2. Grassmann prolongations of a manifold. Let $r \geq 0$, $m, n \geq 1$ be integers, and let Y be a smooth manifold of dimension $n+m$.

Lemma 1. *Let W be a neighborhood of the origin $0 \in \mathbf{R}^n$, and let $\zeta, \chi : W \to Y$ be two C^r-mappings. The following conditions are equivalent:*
(a) *ζ and χ have contact to order r at* 0.
(b) *There exists an element $J_0^r \alpha \in L_n^r$ such that*

(2) $$J_0^r \zeta = J_O^r \chi \circ J_0^r \alpha.$$

Proof. This is immediate.

In this section, we consider contact elements of *immersions* with source $0 \in \mathbf{R}^n$ and target in Y. Lemma 1 shows that such a contact element belongs to the quotient of the manifold of regular n-velocities of order r with target in Y, $\operatorname{imm} T_n^r Y$, by the differential group L_n^r, i.e., to the *orbit space*

(3) $$Gr_n^r Y = \operatorname{imm} T_n^r Y / L_n^r$$

(Section 2.3, Theorem 2). We denote by $\pi_n^r : \operatorname{imm} T_n^r Y \to Gr_n^r Y$ the *canonical* quotient projection.

For every s, $0 \leq s \leq r$, we also have the *canonical projection* of $Gr_n^r Y$ onto $Gr_n^s Y$ defined by

(4) $$\rho_n^{r,s}(G_0^r \zeta) = G_0^s \zeta.$$

If $\tau_n^{r,s}$ is the canonical jet projection of $\operatorname{imm} T_n^r Y$ onto $\operatorname{imm} T_n^s Y$, we have the commutative diagram

(5) $$\begin{array}{ccc} \operatorname{imm} T_n^r Y & \xrightarrow{\pi_n^r} & Gr_n^r Y \\ \Big\downarrow \tau_n^{r,s} & & \Big\downarrow \rho_n^{r,s} \\ \operatorname{imm} T_n^s Y & \xrightarrow{\pi_n^s} & Gr_n^s Y. \end{array}$$

If $Y = \mathbf{R}^{n+m}$, we have the commutative diagram

(6) $$\begin{array}{ccc} \operatorname{imm} T_n^r \mathbf{R}^{n+m} & \xrightarrow{\pi_n^r} & Gr_n^r \mathbf{R}^{n+m} \\ \Big\downarrow \tau_n^{r,s} & & \Big\downarrow \rho_n^{r,s} \\ \operatorname{imm} T_n^s \mathbf{R}^{n+m} & \xrightarrow{\pi_n^r} & Gr_n^s \mathbf{R}^{n+m}. \end{array}$$

In particular, if $s = 0$, we have

(7) $$\begin{array}{ccc} \operatorname{imm} T_n^r \mathbf{R}^{n+m} & \xrightarrow{\pi_n^r} & Gr_n^r \mathbf{R}^{n+m} \\ \Big\downarrow \tau_n^{r,0} & & \Big\downarrow \rho_n^{r,0} \\ \mathbf{R}^{n+m} & \xrightarrow{\mathrm{id}_{\mathbf{R}^{n+m}}} & \mathbf{R}^{n+m}. \end{array}$$

Clearly, $\operatorname{imm} L_{n,n+m}^r = (\tau_n^{r,0})^{-1}(0)$ (see Section 1.2, (13)). We define

(8) $$Gr_{n,n+m}^r = (\rho_n^{r,0})^{-1}(0) = \operatorname{imm} L_{n,n+m}^r / L_n^r.$$

As fibers of surjective submersions, both imm $L^r_{n,n+m}$ and $Gr^r_{n,n+m}$ are closed submanifolds. We call $Gr^r_{n,n+m}$ the *n-grassmannian* of order r over $\mathbf{R}^{n+m}$, or simply a *higher order grassmannian.*

Note that $Gr^r_{n,n+m}$ is endowed with a *left action* of the differential group L^r_{n+m},

(9) $$L^r_{n+m} \times Gr^r_{n,n+m} \ni (A, [P]) \to [A \circ P] \in Gr^r_{n,n+m}.$$

If F^rY is the *bundle of r-frames* over Y, the product $F^rY \times Gr^r_{n,n+m}$ is endowed with the *right action* of L^r_{n+m}

(10) $$\begin{aligned} &L^r_{n+m} \times F^rY \times Gr^r_{n,n+m} \ni (A, (F, [P])) \\ &\quad \to (F \circ A, [A^{-1} \circ P]) \in F^rY \times Gr^r_{n,n+m}. \end{aligned}$$

We have the following result.

Theorem 1. (a) *Let Y be Hausdorff. The orbit space Gr^r_nY has a unique smooth structure such that the canonical quotient projection τ^r_n of* imm T^r_nY *onto Gr^r_nY is a submersion, and*

(11) $$\dim Gr^r_nY = m\binom{n+r}{n} + n.$$

(b) *The mapping*

(12) $$F^rY \times Gr^r_{n,n+m} \ni (F, [P]) \to [F \circ P] \in Gr^r_nY$$

is a frame mapping.

Proof. (a) This is a direct reformulation of Theorem 2, Section 2.3.

(b) It is sufficient to prove that (10) is L^r_{n+m}-invariant. Let $J^r_0\alpha \in L^r_{n+m}$ be a point. Then (12) assigns to the point $(J^r_0\mu \circ J^r_0\alpha, [J^r_0\alpha^{-1} \circ J^r_0\zeta]) \in F^r_Y \times Gr^r_{n.n+m}$ the point $[J^r_0(\mu\alpha) \circ J^r_0(\alpha^{-1}\zeta)] = [J^r_0\mu \circ J^r_0\zeta] \in Gr^r_nY$ proving invariance.

The frame mapping (12) defines on Gr^r_nY the structure of a *fiber bundle* over Y with *fiber $Gr^r_{n,n+m}$*, associated with the bundle or framesF^rY. With this structure, Gr^r_nY is called the *n-Grassmann prolongation of order r* of Y, or simply the *Grassmann prolongation* of Y.

Let $\zeta : U \to Y$ be an immersion of a neighborhood U of the origin $0 \in \mathbf{R}^n$ into Y, and let $T^r_n\zeta$ be its r-prolongation (Section 2.2, 11). By the *r-contact prolongation* of ζ we mean the mapping

(13) $$U \ni t \to G^r_n\gamma(t) = \pi^r_n(T^r_n\gamma(t)) \in G^r_nY.$$

An explicit description of the smooth structure of the Grassmann prolongation G^r_nY follows immediately from the analysis of the canonical group action of the differential group L^r_n on imm T^r_nY (Section 2.3, Lemma 2, Theorem 1, Theorem 2).

Consider a chart (V, ψ), $\psi = (y^K)$ on Y, the associated chart (V^r_n, ψ^r_n), $\psi^r_n = (y^K, y^K_{i_1}, y^K_{i_1i_2}, \dots, y^K_{i_1i_2\cdots i_r})$ on imm T^r_nY, an n-subsequence (i) of the sequence $(1, 2, \dots, n+m)$, and the adapted chart $(W^{(i)}, \Psi^{(i)})$, $\Psi^{(i)} = (y^i, y^i_{j_1}, y^i_{j_1j_2}, \dots, y^i_{j_1j_2\cdots j_r}, w^\sigma, w^\sigma_{i_1}, w^\sigma_{i_1i_2}, \dots, w^\sigma_{i_1i_2\cdots i_r})$, where

(14) $$w^\sigma = y^\sigma, \qquad w^\sigma_{i_1i_2\cdots i_k} = \Delta_{i_k} \cdots \Delta_{i_2}\Delta_{i_1}w^\sigma,$$

and

$$\Delta_i(P)$$

$$= z_i^s(P)\left(\sum_{k=0}^{r-1}\sum_{q_1\le q_2\le\cdots\le q_k}\sum_{l=0}^{r-1}\sum_{i_1\le i_2\le\cdots\le i_l}\left(\frac{\partial w^\sigma_{i_1i_2\cdots i_l}}{\partial y^K_{q_1q_2\cdots q_k}}\right)_P y^K_{q_1q_2\cdots q_k s}(P)\left(\frac{\partial}{\partial w^\sigma_{i_1i_2\cdots i_l}}\right)_P\right.$$

$$(15)\qquad \left.+\sum_{k=0}^{r-1}\sum_{q_1\le q_2\le\cdots\le q_k}\sum_{l=0}^{r-1}\sum_{i_1\le i_2\le\cdots\le i_l}\left(\frac{\partial y^s_{i_1i_2\cdots i_l}}{\partial y^K_{q_1q_2\cdots q_k}}\right)_P y^K_{q_1q_2\cdots q_k s}(P)\left(\frac{\partial}{\partial y^s_{i_1i_2\cdots i_l}}\right)_P\right)$$

$$= \sum_{l=0}^{r-1}\sum_{i_1\le i_2\le\cdots\le i_l}\left(w^\sigma_{i_1i_2\cdots i_l i}(P)\left(\frac{\partial}{\partial w^\sigma_{i_1i_2\cdots i_l}}\right)_P + z_i^s(P)y^k_{i_1i_2\cdots i_l s}(P)\left(\frac{\partial}{\partial y^k_{i_1i_2\cdots i_l}}\right)_P\right),$$

thus,

$$(16)\qquad \Delta_i = \frac{\partial}{\partial y^i} + \sum_{l=0}^{r-1}\sum_{i_1\le i_2\le\cdots\le i_l} w^\sigma_{i_1i_2\cdots i_l i}\frac{\partial}{\partial w^\sigma_{i_1i_2\cdots i_l}} + \sum_{l=1}^{r-1}\sum_{i_1\le i_2\le\cdots\le i_l} z_i^s y^k_{i_1i_2\cdots i_l s}\frac{\partial}{\partial y^k_{i_1i_2\cdots i_l}}.$$

Recall that a chart on Y, (V,ψ), $\psi=(y^K)$, induces the *associated chart* (V^r_n,ψ^r_n), $\psi^r_n=(y^K,y^K_{i_1},y^K_{i_1i_2},\dots,y^K_{i_1i_2\cdots i_r})$, on $\operatorname{imm} T^r_nY$, and for any n-subsequence (i) of the sequence $\{1,2,\dots,n+m\}$, whose complementary subsequence is denoted by (σ), the chart $(W^{(i)},\Psi^{(i)})$, $\Psi^{(i)}=(y^i,y^i_{p_1},y^i_{p_1p_2},\dots,y^i_{p_1j_2\cdots p_r},w^\sigma,w^\sigma_{i_1},w^\sigma_{i_1i_2},\dots,w^\sigma_{i_1i_2\cdots i_r})$, on $\operatorname{imm} T^r_nY$, *adapted* to the canonical action of the group L^r_n (Section 2.3, Theorem 1); in this chart, $i,i_1,i_2,\dots,i_r\in(i)$, $i_1\le i_2\le\cdots\le i_r$, $p_1\le p_2\le\cdots\le p_r\le n$. Denoting

$$(17)\qquad W^{(i)}_G=\pi^r_n\big(W^{(i)}\big),\qquad \Psi^{(i)}_G=\big(y^i,w^\sigma,w^\sigma_{i_1},w^\sigma_{i_1i_2},\dots,w^\sigma_{i_1i_2\cdots i_r}\big),$$

we obtain the *associated* chart $(W^{(i)}_G,\Psi^{(i)}_G)$ on Gr^r_nY.

Assume that we have another chart, $(\bar V,\bar\psi)$, $\psi=(\bar y^K)$, on Y such that $V\cap\bar V\ne\emptyset$, and an n-subsequence (j) of the sequence $\{1,2,\dots,n+m\}$. Denote by (ν) the complementary subsequence. Then on $W^{(i)}\cap\overline{W}^{(j)}$

$$(18)\qquad \overline{\Delta}_j=\bar z^s_j\bar d_s=\bar z^s_jd_s=\bar z^s_j\delta^p_sd_p=\bar z^s_jy^i_sz^p_id_p=\bar z^s_jy^i_s\Delta_i,$$

where $i\in(i)$, $j\in(j)$. Consider the factor $\bar z^s_jy^i_s$. If $P\in W^{(i)}\cap\overline{W}^{(i)}$ and $A\in L^r_n$, we have $\bar z^s_j(P\circ A)\bar y^k_s(P\circ A)=\delta^k_j$ and, in the canonical coordinates on the differential group L^r_n, $\bar y^k_s(P\circ A)=\bar y^k_t(P)a^t_s(A)$. This implies that $\bar z^s_j(P\circ A)=\bar z^t_j(P)a^s_t(A^{-1})$, hence $\bar z^s_j(P\circ A)y^i_s(P\circ A)=\bar z^s_j(P)y^i_s(P)$. In particular, the function

$$(19)\qquad \Psi^i_j=\bar z^s_jy^i_s$$

defined on $W^{(i)}\cap\overline{W}^{(i)}$, depends only on $[P]\in\pi^r_n(W^{(i)})\cap\pi^r_n(\overline{W}^{(i)})$ (in fact, Ψ^i_j depends on $\rho^{r,1}_n([P])$ only).

Now we discuss transformation properties of the functions w^σ, $w^\sigma_{i_1}$, $w^\sigma_{i_1i_2}$, …, $w^\sigma_{i_1i_2\cdots i_r}$ belonging to the associated charts on the Grassmann prolongation Gr^r_nY of Y.

Theorem 2. *Let* (V, ψ), $\psi = (y^A)$, *and* $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\bar{y}^A)$ *be two charts on* Y *such that* $V \cap \overline{V} \neq \emptyset$, *let* $(W_G^{(i)}, \Psi_G^{(i)})$, $\Psi_G^{(i)}) = (y^i, w^\sigma, w^\sigma_{i_1}, w^\sigma_{i_1 i_2}, \dots, w^\sigma_{i_1 i_2 \cdots i_r})$ *and* $(\overline{W}_G^{(j)}, \overline{\Psi}_G^{(j)})$, $\overline{\Psi}_G^{(j)} = (y^j, w^\nu, w^\nu_{j_1}, w^\nu_{j_1 j_2}, \dots, w^\nu_{j_1 j_2 \cdots j_r})$ *be the associated charts on* $Gr^r_n Y$. *Let the transformation equations from* (V, ψ) *to* $(\overline{V}, \overline{\psi})$ *have the form*

$$\bar{y}^i = F^i(y^k, w^\nu), \qquad \overline{w}^\sigma = F^\sigma(y^k, w^\nu). \tag{20}$$

Then

$$\overline{w}^\nu_{j_1} = \Psi^i_{j_1} \Delta_i \overline{w}^\nu = \Psi^i_{j_1} \left(\frac{\partial F^\nu}{\partial y^i} + w^\sigma_i \frac{\partial F^\nu}{\partial w^\sigma} \right), \tag{21}$$

and the functions $w^\nu_{i_1 i_2 \cdots i_k i_{k+1}}$ *obey the recurrent transformation formulas*

$$\overline{w}^\nu_{j_1 j_2 \cdots j_k j_{k+1}} = \Psi^i_{j_{k+1}} \Delta_i \overline{w}^\nu_{j_1 j_2 \cdots j_k}. \tag{22}$$

Proof. (22) follows from (17) and (14). Then using (14), (19), and (20) we get

$$\overline{w}^\nu_{j_1 j_2 \cdots j_k j_{k+1}} = \overline{\Delta}_{j_{k+1}} \overline{w}^\nu_{j_1 j_2 \cdots j_k} = \Psi^i_{j_{k+1}} \Delta_i \overline{w}^\nu_{j_1 j_2 \cdots j_k}. \tag{23}$$

proving (23).

3.3. Prolongations of a fibered manifold. In this section, Y is a fibered manifold with base X and projection π. We denote $n = \dim X$, $\dim Y = n + m$.

Let $r \geq 0$ be an integer. Let $y \in Y$ be a point, let $x = \pi(y)$, and let $\mathrm{Sec}^r_{x,y} Y$ be the set of C^r sections γ of Y defined at x, such that $\gamma(x) = y$. We say that two sections $\gamma, \delta \in \mathrm{Sec}^r_{x,y} Y$ are *tangent to order* r *at* x, if there exists a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, at y, whose associated chart on X is denoted by (U, φ), $\varphi = (x^i)$, such that

$$D_{i_1} D_{i_2} \cdots D_{i_s} (y^\sigma \gamma \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \cdots D_{i_s} (y^\sigma \delta \varphi^{-1})(\varphi(x)) \tag{1}$$

for all $s = 1, 2, \dots, r$ and all $i_1, i_2, \dots, i_s$ such that $1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq n$.

The binary relation " γ, δ *are tangent to order* r *at* x" is obviously an equivalence on the set $\mathrm{Sec}^r_{x,y} Y$. The class, containing a section $\gamma \in \mathrm{Sec}^r_{x,y} Y$ is called the r*-jet* of γ at x, and is denoted by $J^r_x \gamma$. The set of classes with respect to this equivalence relation is denoted by $J^r_{(x,y)} Y$. We define

$$J^r Y = \bigcup_{(x,y)} J^r_{(x,y)} Y. \tag{2}$$

The *canonical jet projections* are the mappings $\pi^{r,s} : J^r Y \to J^s Y$, where $1 \leq s \leq r$, $\pi^{r,0} : J^r Y \to Y$ and $\pi^r : J^r Y \to X$ defined by

$$\pi^{r,s}(J^r_x \gamma) = J^s_x \gamma, \qquad \pi^{r,0}(J^r_x \gamma) = y, \qquad \pi^r(J^r_x \gamma) = x. \tag{3}$$

Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y, and let (U, φ), $\varphi = (x^i)$ be the associated chart on X. We define the *associated chart* (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{i_1}, y^\sigma_{i_1 i_2}, \dots, y^\sigma_{i_1 i_2 \cdots i_r})$, on J^Y by the following condition:

$$V^r = (\pi^{r,0})^{-1}(V), \tag{4}$$

and, if $J^r_x\gamma \in V^r$, then

$$x^i(J^r_x\gamma) = x^i(x), \qquad y^\sigma(J^r_x\gamma) = y^\sigma(y),$$
$$(5) \qquad y^\sigma_{i_1 i_2\cdots i_r}(J^r_x\gamma) = D_{i_1}D_{i_2}\cdots D_{i_r}(y^\sigma\gamma\varphi^{-1})(\varphi(x)),$$
$$1 \le i_1 \le i_2 \le \cdots \le i_r \le n.$$

where $1 \le i \le n$, $1 \le \sigma \le m$, $1 \le s \le r$, and $1 \le i_1 \le i_2 \le \cdots \le i_s \le n$.

Lemma 2. *There exists a unique smooth structure on J^rY such that for every fibered chart (V, ψ) on Y, (V^r, ψ^r) is a chart on J^rY. The dimension of J^rY is*

$$(6) \qquad \dim J^rY = n + m\binom{n+r}{n}.$$

Proof. We want to show that if we have an atlas on Y, consisting of fibered charts (V, ψ), then the associated charts (V^r, ψ^r) form an atlas on J^rY. To this purpose it is obviously sufficient to verify smoothness of the transformations between two charts. If (V, ψ) and $(\overline{V}, \overline{\psi})$ are two charts such that $V \cap \overline{V} \neq \emptyset$, then writing $\bar{y}^\sigma\gamma\bar{\varphi}^{-1} = \bar{y}^\sigma\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\bar{\varphi}^{-1}$, we get, using the chain rule, the *transformation formula*

$$(7) \qquad \begin{aligned} \bar{y}^\sigma_{i_1 i_2\cdots i_s} &= D_{i_1}D_{i_2}\cdots D_{i_s}(\bar{y}^\sigma\gamma\bar{\varphi}^{-1})(\bar{\varphi}(x)) \\ &= D_{i_1}D_{i_2}\cdots D_{i_s}(\bar{y}^\sigma\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\bar{\varphi}^{-1})(\bar{\varphi}(x)). \end{aligned}$$

Thus, the transformation equations are polynomial hence smooth.

Now it is easy to compute the dimension. We get

$$(8) \qquad \begin{aligned} \dim J^rY &= n + m + mn + m\binom{n+1}{2} + \cdots + m\binom{n+r-1}{r} \\ &= n + m\binom{n+r}{r}. \end{aligned}$$

Example 1. If $r = 2$, formula (7) gives

$$(9) \qquad \begin{aligned} \bar{y}^\sigma_{i_1} &= \left(\frac{\partial\bar{y}^\sigma}{\partial x^{j_1}} + \frac{\partial\bar{y}^\sigma}{\partial y^\nu}y^\nu_{j_1}\right)\frac{\partial x^{j_1}}{\partial\bar{x}^{i_1}}, \\ \bar{y}^\sigma_{i_1 i_2} &= \left(\frac{\partial^2\bar{y}^\sigma}{\partial x^{j_1}\partial x^{j_2}} + \frac{\partial^2\bar{y}^\sigma}{\partial y^\nu\partial x^{j_2}}y^\nu_{j_1}\right)\frac{\partial x^{j_2}}{\partial\bar{x}^{i_2}}\frac{\partial x^{j_1}}{\partial\bar{x}^{i_1}} \\ &\quad + \left(\frac{\partial\bar{y}^\sigma}{\partial x^{j_1}} + \frac{\partial\bar{y}^\sigma}{\partial y^\nu}y^\nu_{j_1}\right)\frac{\partial^2 x^{j_1}}{\partial\bar{x}^{i_1}\partial\bar{x}^{i_2}}. \end{aligned}$$

The concepts of the r-jet of a C^r section of a fibered manifold Y, and of the r-jet prolongation J^rY of Y, have been introduced in full analogy with the concepts of the r-jet of an arbitrary C^r mapping, and of the manifold of r-jets $J^r(X, Y)$.

J^rY can also be defined as a submanifold of $J^r(X, Y)$, and of the Grassmann prolongation Gr^r_nY. We denote

$$(10) \qquad \operatorname{imm}_\pi T^r_nY = \left\{J^r_0\zeta \in \operatorname{imm} T^r_nY \,|\, J^r_0\pi \circ J^r_0\zeta \in F^rX\right\}.$$

Theorem 3. (a) *is a closed submanifold of* $J^r(X, Y)$.

(b) $\operatorname{imm}_\pi T^r_n Y$ *is an open,* L^r_n*-invariant set in* $\operatorname{imm} T^r_n Y$, $J^r X = (\operatorname{imm}_\pi T^r_n Y)/L^r_n$ *is an open set in* $Gr^r_n Y$, *and the diagram*

$$\begin{tikzcd} \operatorname{imm}_\pi T^r_n Y \arrow[r] \arrow[d] & \operatorname{imm} T^r_n Y \arrow[d] \\ J^r Y \arrow[r] & Gr^r_n Y \end{tikzcd} \tag{11}$$

in which the horizontal arrows are the canonical inclusions, and the vertical arrows are the quotient projections, commutes.

Proof. (a) Let (V, ψ), $\psi = (x^i, y^I)$, be a fibered chart on Y, (U, φ), $\varphi = (x^i)$, the associated chart on X. (V, ψ) and (U, φ) define the *associated* chart $(W^r, \chi^r_{\varphi,\psi})$ on $J^r(X, Y)$ (Section 3.1). Recall that

$$W^r = (\rho^{r,0})^{-1}(U \times V), \qquad \chi^r_{\varphi,\psi} = (x^i, y^I, \chi^I_{i_1 i_2 \cdots i_k}), \tag{12}$$

where $1 \le k \le r$, $1 \le \sigma \le m$, $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$, $\rho^{r,0} : J^r(X, Y) \to X \times Y$ is the canonical jet projection, and $\chi^\sigma_{i_1 i_2 \cdots i_k}$ are real-valued functions on W^r defined by $\chi^I_{i_1 i_2 \cdots i_k}(J^r_x f) = D_{i_1} D_{i_2} \cdots D_{i_k}(y^I f \varphi^{-1})(\varphi(x))$, where $I = i, \sigma$. More precisely,

$$\chi^I_{i_1 i_2 \cdots i_k} = \left(\chi^i_{i_1 i_2 \cdots i_k}, \chi^\sigma_{i_1 i_2 \cdots i_k}\right). \tag{13}$$

Since every section γ satisfies $x^i \circ \gamma = x^i$, $J^r Y \cap W^r$ is expressed by the equations

$$\chi^i_{i_1} = \delta^i_{i_1}, \ \chi^i_{i_1 i_2} = 0, \ldots, \chi^i_{i_1 i_2 \cdots i_r} = 0. \tag{14}$$

Now it is immediate that $J^r Y$ is a submanifold, and a closed subset of $J^r(X, Y)$.

(b) We have a smooth mapping

$$\operatorname{imm} T^r_n Y \ni J^r_0 \zeta \to J^r_0 \pi \circ J^r_0 \zeta = J^r_0(\pi \circ \zeta) \in T^r_n X. \tag{15}$$

Since the manifold of r-frames $F^r X$ is an open set in $T^r_n X$, the preimage of $F^r X$ in $\operatorname{imm} T^r_n Y$ by the mapping (15), i.e., the set $\operatorname{imm}_\pi T^r_n Y$, is open. If $J^r_0 \alpha \in L^r_n$ and $J^r_0 \zeta \in \operatorname{imm}_\pi T^r_n Y$, then obviously, $J^r_0 \zeta \circ J^r_0 \alpha = J^r_0(\zeta \alpha) \in \operatorname{imm}_\pi T^r_n Y$, that is, $\operatorname{imm}_\pi T^r_n Y$ is an L^r_n-invariant subset.

Since by definition, $Gr^r_n Y = \operatorname{imm} T^r_n Y / L^r_n$, the set $\operatorname{imm}_\pi T^r_n Y / L^r_n$ is obviously open in $Gr^r_n Y$, and we have the commutative diagram

$$\begin{tikzcd} \operatorname{imm}_\pi T^r_n Y \arrow[r] \arrow[d] & \operatorname{imm} T^r_n Y \arrow[d] \\ \operatorname{imm}_\pi T^r_n Y / L^r_n \arrow[r] & Gr^r_n Y \end{tikzcd} \tag{16}$$

in which the horizontal arrows are the canonical inclusions, and the vertical arrows are the quotient projections. It remains to show that $J^r Y$ can be considered as the quotient $\operatorname{imm}_\pi T^r_n Y / L^r_n$.

If $J^r_0 \zeta \in \operatorname{imm}_\pi T^r_n Y$, then the formula

$$\gamma = \zeta \circ (\pi \zeta)^{-1} \tag{17}$$

defines a section of Y over a neighborhood of $x = \pi(\zeta(0))$. Thus, we have a mapping

$$\text{(18)} \qquad \operatorname{imm}_\pi T_n^r Y \ni J_0^r \zeta \to J^r_{\pi(\zeta(0))}(\zeta \circ (\pi\zeta)^{-1}) \in J^r Y.$$

It is easily seen that this mapping is surjective, and its fibers coincide with L_n^r orbits.

Let $J_x^r \gamma \in J^r Y$ be any element. If γ is a representative of $J_x^r\gamma$, defined on a neighborhood of x, then for any chart (U, φ) at x, $\zeta = \gamma \circ \varphi^{-1} \operatorname{tr}_{-\varphi(x)}$ is an immersion of a neighborhood of $0 \in \mathbf{R}^n$ into Y, and $\pi\zeta = \varphi^{-1} \operatorname{tr}_{-\varphi(x)}$ is an immersion of a neighborhood of $0 \in \mathbf{R}^n$ into X. Thus, (U, φ) defines an element $J_x^r \zeta \in \operatorname{imm}_\pi T_n^r Y$. The mapping (18) obviously sends $J_x^r\zeta$ to $J_x^r\gamma$, proving surjectivity. Moreover, for every $J_0^r \alpha \in L_n^r$,

$$\text{(19)} \qquad \begin{aligned} J^r_{\pi(\zeta\alpha(0))}\big(\zeta\alpha \circ (\pi\zeta\alpha)^{-1}\big) &= J^r_{\pi(\zeta(0))}\big(\zeta\alpha \circ \alpha^{-1} \circ (\pi\zeta)^{-1}\big) \\ &= J^r_{\pi(\zeta(0))}\big(\zeta \circ (\pi\zeta)^{-1}\big). \end{aligned}$$

Thus, (18) is constant on L_n^r orbits. If $J^r_{\pi(\zeta(0))}(\zeta \circ (\pi\zeta)^{-1}) = J^r_{\pi(\chi(0))}(\chi \circ (\pi\chi)^{-1})$, then $J_0^r\zeta = J_0^r \chi \circ J_0^r((\pi\chi)^{-1} \circ \pi\zeta)$ proving that any two elements of a fiber of (18) belong to the same orbit.

Let Y_1 (resp. Y_2) be a fibered manifold with base X_1 (resp. X_2) and projection π_1 (resp. π_2), and let $\alpha : Y_1 \to Y_2$ be a morphism of fibered manifolds. Denote by $\alpha_0 : X_1 \to X_2$ the *projection* of α. If α_0 is a diffeomorphism, then for any section γ of Y_1, $\alpha\gamma\alpha_0^{-1}$ is a section of Y_2. We define a mapping $J^r\alpha : J^r Y_1 \to J^r Y_2$ by

$$\text{(20)} \qquad J^r\alpha(J_x^r\gamma) = J^r_{\alpha_0(x)} \alpha\gamma\alpha_0^{-1}.$$

If α is smooth then $J^r\alpha$ is also smooth. Obviously,

$$\text{(21)} \qquad \pi^{r,s} \circ J^r\alpha = J^s\alpha \circ \pi^{r,s}, \quad \pi^r \circ J^r\alpha = \alpha_0 \circ \pi^r, \quad J^r \operatorname{id}_Y = \operatorname{id}_{J^rY}.$$

for every s, $0 \le s < r$, and every fibered manifold Y.

$J^r\alpha$ is called the *r-jet prolongation,* or simply the *prolongation*, of α.

Lemma 3. *If $\alpha : Y_1 \to Y_2$ and $\beta : Y_2 \to Y_3$ are morphisms of fibered manifolds, whose projections are diffeomorphisms, then*

$$\text{(22)} \qquad J^r(\beta \circ \alpha) = J^r\beta \circ J^r\alpha.$$

Proof. This is an immediate consequence of definitions.

Remark 1. One can easily determine the chart expression of $J^r\alpha$. Consider for simplicity the case $r = 1$. Let (V, ψ), $\psi = (x^i, y^\sigma)$, (resp. $(\tilde V, \tilde\psi)$, $\tilde\psi = (\tilde x^j, \tilde y^J)$) be a fibered chart on Y_1 (resp. Y_2), let (U, φ), $\varphi = (x^i)$ (resp. $(\tilde U, \tilde\varphi)$, $\tilde\varphi = (\tilde x^j)$) be the associated chart on X_1 (resp. X_2). Assume that $\alpha(V) \subset \tilde V$. Let us denote

$$\text{(23)} \qquad \tilde x^i\alpha = f^i, \qquad \tilde y^J\alpha = F^J, \qquad x^k\alpha^{-1} = g^k.$$

Then we get

$$\text{(24)} \qquad \begin{aligned} \tilde y^J_j \circ J^1\alpha(J_x^r\gamma) &= \tilde y^J_j\big(J^1_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}\big) \\ &= D_j\big(\tilde y^J\alpha\psi^{-1} \circ \psi\gamma\varphi^{-1} \circ \varphi\alpha_0^{-1}\tilde\varphi^{-1}\big)\big(\tilde\varphi\alpha_0\varphi^{-1}(\varphi(x))\big) \\ &= D_k\big(\tilde(y)^J\alpha\psi^{-1}\big)\big(\psi(\gamma(x))\big)\, D_j\big(x^k\alpha_0^{-1}\tilde\varphi^{-1}\big)\big(\tilde\varphi(\alpha_0(x))\big) \\ &\quad + D_\sigma\big(\tilde y^J\alpha\psi^{-1}\big)\big(\psi(\gamma(x))\big)\, y^\sigma_k(J^1_x\gamma) D_j\big(x^k\alpha_0^{-1}\tilde\varphi^{-1}\big)\big(\tilde\varphi(\alpha_0(x))\big). \end{aligned}$$

Thus,

$$\tilde{y}^J_j \circ J^r\alpha = \frac{\partial F^J}{\partial x^k}\frac{\partial g^k}{\partial \tilde{x}^j} + \frac{\partial F^J}{\partial y^\sigma} y^\sigma_k \frac{\partial g^k}{\partial \tilde{x}^j} = d_k F^J \frac{\partial g^k}{\partial \tilde{x}^j}, \tag{25}$$

where d_k stands for the formal derivative operator.

Remark 2. A manifold X can naturally be viewed as a fibered manifold over X, with projection id_X. If $\gamma : U \to X$ is smooth section of this fibered manifold over an open set U, then $\gamma(x) = x$ for every $x \in U$. Thus, $\gamma = \mathrm{id}_U$. In particular, $J^r_x\gamma = J^r_x\,\mathrm{id}_U$. The mapping $J^rX \ni J^r_x\,\mathrm{id}_U \to x \in X$ is a diffeomorphism called the *canonical identification*. Using the canonical identification, we always identify J^rX with X. If $\alpha : X_1 \to X_2$ is a morphism viewed as fibered manifolds, then α is a diffeomorphism, and $J^r\alpha$ is canonically identified with α.

Remark 3. A section γ of a fibered manifold Y over X, with projection π, can naturally be viewed as a morphism of fibered manifolds. Indeed, we have the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Y \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\mathrm{id}_X} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{J^r\gamma} & J^rY \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle \pi^r} \\ X & \xrightarrow{\mathrm{id}_X} & X. \end{array} \tag{26}$$

Remark 4. Note that, with the convention of Remark 2,

$$J^r\alpha \circ J^r\gamma \circ \alpha_0^{-1} = J^r\alpha\gamma\alpha_0^{-1}. \tag{27}$$

Remark 5. Let (V, ψ) be a fibered chart on a fibered manifold Y with base X, and let (U, φ) be the associated chart on X. Then using the notation of Remark 1, and applying (22) and (Section 3.2, (10)) to ψ, we get $J^r\psi(J^r_x\gamma) = J^r_{\varphi(x)}\psi\gamma\varphi^{-1}$, i.e.,

$$\psi^r = J^r\psi. \tag{28}$$

3.4. Prolongations of fibrations. Let Y be a *fibration* with base X, projection π, and fiber Q, and consider the r-jet prolongation J^rY.

Lemma 4. *J^rY has the structure of a fibration with base X, projection π^r, and fiber T^r_nQ.*

Proof. Let (U, φ) be a chart on X, and let $\Phi : \pi^{-1}(U) \to U \times Q$ be a trivialization. Define $\tilde{\Phi}$ by the condition $\Phi(y) = (\pi(y), \tilde{\Phi}(y))$, and consider the morphism of fibered manifolds $\Phi_\varphi : \pi^{-1}(U) \to \varphi(U) \times Q$, defined by $\Phi_\varphi(y) = (\varphi(\pi(y)), \tilde{\Phi}(y))$. The r-jet prolongation $J^r\Phi_\varphi : (\pi^r)^{-1}(U) \to J^r(\varphi(U) \times Q)$ of Φ_φ is defined by $J^r\Phi^r_\varphi(J^r_x\gamma) = J^r_{\varphi(x)}\Phi_\varphi\gamma\varphi^{-1}$, where $J^r_x\gamma \in (\pi^r)^{-1}(U)$. But $\Phi_\varphi\gamma\varphi^{-1}$ is of the form $(\Phi_\varphi\gamma\varphi^{-1})(x') = (x', \Phi\gamma\varphi^{-1}(x'))$, i.e.,

$$\Phi_\varphi\gamma\varphi^{-1} = \left(\mathrm{id}_{\varphi(U)}, \tilde{\Phi}\gamma\varphi^{-1}\right), \tag{1}$$

where $\tilde{\Phi}\gamma\varphi^{-1}$ is a mapping of $\varphi(U)$ into Q. Thus, identifying $J^r_{\varphi(x)}(\Phi_\varphi\gamma\varphi^{-1})$ with the point $(\varphi(x), J^r_0(\tilde{\Phi}\gamma\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)}))$ of $\varphi(U)\times T^r_n Q$, and setting

(2) $$\Phi^r_\varphi(J^r_x\gamma) = \big(x, J^r_0(\tilde{\Phi}\gamma\varphi^{-1}\circ\mathrm{tr}_{-\varphi(x)})\big),$$

we get the commutative diagram

(3) $$\begin{array}{ccc} (\pi^r)^{-1}(U) & \xrightarrow{\Phi^r_\varphi} & U\times T^r_n Q \\ \Big\downarrow{\scriptstyle \pi^r} & & \Big\downarrow \\ U & \longrightarrow & U \end{array}$$

Φ^r_φ is obviously a trivialization.

Note that to define trivializations of J^rY, we need not only trivializations of Y, but also charts on the base X of Y. The trivialization Φ^r_φ is said to be *associated* with the pair (Φ, φ).

Remark 6. Formula (2) can be applied to special cases of prolongations of principal and associated fiber bundles.

3.5. Prolongations of Lie groups. Let G be a Lie group, and let $T^r_n G$ be the manifold of r-jets with source $O\in\mathbf{R}^n$ and target in G. Let $S, T\in T^r_n G$, $S = J^r_0 f$, $T = J^r_0 g$, be any elements. We define a group operation in $T^r_n G$ by

(1) $$S\cdot T = J^r_0(f\cdot g),$$

where $(f\cdot g)(x) = f(x)\cdot g(x)$ is defined by the group operation in G. The *unity* of $T^r_n G$ is the r-jet $e_{T^r_n G} = J^r_0 e_G$, where e_G denotes the unity of G, and also the constant mapping of $\mathbf{R}^n$ with value e_G. The *inverse* of $S = J^r_0 f$ is the r-jet $S^{-1} = J^r_0 f$, where $f^{-1}(x) = (f(x))^{-1}$, and the inversion is taken in the group G.

Denoting for a moment the group operation in G by Ψ, we can write $f\cdot g = \Psi\circ(f\times g)$. Then $S\cdot T = J^r_{(f(0),g(0))}\Psi\circ J^r_0(f\times g)$ which shows that the group operation (1) is smooth. In particular, $T^r_n G$ is a Lie group.

An element $A\in L^r_n$ defines a mapping $\varphi(A): T^r_n G\to T^r_n G$ by the formula

(2) $$\varphi(A)(S) = S\circ A^{-1}.$$

Since for every $S, T\in T^r_n G$, $A, B\in L^r_n$, $\varphi(A)(S\cdot T) = (S\cdot T)\circ A^{-1} = (S\circ A^{-1})\cdot(T\circ A^{-1}) = \varphi(A)(S)\cdot\varphi(A)(T)$ and $\varphi(A\cdot B)(S) = S\circ(A\cdot B)^{-1} = (S\circ B^{-1})\circ A^{-1} = \varphi(A)(\varphi(B)(S)) = \varphi(A)\circ\varphi(B)(S)$, $\varphi(A)$ is an automorphism of the Lie group $T^r_n G$, and the mapping $A\to\varphi(A)$ is a homomorphism of L^r_n into the group $\mathrm{aut}\,T^r_n G$ of automorphisms of $T^r_n G$. The mapping $(A, S)\to\varphi(A)(S)$ is obviously smooth. Thus, (2) defines the *exterior semi-direct product*

(3) $$G^r_n = L^r_n\times_s T^r_n G.$$

Recall that the group operation in G^r_n is given by

(4) $$(A, S)\cdot(B, T) = \big(A\cdot B, S\cdot(T\circ A^{-1})\big).$$

The Lie group G^r_n is called the (r, n)-*prolongation*, or simply the *prolongation* of G. Note that

$$(5)\qquad e_{G^r_n} = (e_{L^r_n}, e_{T^r_n G}), \quad (A, S)^{-1} = (A^{-1}, S^{-1} \circ A).$$

3.6. Prolongations of Lie group actions. Let G be a Lie group, and let Y be a *right* G-manifold. Let $p \in T^r_n Y$ and $(A, S) \in G^r_n$. If $p = J^r_0 \tau$ and $A = J^r_0 \alpha$, $S = J^r_0 \sigma$, then the representatives of these r-jets define the mapping $x \to \tau(\alpha(x)) \cdot \sigma(\alpha(x))$, whose r-jet is denoted by $(p \cdot S) \circ A$. We define

$$(1)\qquad p \cdot (A, S) = (p \cdot S) \circ A.$$

We claim that the mapping $T^r_n Y \times G^r_n \ni (p, (A, S)) \to p \cdot (A, S) \in T^r_n Y$ is a right action of G^r_n on $T^r_n Y$. Indeed, using (1) and Section 3.5, (4), we get

$$(2)\qquad \begin{aligned} p \cdot \big((A, S) \cdot (B, T)\big) &= p \cdot \big(A \cdot B, S \cdot (T \circ A^{-1})\big) \\ &= \big((p \cdot S) \circ A\big) \cdot (B, T) = \big(p \cdot (A, S)\big) \cdot (B, T). \end{aligned}$$

$T^r_n Y$ is therefore a right G^r_n-manifold called the *r-jet prolongation* of the right G-manifold Y.

Let G be a Lie group, and let Y be a *left* G-manifold. Writing $y \cdot g = g^{-1} \cdot y$ we obtain the corresponding *right* action of G on Y. Prolonging this right action, using formula (1), we obtain a right action of G^r_n on $T^r_n Y$. Our aim now will be to determine the corresponding formula for the associated *left* action of G^r_n on $T^r_n Y$.

The inverse of an element $(A, S) \in G^r_n$ is given by $(A, S)^{-1} = (A^{-1}, S^{-1} \circ A)$ (Section 3.5, (5)), Thus, $(A, S) \cdot p = p \cdot (A, S)^{-1} = (p \cdot (S^{-1} \circ A)) \circ A^{-1} = (p \circ A^{-1}) \cdot S^{-1}$. Therefore, if $A = J^r_0 \alpha$, $S = J^r_0 \sigma$, and $p = J^r_0 \tau$, then $(A, S) \cdot p$ is the r-jet of the mapping $x \to (\tau(\alpha^{-1}(x)) \cdot \sigma(x)^{-1} = \sigma(x) \cdot \tau(\alpha^{-1}(x))$ defined by the left action of G on Y. Passing to r-jets we get

$$(3)\qquad (A, S) \cdot p = S \cdot (p \circ A^{-1}).$$

$T^r_n Y$ endowed with this left action of G^r_n is called the *r-jet prolongation* of the left G-manifold Y.

3.7. Prolongations of principal bundles. Now we investigate the structure of the r-jet prolongation $J^r Y$ of a fibration Y, endowed with the structure of a principal G-bundle. We know that $J^r Y$ is a fibration with base X, and with fiber $T^r_n G$. As usual, we denote by π^r the canonical projection of $J^r Y$ onto X.

Our first aim in this section is to determine trivializations and the corresponding transition functions of $J^r Y$ (Section 3.4).

Assume that we have two charts on X, (U, φ) and (V, ψ), such that $U \cap V \neq \emptyset$, and Y is trivializable over U and V. Let $\Phi : \pi^{-1}(U) \to U \times G$ and $\Psi : V \to V \times G$ be G-*equivariant* trivializations. Φ and Ψ define smooth mappings $\tilde{\Phi} : \pi^{-1}(U) \to G$ and $\tilde{\Psi} : \pi^{-1}(V) \to G$ by

$$(1)\qquad \Phi(y) = \big(\pi(y), \tilde{\Phi}(y)\big), \qquad \Psi(y) = \big(\pi(y), \tilde{\Psi}(y)\big).$$

The transition function $\chi : U \cap V \to G$ is defined by

$$(2)\qquad \tilde{\Psi}(y) = \chi(x) \cdot \tilde{\Phi}(y),$$

where $x = \pi(y)$.

The *associated trivializations* $\Phi^r_\varphi : (\pi^r)^{-1}(U) \to U \times T^r_n G$, $\Psi^r_\psi : (\pi^r)^{-1}(V) \to V \times T^r_n G$ of $J^r Y$ are expressed by

$$\Phi^r_\varphi(J^r_x\gamma) = \big(x, \tilde{\Phi}^r_\varphi(J^r_x\gamma)\big), \qquad \Psi^r_\psi(J^r_x\gamma) = \big(x, \tilde{\Psi}^r_\psi(J^r_x\gamma)\big), \tag{3}$$

where

$$\tilde{\Phi}^r_\varphi(J^r_x\gamma) = J^r_0\big(\tilde{\Phi}\gamma\varphi^{-1} \circ \mathrm{tr}_{-\varphi(x)}\big), \qquad \tilde{\Psi}^r_\psi(J^r_x\gamma) = J^r_0\big(\tilde{\Psi}\gamma\psi^{-1} \circ \mathrm{tr}_{-\psi(x)}\big) \tag{4}$$

(see also Section 3.4 (2)). We have the following result.

Lemma 5. *The transition function between the trivializations $\tilde{\Phi}^r_\varphi$ and $\tilde{\Psi}^r_\psi$ is defined by*

$$\tilde{\Psi}^r_\psi(J^r_x\gamma) = (A, S) \cdot \tilde{\Phi}^r_\varphi(J^r_x\gamma), \tag{5}$$

where $(A, S) \in G^r_n$,

$$A = J^r_0\big(\mathrm{tr}_{\psi(x)} \circ \psi\varphi^{-1} \circ \mathrm{tr}_{-\varphi(x)}\big), \qquad S = J^r_0\big(\chi\psi^{-1} \circ \mathrm{tr}_{-\psi(x)}\big). \tag{6}$$

Proof. Writing $T = J^r_0(\tilde{\Phi}\gamma\varphi^{-1} \circ \mathrm{tr}_{-\varphi(x)})$ and $U = J^r_0(\tilde{\Psi}\gamma\psi^{-1} \circ \mathrm{tr}_{-\psi(x)})$, we have to show that $R = (A, S) \cdot T = S \cdot (T \circ A^{-1})$. But

$$\begin{aligned} T \circ A^{-1} &= J^r_0\big(\tilde{\Phi}\gamma\varphi^{-1} \circ \mathrm{tr}_{-\varphi(x)}\big) \circ J^r_0\big(\mathrm{tr}_{\varphi(x)} \circ \varphi\psi^{-1} \circ \mathrm{tr}_{-\psi(x)}\big) \\ &= J^r_0\big(\tilde{\Phi}\gamma\psi^{-1} \circ \mathrm{tr}_{-\psi(x)}\big), \end{aligned} \tag{7}$$

hence

$$\begin{aligned} S \cdot (T \circ A^{-1}) &= J^r_0\big(\chi\psi^{-1} \circ \mathrm{tr}_{-\psi(x)}\big) \cdot J^r_0\big(\tilde{\Phi}\gamma\psi^{-1} \circ \mathrm{tr}_{-\psi(x)}\big) \\ &= J^r_0\big((\chi\psi^{-1} \circ \mathrm{tr}_{-\psi(x)}) \cdot (\tilde{\Phi}\gamma\psi^{-1} \circ \mathrm{tr}_{-\psi(x)})\big) \\ &= J^r_0\big((\chi \cdot \tilde{\Phi}\gamma) \circ \psi^{-1} \circ \mathrm{tr}_{-\psi(x)}\big) = J^r_0\big(\tilde{\Psi}\gamma\psi^{-1} \circ \mathrm{tr}_{-\psi(x)}\big) = R \end{aligned} \tag{8}$$

as required.

Remark 7. Formula (5) defines the *transition function* with values in the group $G^r_n = L^r_n \times T^r_n G$,

$$\chi^r : U \cap V \ni x \to \big(J^r_0(\mathrm{tr}_{\psi(x)} \circ \psi\varphi^{-1} \circ \mathrm{tr}_{-\varphi(x)}), J^r_0(\chi\psi^{-1} \circ \mathrm{tr}_{-\psi(x)})\big) \in G^r_n. \tag{9}$$

The structure of (5) coincides with the prolongation of the left translation on the structure group G of Y (Section 3.6, (3)).

Consider the *bundle of r-frames* $F^r X$, and the fiber product

$$W^r Y = F^r X \oplus J^r Y. \tag{10}$$

$W^r Y$ is a fibration over X with fiber G^r_n. The transition functions are easily determined by means of Lemma 5, (5). If $(J^r_0\mu, J^r_x\gamma) \in W^r Y$, then $\mu(0) = x$, and if (U, φ) is a chart at x such that Y is trivializable over U, then we have a trivialization

$$\begin{aligned} (J^r_0\mu, J^r_x\gamma) &\to \big(x, (\tilde{\varphi}^r(J^r_0\mu), \tilde{\Phi}^r_\varphi(J^r_x\gamma))\big) \\ &= \big(x, (J^r_0(\mathrm{tr}_{\varphi(x)}\varphi\mu), J^r_0(\tilde{\Phi}\gamma\varphi^{-1} \circ \mathrm{tr}_{-\varphi(x)}))\big), \end{aligned} \tag{11}$$

where the same notation as in (1) is applied to $F^r X$. (11) is said to be *associated* with (Φ, φ). The corresponding transition function is expressed by the equations

$$\tilde{\psi}^r(J_0^r \mu) = A \cdot \tilde{\varphi}^r(J_0^r \mu), \qquad \tilde{\Psi}_\varphi^r(J_x^r \gamma) = S \cdot \big(\tilde{\Phi}_\varphi^r(J_x^r \gamma) \circ A^{-1}\big). \tag{12}$$

(see (5)). Note that (12) corresponds with the *left translation* on the group G_n^r (Section 3.5, (4)).

Lemma 6. *If Y is a right principal G bundle, then there exists a unique structure of a right principal G_n^r-bundle on $W^r Y$ such that all induced trivializations are G_n^r-equivariant.*

Proof. Only existence needs proof. If $(p, Z) \in W^r Y$, $(A, S) \in G_n^r$, we define

$$(p, Z) \cdot (A, S) = \big(p \cdot A, Z \cdot (S \circ p^{-1})\big). \tag{13}$$

It is directly verified that (13) is a right action of G_n^r on $W^r Y$. Indeed, if $(B, T) \in G_n^r$ is another element, we have, using the group operation in G_n^r (Section 3.5, (4)),

$$\begin{aligned} &(p, Z) \cdot \big((A, S) \cdot (B, T)\big) = (p, Z) \cdot \big(A \cdot B, S \cdot (T \circ A^{-1})\big) \\ &\quad = \big(p \cdot A \cdot B, Z \cdot ((S \circ p^{-1}) \cdot (T \circ A^{-1}) \circ p^{-1})\big). \end{aligned} \tag{14}$$

Thus,

$$\begin{aligned} &\big((p, Z) \cdot (A, S)\big) \cdot (B, T) = \big(p \cdot A, Z \cdot (S \circ p^{-1})\big) \cdot (B, T) \\ &\quad = \big(p \cdot A \cdot B, (Z \cdot (S \circ p^{-1})) \cdot T \circ (p \cdot A)^{-1}\big) \\ &\quad = (p, Z) \cdot \big((A, S) \cdot (B, T)\big). \end{aligned} \tag{15}$$

Let X be the base, and let π be the projection of Y. Assume that we have a chart (U, φ) on X, and a trivialization $\Phi : \pi^{-1}(U) \to U \times G$, and consider the associated trivialization (11) of $W^r Y$. This trivialization sends $(p, Z) = (J_0^r \mu, J_x^r \gamma)$ to $(x, (J_0^r \operatorname{tr}_{\varphi(x)} \varphi\mu, J_0^r(\tilde{\Phi}\gamma\varphi^{-1} \circ \operatorname{tr}_{-\varphi(x)})))$, and the element

$$\begin{aligned} &(p, Z) \cdot (A, S) = (J_0^r \mu, J_x^r \gamma) \cdot (J_0^r \alpha, J_0^r \sigma) = \big(p \cdot A, Z \cdot (S \circ p^{-1})\big) \\ &\quad = \big(J_0^r \mu\alpha, J_x^r(\gamma \cdot (\sigma\mu^{-1})\big) \end{aligned} \tag{16}$$

is sent to

$$\big(x, (J_0^r \operatorname{tr}_{\varphi(x)} \varphi\mu\alpha, J_0^r(\tilde{\Phi} \circ (\gamma \cdot (\sigma\mu^{-1})) \circ \varphi^{-1} \circ \operatorname{tr}_{-\varphi(x)}))\big). \tag{17}$$

The first component yields

$$J_0^r \operatorname{tr}_{\varphi(x)} \varphi\mu\alpha = J_0^r \operatorname{tr}_{\varphi(x)} \varphi\mu \circ J_0^r \alpha = J_0^r \operatorname{tr}_{\varphi(x)} \varphi\mu \circ A. \tag{18}$$

Consider the second component. We have at a point t, because $\tilde{\Phi}$ is G-equivariant,

$$\begin{aligned} &\tilde{\Phi} \circ \big(\gamma \cdot (\sigma\mu^{-1})\big) \circ \varphi^{-1} \circ \operatorname{tr}_{-\varphi(x)}(t) \\ &\quad = \tilde{\Phi}\big((\gamma\varphi^{-1} \operatorname{tr}_{-\varphi(x)})(t) \cdot (\sigma\mu^{-1}\varphi^{-1} \operatorname{tr}_{-\varphi(x)})(t)\big) \\ &\quad = \tilde{\Phi}\big(\gamma\varphi^{-1} \operatorname{tr}_{-\varphi(x)}(t)\big) \cdot (\sigma\mu^{-1}\varphi^{-1} \operatorname{tr}_{-\varphi(x)})(t). \end{aligned} \tag{19}$$

Consequently,

$$
(20)\qquad
\begin{aligned}
&J_0^r\big(\tilde{\Phi}\circ(\gamma\cdot(\sigma\mu^{-1}))\circ\varphi^{-1}\circ \mathrm{tr}_{-\varphi(x)}\big)\\
&\quad= J_0^r\tilde{\Phi}((\gamma\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)})\cdot(\sigma\mu^{-1}\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)}))\\
&\quad= J_0^r\tilde{\Phi}\gamma\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)}\cdot(J_0^r\sigma\circ J_0^r(\mu^{-1}\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)}))\\
&\quad= J_0^r\tilde{\Phi}\gamma\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)}\cdot(S\circ J_0^r(\mu^{-1}\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)})).
\end{aligned}
$$

Altogether, (18) is expressed, with the help of the group operation in Gr_n^r, by

$$
(21)\qquad
\begin{aligned}
&\big(x,(J_0^r\,\mathrm{tr}_{\varphi(x)}\,\varphi\mu\alpha,\ J_0^r(\tilde{\Phi}\circ(\gamma\cdot(\sigma\mu^{-1}))\circ\varphi^{-1}\circ\mathrm{tr}_{-\varphi(x)}))\big)\\
&\quad=\big(x,(J_0^r\,\mathrm{tr}_{\varphi(x)}\,\varphi\mu\circ A,\ J_0^r\tilde{\Phi}\gamma\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)}\cdot(S\circ J_0^r(\mu^{-1}\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)})))\big)\\
&\quad=\big(x,(J_0^r\,\mathrm{tr}_{\varphi(x)}\,\varphi\mu,\ J_0^r(\tilde{\Phi}\gamma\varphi^{-1}\,\mathrm{tr}_{-\varphi(x)}))\cdot(A,S)\big).
\end{aligned}
$$

Thus, the associated trivialization is G_n^r-equivariant with respect to the group action (13).

W^rY is called the *r-th principal prolongation* of the principal G-bundle Y.

3.8. Principal prolongations of frame bundles. Consider the bundle of s-frames F^sX over an n-dimensional manifold X (Section 2.4). The principal prolongation W^rF^sX (10) is a right principal $(L_n^s)_n^r$-bundle, where $(L_n^s)_n^r = L_n^r\times_s T_n^rL_n^s$ is the r-the principal prolongation of L_n^s. We show that $W^rF^sX = F^sX\oplus J^rF^sX$ is reducible to the bundle of frames $F^{r+s}X$.

Let r, and s be positive integers, and consider an $(r+s)$-jet $A\in L_n^{r+s}$, $A=J_0^{r+s}\alpha$. The representative $\alpha : U\to\mathbf{R}^n$ of A defines the morphism

$$
(1)\qquad U_x\ni t\to\alpha_x(t)=\big(\mathrm{tr}_x\circ\alpha\circ\mathrm{tr}_{-\alpha^{-1}(x)}\big)(t)\in\mathbf{R}^n,
$$

where U_x is a neighborhood of the origin $0\in\mathbf{R}^n$. Obviously, $\alpha_x(0)=0$, which implies that for every $x\in U$,

$$
(2)\qquad \alpha^{(s)}(x)=J_0^s\alpha_x,
$$

is an element of the differential group L_n^s. Thus, formulas (1) and (2) define a mapping $U\ni x\to\alpha^{(s)}(x)\in L_n^s$. By the chain rule,

$$
(3)\qquad
\begin{aligned}
&D\alpha_x(t)=D\,\mathrm{tr}_x\big((\alpha\circ\mathrm{tr}_{-\alpha^{-1}(x)})(t)\big)\circ D\alpha\big((\mathrm{tr}_{-\alpha^{-1}(x)})(t)\big)\circ D\,\mathrm{tr}_{\alpha^{-1}(x)}(t)\\
&\quad=\big(D\alpha\circ\mathrm{tr}_{-\alpha^{-1}(x)}\big)(t),\\
&D^2\alpha_x(t)=D^2\alpha\big((\mathrm{tr}_{-\alpha^{-1}(x)}(t)\big)\circ D\,\mathrm{tr}_{-\alpha^{-1}(x)}(t)=\big(D^2\alpha\circ\mathrm{tr}_{-\alpha^{-1}(x)}\big)(t),\\
&\dots\\
&D^s\alpha_x(t)=D\big(D^{s-1}\alpha\circ\mathrm{tr}_{-\alpha^{-1}(x)}\big)(t)=D^s\alpha\big((\mathrm{tr}_{-\alpha^{-1}(x)}(t)\big)\circ D\,\mathrm{tr}_{-\alpha^{-1}(x)}(t)\\
&\quad=\big(D^s\alpha\circ\mathrm{tr}_{-\alpha^{-1}(x)}\big)(t),
\end{aligned}
$$

hence

$$
(4)\qquad
\begin{aligned}
&D\alpha_x(0)=D\alpha\big(\alpha^{-1}(x)\big),\ D\alpha_x(0)=D^2\alpha\big(\alpha^{-1}(x)\big),\ \dots,\\
&\quad D^s\alpha_x(t)=D^s\alpha\big(\alpha^{-1}(x)\big).
\end{aligned}
$$

Thus, we get a smooth mapping $U \ni x \to \alpha^{(s)}(x) \in L^s_n$, whose coordinate expression is determined by (4).

Analogously, let $p \in F^{r+s}X$, $J^{r+s}_0\mu \in F^{r+s}X$. The representative μ of p defines the morphism

$$U_x \ni t \to \mu_x(t) = \left(\mu \circ \mathrm{tr}_{-\mu^{-1}(x)}\right)(t) = \mu\left(t + \mu^{-1}(x)\right) \in \mathbf{R}^n, \tag{5}$$

where U_x is a neighborhood of the origin $0 \in \mathbf{R}^n$. Obviously, $\mu_x(0) = x$, which implies that for every $x \in U$,

$$\mu^{(s)}(x) = J^s_0\mu_x. \tag{6}$$

is an s-frame at $x \in X$. Thus, (5) and (6) define a mapping $U \ni x \to \mu^{(s)}(x) \in F^sX$.

Theorem 4. *Let X be an n-dimensional manifold.*

(a) *The mapping*

$$L^{r+s}_n \ni J^{r+s}_0\alpha \to \nu\left(J^{r+s}_0\alpha\right) = \left(J^r_0\alpha, J^r_0\alpha^{(s)}\right) \in (L^s_n)^r_n. \tag{7}$$

is a morphism of Lie groups, and an injective immersion. The set $\nu(L^{r+s}_n)$ is closed in $(L^s_n)^r_n$.

(b) *The mapping*

$$F^{r+s}X \ni J^{r+s}_0\mu \to \nu_X\left(J^{r+s}_0\mu\right) = \left(J^r_0\mu, J^r_{\mu(0)}\mu^{(s)}\right) \in W^rF^sX \tag{8}$$

is a ν-morphism of principal bundles, and an injective immersion.

Proof. (a) If $J^{r+s}_0\alpha$, $J^{r+s}_0\beta \in L^{r+s}_n$, then

$$\nu\left(J^{r+s}_0\alpha \circ J^{r+s}_0\beta\right) = \nu\left(J^{r+s}_0\alpha\beta\right) = \left(J^r_0\alpha\beta, J^r_0(\alpha\beta)^{(s)}\right). \tag{9}$$

But

$$\begin{aligned}(\alpha\beta)^{(s)}(x) &= J^s_0(\alpha\beta)_x = J^s_0\left(\mathrm{tr}_x \circ \alpha\beta \circ \mathrm{tr}_{-(\alpha\beta)^{-1}(x)}\right)\\ &= J^s_0\left((\mathrm{tr}_x \circ \alpha \circ \mathrm{tr}_{-\alpha^{-1}(x)})\right) \circ J^s_0\left(\mathrm{tr}_{\alpha^{-1}(x)}\,\beta \circ \mathrm{tr}_{-\beta^{-1}(\alpha^{-1}(x))}\right)\\ &= J^s_0\alpha_x \circ J^s_0\beta_{\alpha^{-1}(x)} = \alpha^{(s)}(x) \circ \beta^{(s)}\left(\alpha^{-1}(x)\right).\end{aligned} \tag{10}$$

thus

$$\nu\left(J^{r+s}_0\alpha \circ J^{r+s}_0\beta\right) = \left(J^r_0\alpha\beta, J^r_0\alpha^{(s)} \circ J^r_0(\beta^{(s)} \circ \alpha^{-1})\right). \tag{11}$$

On the other hand, setting $(A, S) = (J^r_0\alpha, J^r_0\alpha^{(s)})$, $(B, T) = (J^r_0\beta, J^r_0\beta^{(s)})$, and multiplying these elements in $(L^s_n)^r_n$, we get

$$\begin{aligned}(A, S) \cdot (B, T) &= \left(A \cdot B, S \cdot (T \circ A^{-1})\right)\\ &= \left(J^r_0\alpha\beta, J^r_0\alpha^{(s)} \circ (J^r_0\beta^{(s)} \circ J^r_0\alpha^{-1})\right).\end{aligned} \tag{12}$$

Since (11) and (12) coincide, we see that ν is a group morphism. Since ν is smooth, it is a morphism of Lie groups.

We find the chart expression of ν in the canonical coordinates. To this purpose we use the *second canonical coordinates* $b^i_{j_1 j_2 \cdots j_k}$ on L^{r+s}_n (Section 2.1, Remark 1), and the *second canonical coordinates* $b^i_{j_1 j_2 \cdots j_k, p_1 p_2 \cdots p_l}$ on $(L^s_n)^r_n$ defined as follows. Recall that if $A \in L^{r+s}_n$, $A = J^{r+s}_0 \alpha$, we have

$$b^i_{j_1 j_2 \cdots j_k}(A) = a^i_{j_1 j_2 \cdots j_k}(A^{-1}) = D_{j_1} D_{j_2} \cdots D_{j_k} (\alpha^{-1})^i (0), \tag{13}$$

where $a^i_{j_1 j_2 \cdots j_k}$ are the *first canonical coordinates* on L^{r+s}_n, and in components, $\alpha^{-1} = ((\alpha^{-1}), (\alpha^{-1})^2, \dots, (\alpha^{-1})^n)$. If $S \in (L^s_n)^r_n$, $S = J^r_n \eta$, we set

$$\begin{aligned} &b^i_{j_1 j_2 \cdots j_k, p_1 p_2 \cdots p_l}(S) = D_{p_1} D_{p_2} \cdots D_{p_k} \big(b^i_{j_1 j_2 \cdots j_k} \circ \eta\big)(0), \\ &\quad 1 \le k \le s, 0 \le l \le r. \end{aligned} \tag{14}$$

Then by definition, for every $A \in L^{r+s}_n$, $A = J^{r+s}_0 \alpha$

$$\begin{aligned} b^i_{j_1 j_2 \cdots j_k, p_1 p_2 \cdots p_l}\big(\nu(A)\big) &= D_{p_1} D_{p_2} \cdots D_{p_k} \big(J^r_0 \alpha^{(s)}\big) \\ &= D_{p_1} D_{p_2} \cdots D_{p_k} \big(b^i_{j_1 j_2 \cdots j_k} \circ \alpha^{(s)}\big)(0), \end{aligned} \tag{15}$$

where

$$\alpha^{(s)}(x) = J^s_0 \alpha_x, \qquad \alpha_x = \mathrm{tr}_x \, \alpha \, \mathrm{tr}_{-\alpha^{-1}(x)} \, . \tag{16}$$

Thus,

$$\big(b^i_{j_1 j_2 \cdots j_k} \circ \alpha^{(s)}\big)(x) = b^i_{j_1 j_2 \cdots j_k}\big(J^s_0 \alpha_x\big) = a^i_{j_1 j_2 \cdots j_k}\big(J^s_0 \alpha^{-1}_x\big). \tag{17}$$

But $\alpha^{-1}_x = \mathrm{tr}_{\alpha^{-1}(x)} \, \alpha^{-1} \, \mathrm{tr}_{-x}$, so we have, computing the canonical coordinates $a^i_{j_1 j_2 \cdots j_k}$ $(J^s_0 \alpha^{-1}_x)$,

$$\begin{aligned} &D_{j_1} (\alpha^{-1}_x)^i (t) \\ &\quad = D_p (\mathrm{tr}_{\alpha^{-1}(x)})^i \big(\alpha^{-1} \, \mathrm{tr}_{-x}(t)\big) D_q (\alpha^{-1})^p \big(\mathrm{tr}_{-x}(t)\big) \, D_{j_1} (\mathrm{tr}_{-x})^q (t) \\ &\quad = \delta^i_p D_q (\alpha^{-1})^p \big(\mathrm{tr}_{-x}(t)\big) \, \delta^q_{j_1} = \big(D_{j_1} (\alpha^{-1})^i \circ \mathrm{tr}_{-x}\big)(t), \\ &D_{j_1} D_{j_2} \alpha^{-1}_x (t) = D_p D_{j_1} \big(\alpha^{-1} (\mathrm{tr}_{-x}(t)\big) \, D_{j_2} (\mathrm{tr}_{-x})^p (t) \\ &\quad = \big(D_{j_1} D_{j_2} (\alpha^{-1})^i \circ \mathrm{tr}_{-x}\big)(t) \\ &\dots \\ &D_{j_1} D_{j_2} \cdots D_{j_s} \alpha^{-1}_x (t) = D_p D_{j_1} D_{j_2} (\alpha^{-1})^i \big(\mathrm{tr}_{-x}(t)\big) D_{j_s} (\mathrm{tr}_{-x})^p (t) \\ &\quad = \big(D_{j_1} D_{j_2} \cdots D_{j_s} (\alpha^{-1})^i \circ \mathrm{tr}_{-x}\big)(t). \end{aligned} \tag{18}$$

Setting $t = 0$, we obtain

$$\begin{aligned} \big(b^i_{j_1 j_2 \cdots j_k} \circ \alpha^{(s)}\big)(x) &= \big(D_{j_1} D_{j_2} \cdots D_{j_k} (\alpha^{-1})^i \circ \mathrm{tr}_{-x}\big)(0) \\ &= D_{j_1} D_{j_2} \cdots D_{j_k} (\alpha^{-1})^i (x). \end{aligned} \tag{19}$$

Now we are in a position to determine (15). We have

$$\begin{aligned} \big(b^i_{j_1 j_2 \cdots j_k, p_1 p_2 \cdots p_l} \circ \nu\big)(A) &= D_{p_1} D_{p_2} \cdots D_{p_k} \big(b^i_{j_1 j_2 \cdots j_k} \circ \alpha^{(s)}\big)(0) \\ &= D_{p_1} D_{p_2} \cdots D_{p_k} D_{j_1} D_{j_2} \cdots D_{j_k} (\alpha^{-1})^i (0) = b^i_{j_1 j_2 \cdots j_k, p_1 p_2 \cdots p_l}(A). \end{aligned} \tag{20}$$

This is the desired chart expression for ν.

Now it is trivial to conclude that ν is an injective immersion, and $\nu(L^{r+s}_n)$ is a closed subset of $T^r_n L^s_n$. Replace the canonical coordinates on $T^r_n L^s_n$ by new coordinates

$s^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l}$, $t^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l}$ where $s^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l}$ are defined by symmetrization of $b^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l}$ in the subscripts, and $t^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l}$ are defined by

$$b^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l} = s^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l} + t^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l}. \tag{21}$$

Then (20) is equivalent with the equations

$$s^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l} \circ \nu = b^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l}, \; t^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l} \circ \nu = 0, \tag{22}$$

and the set $\nu(L^{r+s}_n) \subset T^r_n L^s_n$ is expressed by the equations

$$t^i_{j_1 j_2\cdots j_k, p_1 p_2\cdots p_l} = 0, \tag{23}$$

so is obviously closed.

(b) If $J^{r+s}_0 \mu \in F^{r+s}_n X$, and $J^{r+s}_0 \alpha \in L^{r+s}_n$, we have

$$\begin{aligned} \nu_X\big(J^{r+s}_0 \mu \cdot J^{r+s}_0 \alpha\big) &= \big(J^r_0(\mu \circ \alpha), J^r_{(\mu\circ\alpha)(0)}(\mu \circ \alpha)^{(s)}\big) \\ &= (J^r_0 \mu \circ J^r_0 \alpha, J^r_{\mu(0)}(\mu \circ \alpha)^{(s)}). \end{aligned} \tag{24}$$

But by (6) and (5), $(\mu \circ \alpha)^{(s)}(x) = J^s_0(\mu \circ \alpha)_x$ and

$$\begin{aligned} (\mu \circ \alpha)_x &= \mu \circ \alpha \circ \mathrm{tr}_{-(\mu\circ\alpha)^{-1}(x)} \\ &= \mu \circ \mathrm{tr}_{-\mu^{-1}(x)} \circ \mathrm{tr}_{\mu^{-1}(x)} \alpha \, \mathrm{tr}_{-\alpha^{-1}(\mu^{-1}(x))} = \mu_x \circ \alpha_{\mu^{-1}(x)}, \end{aligned} \tag{25}$$

so that

$$\begin{aligned} (\mu \circ \alpha)^{(s)}(x) &= J^r_0\big(\mu_x \circ \alpha_{\mu^{-1}(x)}\big) = J^r_0 \mu_x \circ J^r_0 \alpha_{\mu^{-1}(x)} \\ &= \mu(s)(x) \cdot \alpha^{(s)}\big(\mu^{-1}(x)\big), \end{aligned} \tag{26}$$

where the dot is used for the group operation in L^s_n (in fact, this is the composition of jets). Thus, passing to r-jets, we get

$$\begin{aligned} J^r_{\mu(0)}(\mu \circ \alpha)^{(s)} &= J^r_{\mu(0)}\big(\mu^{(s)} \cdot (\alpha^{(s)} \circ \mu^{-1})\big) \\ &= J^r_{\mu(0)}\mu^{(s)} \cdot J^r_{\mu(0)}\big(\alpha^{(s)} \circ \mu^{-1}\big) = J^r_{\mu(0)}\mu^{(s)} \cdot \big(J^r_0 \alpha^{(s)} \circ J^r_{\mu(0)}\mu^{-1}\big). \end{aligned} \tag{27}$$

To summarize, denote $A = J^{r+s}_0 \alpha$, $p = J^{r+s}_0 \mu$. Then, $\nu(A) = (J^r_0 \alpha, J^r_0 \alpha^{(s)})$, and, $\nu_X(p) = (J^r_0 \mu, J^r_{\mu(0)}\mu^{(s)})$. We have

$$\nu_X(p \cdot A) = \big(J^r_0 \mu \circ J^r_0 \alpha, J^r_{\mu(0)}\mu^{(s)} \cdot (J^r_0 \alpha(s) \circ J^r_{\mu(0)}\mu^{-1})\big). \tag{28}$$

On the other hand,

$$\begin{aligned} \nu_X(p) \cdot \nu(A) &= (J^r_0 \mu, J^r_{\mu(0)}\mu^{(s)}) \cdot (J^r_0 \alpha, J^r_0 \alpha^{(s)}) \\ &= \big(J^r_0 \mu \cdot J^r_0 \alpha, J^r_{\mu(0)}\mu^{(s)} \cdot (J^r_0 \alpha^{(s)} \circ J^r_{\mu(0)}\mu^{-1})\big) = \nu_X(p \cdot A), \end{aligned} \tag{29}$$

which proves (b).

Corollary 1. *L^{r+s}_n is a Lie subgroup of $(L^s_n)^r_n$.*

3.9. Prolongations of associated bundles. Now we study the structure of r-jet prolongations of associated fiber bundles. To this purpose we construct a *frame mapping* for the prolonged fiber bundles by means of a frame mapping on the initial fiber bundles.

Theorem 5. *If Y_Q is a fiber bundle with fiber Q, associated with a principal G-bundle Y, then the r-prolongation J^rY_Q has the structure of a fiber bundle with fiber T^r_nQ, associated to the principal G^r_n-bundle W^rY.*

Proof. Assume that we have a *frame mapping* $\rho : Y \times Q \to Y_Q$. We want to construct a frame mapping $\rho^r : W^rY \times T^n_nQ \to J^rY_Q$.

Let X be the base of Y_G, and let $x_0 \in X$ be a point. Let $(p, Z) \in W^rY$, $p = J^r_0\mu$, $Z = J^r_{x_0}\gamma$ and $q \in T^n_nQ$, $q = J^r_0\zeta$. Since $W^rY = F^rX \oplus J^rY$, we have $\mu(0) = x_0$. Let U be a neighborhood of x_0. Assume that U is chosen in such a way that the representatives $\mu : \mu^1(U) \to X$, $\gamma : U \to Y$, and $\zeta : \mu^{-1}(U) \to Q$ are defined. These representatives define a *section* $\delta : U \to Y_Q$ by

$$\delta = \rho \circ (\gamma \times \zeta\mu^{-1}). \tag{1}$$

Then the r-jet $J^r_{x_0}\delta$ depends only on $J^r_{x_0}\gamma$, $J^r_{x_0}\mu^{-1}$, and $J^r_0\zeta$, i.e., on Z, p, and q. We define a mapping $\rho^r : W^rY \times T^r_nQ \to J^rY$ by

$$\rho^r\big((p, Z), q\big) = J^r_{x_0}\big(\rho \circ (\gamma \times \zeta\mu^{-1})\big). \tag{2}$$

We claim that ρ^r is a frame mapping. If $(A, S) \in G^r_n$, $A = J^r_0\alpha$, $S = J^r_0\sigma$ and $q \in T^n_nQ$, $q = J^r_0\zeta$, we have, by Section 3.7, (13), $(p, Z)\cdot(A, S) = (p\cdot A, Z\cdot(S\circ p^{-1}))$. By Section 3.5, (5), $(A, S)^{-1} = (A^{-1}, S^{-1} \circ A)$ and by Section 3.6, (3), $(A, S) \cdot q = S \cdot (q \circ A^{-1})$, and $(A, S)^{-1} \cdot q = (A^{-1}, S^{-1} \circ A) \cdot q = (S^{-1} \circ A) \cdot q \circ A)$. Thus,

$$\begin{aligned} &\rho^r\big((p, Z) \cdot (A, S), (A, S)^{-1} \cdot q\big) \\ &\quad = \rho^r\big((p \cdot A, Z \cdot (S \circ p^{-1})), (S^{-1} \cdot q) \circ A\big). \end{aligned} \tag{3}$$

But $p\cdot A = J^r_0\mu\alpha$, $Z\cdot(S\circ p^{-1}) = J^r_{x_0}(\gamma\cdot(\sigma\mu^{-1}))$, and $(S^{-1}\cdot q)\circ A = J^r_0((\sigma^{-1}\cdot\zeta)\circ\alpha)$, so that, using (2),

$$\begin{aligned} &\rho^r\big((p, Z) \cdot (A, S), (A, S)^{-1} \cdot q\big) \\ &\quad = \rho^r\big((p \cdot A, Z \cdot (S \circ p^{-1})), (S^{-1} \cdot q) \circ A\big) \\ &\quad = J^r_{x_0}(\rho \circ (\gamma \cdot (\sigma\mu^{-1})) \times (\sigma^{-1} \cdot \zeta) \circ \mu^{-1}). \end{aligned} \tag{4}$$

But

$$\begin{aligned} &\big(\rho \circ (\gamma \cdot (\sigma\mu^{-1})) \times (\sigma^{-1} \cdot \zeta) \circ \mu^{-1}\big)(x) \\ &\quad = \rho\big(\gamma(x) \cdot \sigma\mu^{-1}(x), \sigma^{-1}\mu^{-1}(x) \cdot \zeta\mu^{-1}(x)\big) \\ &\quad = \rho\big((\gamma(x), \zeta\mu^{-1}(x)\big) = \rho \circ (\gamma \times \zeta\mu^{-1})(x), \end{aligned} \tag{5}$$

because ρ is the frame mapping for Y_Q. Therefore, we get finally,

$$\begin{aligned} &\rho^r\big((p, Z) \cdot (A, S), (A, S)^{-1} \cdot q\big) = J^r_{x_0}\big(\rho \circ (\gamma \times \zeta\mu^{-1})\big) \\ &\quad = \rho^r\big((p, Z), q\big). \end{aligned} \tag{6}$$

This proves that ρ^r is a frame mapping.

References

[1] J. Dieudonné, *Treatise on Analysis*, Vol. III, Academic Press, New York–London, 1972.

[2] N. Bourbaki, *Variétés Différentielles et Analytiques, Fascicule de Résultats*, Hermann, Paris, 1967, 1971.

[3] C. Ehresmann, Les prolongements d'une variete différentiable I–V, C. R. Acad. Sc. Paris 223 (1951), 598–600, 777–779, 1081–1083, 234 (1952), 1028–1030, 1424–1426.

[4] D. R. Grigore, D. Krupka, Invariants of velocities and higher-order Grassmann bundles, Journ. Geom. Phys. 24 (1998) 244–264.

[5] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag Berlin Heidelberg, 1993.

[6] D. Krupka, A setting for generally invariant Lagrangian structures in tensor bundles, Bull. Acad. Polon. Sci., Sér Math. Astronom. Phys. 22 (1974) 967–972.

[7] D. Krupka, *Natural Lagrangian Structures* (Semester on Differential Geometry, Banach Center, Warsaw, 1979), Banach Center Publications 12 (1984) 185–210.

[8] D. Krupka, J. Janyška, *Lectures on Differential Invariants*, J. E. Purkyně University, Brno, Czechia, 1990.

[9] D. Krupka, M. Krupka, Elements of the Theory of Jets I., preprint series in Global Analysis GA 4 / 1999, Mathematical Institute, Silesian University in Opava, 1999.

[10] D. Saunders, *The Geometry of Jet Bundles*, Cambridge Univ. Press, 1989.

[11] A. M. Vinogradov, I. S. Krasil'shchik, V. V. Lychagin, *Introduction to the Geometry of Non-linear Differential Equations* (Russian), Nauka, Moscow, 1986.

Demeter Krupka
Mathematical Institute
Silesian University in Opava
Bezručovo nám. 13, 746 01 Opava
Czech Republic
E-mail: Demeter.Krupka@math.slu.cz

Received 17 July 2000